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The solutions problems of nonlinear differential equations

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Abstract

In this paper,we investigate the solutions problems of certain type of nonlinear differential equations.The results which we obtain improve the earlier results of Ping Li and Chung-chun Yang.

Keywords: *Differential equation, Transcendental entire solution, differential polynomial.*

1 Introduction

Currently, many scholars study the growth, oscillation, solvability and existence of entire or meromorphic solutions of differential equations in complex domains. Li and Yang mainly study the Existence or Non-Existence of Solutions for nonlinear differential equations. Furthermore, Li and Yang get the following general results, please see[1,2]:

Theorem A. Let $n \geq 4$ be an integer and $P_d(f)$ denote an algebraic differential polynomial f of degree $d \leq n-3$. Let p_1, p_2 be two nonzero polynomials, α_1 and α_2 be two nonzero constants with $\alpha_1/\alpha_2 \neq rational$. Then the differential equation

$$f^n(z) + P_d(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \tag{1}$$

has no transcendental entire solutions.

Theorem B. Let $n \geq 2$ be a positive integer. Let f be a transcendental entire function, $P(f)$ be a differential polynomial in f of degree $\leq n-1$. If

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{2}$$

where $p_i (i = 1, 2)$ are nonvanishing small functions of e^z, $\alpha_i (i = 1, 2)$ are positive numbers satisfying $(n-1)\alpha_2 \geq n\alpha_1 > 0$, then there exists a small function γ of f such that

$$(f - \gamma)^n = p_2 e^{\alpha_2 z}. \tag{3}$$

2 Preliminaries

Let $f(z)$ be a non-constant transcendental meromorphic function in the whole complex plane. We assume familiarity with the Nevanlinna's theory of meromorphic functions and the standard notations such as$T(r, g), N(r, f)$ and $m(r, f)$and so on. Throughout the paper, $S(r, f)$ will be used to denoted any quantity that satisfies $S(r, f) = o(1)T(r, f)$ as $r \to \infty$, outside possibly an exceptional set of r values of finite linear measure. For references, please see[3,4].

Lemma 1(see[2]). Let f be a transcendental meromorphic function, k is a arbitrary positive integer, then

$$m(r, \frac{f^{(k)}}{f}) = S(r, f). \tag{4}$$

Lemma 2(see[5]). Let $f(z)$ be a transcendental meromorphic function, and

$$f^n(z)P(f) = Q(f),$$

where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f)$ is at most n. Then

$$m(r, P(f)) = S(r, f).$$

Lemma 3(see[6]). Let $f(z)$ be a nonconstant meromorphic function,and $F = f^n + Q(f)$, where $Q(f)$ is a differential polynomial in f with degree $\leq n-1$,if $N(r, f) + N(r, 1/F) = S(r, f)$, then

$$F = (f + \gamma)^n,$$

where γ is meromorphic and $T(r, \gamma) = S(r, f)$.

Lemma 4(see[7]) Suppose that h is a meromorphic function which satisfies $\bar{N}(r, \frac{1}{h}) + \bar{N}(r, h) = S(r, h)$. Let $f = a_0 h^p + a_1 h^{p-1} + ... + a_p$ and $g = b_o h^q + b_1 h^{q-1} + ... + b_q$ be polynomials in h with all coefficients being small functions of h and $a_0 b_0 a_p \neq 0$, if $q \leq p$,then $m(r, g/f) = S(r, h)$.

3 Main results

In this paper, we will discuss the solutions of nonlinear differential equations $f^n(z) + P(f) = fe^{\alpha_1 z} + fe^{\alpha_2 z}$, where $P(f)$ be a differential polynomial in f of degree $\leq n-1$. We prove the following results which are improvement or complementarity of Theorems A and B.

Theorem 1. Let $n \geq 4$ be an integer and $P(f)$ be a differential polynomial in f of degree $d \leq n-3$. Let f be a transcendental entire function, α_1 and α_2 be two nonzero constants with $\alpha_1/\alpha_2 \neq d/n$, and $\alpha_1/\alpha_2 = s/t$, where s and t are positive integers satisfying $s/t \geq d/n - 1$. Then the differential equation

$$f^n(z) + P(f) = fe^{\alpha_1 z} + fe^{\alpha_2 z} \tag{5}$$

has no transcendental entire solutions.

proof . Let f be a transcendental entire solution of (5). By differentiating both sides of Eq.(5), we have

$$nf^{n-1}f' + (P(f))' = (f' + f\alpha_1)e^{\alpha_1 z} + (f' + f\alpha_2)e^{\alpha_2 z}. \tag{6}$$

By eliminating $e^{\alpha_1 z}$ from (5) and (6), we have

$$(f' + f\alpha_1)f^n - nf^nf' + G(f) = re^{\alpha_2 z}, \tag{7}$$

and

$$(f' + f\alpha_2)f^n - nf^nf' + H(f) = -re^{\alpha_1 z}, \tag{8}$$

where

$$r = (\alpha_1 - \alpha_2)f^2, \tag{9}$$

$$G = (f' + f\alpha_1)P - fP', \tag{10}$$

$$H = (f' + f\alpha_2)P - fP'. \tag{11}$$

By differentiating (7), we have

$$(f' + f\alpha_1)'f^n + n(f' + f\alpha_1)f^{n-1}f' - n^2f^{n-1}(f')^2 - nf^nf'' + G' = (r' + r\alpha_2)e^{\alpha_2 z}. \tag{12}$$

By eliminating $e^{\alpha_2 z}$ from (7) and (12), we get

$$f^{n-1}\{[(r'+r\alpha_2)(f'+f\alpha_1) - r(f'+f\alpha_1)']f - n(r'+r\alpha_2)ff' - rn(f'+f\alpha_1)f' + rn^2(f')^2 + rnff''\}$$
$$= rG' - (r' + r\alpha_2)G. \tag{13}$$

Set

$$\varphi = [(r'+r\alpha_2)(f'+f\alpha_1) - r(f'+f\alpha_1)']f - n(r'+r\alpha_2)ff' - rn(f'+f\alpha_1)f' + rn^2(f')^2 + rnff'', \tag{14}$$

$$\phi = rG' - (r' + r\alpha_2)G. \tag{15}$$

By Lemma 2, we have $m(r, \varphi) = S(r, f)$. Therefore,$T(r, \varphi) = S(r, f)$.

Case 1. If $\varphi \equiv 0$,then from (13)and (14), we get

$$rG' - (r' + r\alpha_2)G = 0. \tag{16}$$

If $G \neq 0$, from (16), we have

$$\frac{G'}{G} = \frac{r'}{r} + \alpha_2.$$

Therefore

$$G = c_1 r e^{\alpha_2 z}, \tag{17}$$

where c_1 is a nonzero constant. It follows from (7) and (17)that

$$f^n[(f' + f\alpha_1) - nf'] + (1 - \frac{1}{c_1})G = 0. \tag{18}$$

Set

$$\omega = (f' + f\alpha_1) - nf'. \tag{19}$$

By using Lemma 2, we have

$$m(r, \omega) = S(r, f). \tag{20}$$

From (19) and (20), we get

$$m(r, \frac{1}{f}) = S(r, f),$$

So

$$T(r, f) \leq S(r, f),$$

which contradicts. Hence, $G \equiv 0$. From (10), we get

$$(f' + f\alpha_1)P - fP' = 0. \tag{21}$$

From (21), we obtain

$$P = c_2 f e^{\alpha_1 z}. \tag{22}$$

From (5) and (22),we get

$$f^n + (1 - \frac{1}{c_2})P = fe^{\alpha_2 z}.$$

By Lemma 3, we can derive that

$$f^n = fe^{\alpha_2 z}. \tag{23}$$

Thus $c_2 = 1$, and

$$P = fe^{\alpha_1 z}.$$

This time we can deduce that $\alpha_1/\alpha_2 = d/n$, which contradicts the assumption.

Case 2. If $\varphi \neq 0$, since $\alpha_1/\alpha_2 = s/t$, it follows from (7) and (8) that

$$f^{sn}[(f^{'} + f\alpha_1) - nf^{'} + \frac{G}{f^n}] = (-1)^t r^{s-t}[(f^{'} + f\alpha_2)f^n - nf^n f^{'} + H]^t.$$

The right-hand side of the above equation is a differential polynomial in f. Therefore, by Lemma 2

$$m(r, h + \frac{G}{f^n}) = S(r, f), \tag{24}$$

where $h = (f^{'} + f\alpha_1) - nf^{'}$, Since

$$m(r, \frac{G}{f^{n-2}}) = S(r, f). \tag{25}$$

From (24) and (25), we have

$$m(r, h) = S(r, f), m(r, \frac{1}{f^2}) = S(r, f).$$

Then

$$T(r, f) = S(r, f).$$

This yield a contradiction,and completes the proof of Theorem 1.

Theorem 2. Let $n \geq 2$ be a positive integer. Let f be a transcendental entire function which satisfies $\bar{N}(r, \frac{1}{f}) + \bar{N}(r, f) = S(r, f)$, $P(f)$ be a differential polynomial in f of degree $\leq n - 1$. If

$$f^n(z) + P(f) = fe^{\alpha_1 z} + fe^{\alpha_2 z}, \tag{26}$$

where $\alpha_i (i = 1, 2)$ are positive numbers satisfying $(n - 1)\alpha_2 \geq n\alpha_1 > 0$, then there exists a small function γ of f such that

$$(f + \gamma)^{n+1} = f^n e^{\alpha_2 z}. \tag{27}$$

proof. First of all, we write $P(f)$ as the following:

$$P(f) = \sum_{j=0}^{n-1} b_j M_j(f), \tag{28}$$

where b_j are small functions of f,$M_0(f) = 1$, $M_j(f)(j = 1, 2, ..., n-1)$ are differential monomials in f of degree j. Without loss of generality, we assume that $b_0 \neq o$, otherwise, we do the transformation. From (26), we have

$$\frac{1}{fe^{\alpha_1 z} + fe^{\alpha_2 z} - b_0} + \sum_{j=1}^{n-1} \frac{b_j}{fe^{\alpha_1 z} + fe^{\alpha_2 z} - b_0} \frac{M_j(f)}{f^j} (\frac{1}{f})^{n-j} = (\frac{1}{f})^n. \quad (29)$$

Note that $m(r, M_j(f)/f^j) = S(r, f)$, and by Lemma 3, we have

$$m(r, \frac{1}{fe^{\alpha_1 z} + fe^{\alpha_2 z} - b_0}) = S(r, f).$$

Therefore, the left-hand side of (29) is a polynomial in $1/f$ of degree at most $n-1$, with coefficients being small proximate functions of $1/f$. Hence

$$m(r, 1/f) = S(r, f).$$

Taking the derivatives in both sides of (26) gives

$$nf^{n-1}f' + P' = (f' + \alpha_1 f)e^{\alpha_1 z} + (f' + \alpha_2 f)e^{\alpha_2 z}. \quad (30)$$

By eliminating $e^{\alpha_2 z}$ and $e^{\alpha_1 z}$, respectively from (26) and the above equation, we get

$$(f' + \alpha_2 f)f^n + (f' + \alpha_2 f)P - nf^n f' - P'f = (\alpha_2 - \alpha_1)f^n e^{\alpha_1 z}, \quad (31)$$

$$(f' + \alpha_1 f)f^n + (f' + \alpha_1 f)P - nf^n f' - P'f = -(\alpha_2 - \alpha_1)f^n e^{\alpha_2 z}. \quad (32)$$

From (31) and (32), we get

$$m(r, e^{\alpha_j z}) \leq T(r, f) + S(r, f), (j = 1, 2).$$

On the other hand, from (26), we have

$$(n-1)T(r, f) \leq T(r, e^{\alpha_1 z} + e^{\alpha_2 z}) + S(r, f).$$

Therefore, $S(r, e^{\alpha_1 z}) = S(r, e^{\alpha_2 z}) = S(r, f)$. From (29), we have

$$\frac{e^{\alpha_i z}}{fe^{\alpha_1 z} + fe^{\alpha_2 z} - b_0} + \sum_{j=1}^{n-1} \frac{b_j e^{\alpha_i z}}{fe^{\alpha_1 z} + fe^{\alpha_2 z} - b_0} \frac{M_j(f)}{f^j} (\frac{1}{f})^{n-j} = \frac{e^{\alpha_i z}}{f^n}, (i = 1, 2).$$

It follows that

$$m(r, \frac{e^{\alpha_i z}}{f^n}) = S(r, f), (i = 1, 2). \quad (33)$$

It follows from (31) that

$$f^n\varphi = \beta\frac{e^{\alpha_1 z}}{f^n}f^n - R.$$

where $\varphi = \alpha_2 f - (n-1)f'$, and $\beta = (\alpha_2 - \alpha_1)f^n$, and $R = (f' + \alpha_1 f)P - fP'$.

By Lemma 2, we get $m(r,\varphi) = S(r,f)$. Note that φ is entire, we have $T(r,\varphi) = S(r,f)$ i.e.,φ is a small function of f, By the definition of φ, we get

$$f' = \frac{\alpha_2}{n-1}f - \frac{\varphi}{n-1}.$$

Substituting the above equation into (32) gives

$$f^{n+1} + \frac{\varphi}{\alpha_1 - \alpha_2}f^n + \frac{[(\frac{\alpha_2}{n-1} + \alpha_1)f - \frac{\varphi}{n-1}]}{\alpha_1 - \alpha_2}P - \frac{f'}{\alpha_1 - \alpha_2}P' = f^n e^{\alpha_2 z}.$$

By Lemma 3, we see that there exists a small function γ of f such that $(f + \gamma)^{n+1} = f^n e^{\alpha_2 z}$.

This completes the proof of Theorem 2.

4 Open Problem

If $f(z)$ be a non-constant transcendental meromorphic function in the whole complex plane, one can consider the solutions of nonlinear differential equations $f^n(z) + P(f) = fe^{\alpha_1 z} + fe^{\alpha_2 z}$. New results can be obtained.

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References

[1] Ping Li and C.C. Yang, On the nonexistence of entire solutions of certain type of nonlinear differential equations,J. Math. Anal. Appl., 320 (2006), 827-835.

[2] Ping Li, Entire solutions of certain type of differential equations, J. Math. Anal. Appl., 344 (2008), 253-259.

[3] C.C. Yang and H.-X. Yi, Uniqueness Theory of Meromorphic Functions, Science Press.Beijing, 2003.

[4] Yang Lo, Value distribution theory, Science Press,Beijing,1993.

[5] J. Clunie, On integral and meromorphic functions,J. London Math. Soc.,37(1962), 17-27.

[6] W. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[7] P. Li and W.J. Wang, Entire function that share a small function with its derivative, J. Math. Anal. Appl., 328 (2007), 743-751.