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TANGENT SURFACES OF BIHARMONIC $\mathbb{B}$-GENERAL HELICES ACCORDING TO BISHOP FRAME IN HEISENBERG GROUP Heis^3

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Abstract

In this paper, we study tangent surfaces of biharmonic $\mathbb{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. We give necessary and sufficient conditions for $\mathbb{B}$-general helices to be biharmonic according to Bishop frame. We characterize the tangent surfaces of biharmonic $\mathbb{B}$-general helices in terms of Bishop frame in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

Keywords: *Biharmonic curve, Bishop frame, Heisenberg group.*

1 Introduction

Developable surfaces, which can be developed onto a plane without stretching and tearing, form a subset of ruled surfaces, which can be generated by sweeping a line through space. There are three types of developable surfaces: cones, cylinders (including planes) and tangent surfaces formed by the tangents of a space curve, which is called the cuspidal edge of this surface.

In this paper, we study tangent surfaces of biharmonic $\mathbb{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. We give necessary and sufficient conditions for $\mathbb{B}$-general helices to be biharmonic according to Bishop frame. We characterize the tangent surfaces of biharmonic $\mathbb{B}$-general helices in terms of Bishop frame in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

2 The Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}) \qquad (2.1)$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial x},\ \mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\ \mathbf{e}_3=\frac{\partial}{\partial z}, \qquad (2.2)$$

for which we have the Lie products

$$\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,[\mathbf{e}_2,\mathbf{e}_3]=[\mathbf{e}_3,\mathbf{e}_1]=0$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1.$$

We obtain

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1=\nabla_{\mathbf{e}_2}\mathbf{e}_2=\nabla_{\mathbf{e}_3}\mathbf{e}_3=0,$$

$$\nabla_{\mathbf{e}_1}\mathbf{e}_2=-\nabla_{\mathbf{e}_2}\mathbf{e}_1=\frac{1}{2}\mathbf{e}_3,$$

$$\nabla_{\mathbf{e}_1}\mathbf{e}_3=\nabla_{\mathbf{e}_3}\mathbf{e}_1=-\frac{1}{2}\mathbf{e}_2,$$

$$\nabla_{\mathbf{e}_2}\mathbf{e}_3=\nabla_{\mathbf{e}_3}\mathbf{e}_2=\frac{1}{2}\mathbf{e}_1.$$

We adopt the following notation and sign convention for Riemannian curvature operator on Heis^3 defined by

$$R(X,Y)Z=-\nabla_X\nabla_YZ+\nabla_Y\nabla_YZ+\nabla_{X,Y]}Z,$$

while the Riemannian curvature tensor is given by

$$R(X,Y,Z,W)=g(R(X,Y)Z,W),$$

where X,Y,Z,W are smooth vector fields on Heis ^{3}.

The components $\{R_{ijkl}\}$ of R relative to $\{e_1,e_2,e_3\}$ are defined by

$$g\big(R(e_i,e_j)e_k,e_l\big)=R_{ijkl.}$$

The non vanishing components of the above tensor fields are

$$R_{121}=-\frac{3}{4}e_2,\quad R_{131}=\frac{1}{4}e_3,\quad R_{122}=\frac{3}{4}e_1,$$

$$R_{232}=\frac{1}{4}e_3,\quad R_{133}=-\frac{1}{4}e_1,\quad R_{233}=-\frac{1}{4}e_2,$$

and

$$R_{1212}=-\frac{3}{4},\quad R_{1313}=R_{2323}=\frac{1}{4}. \tag{2.3}$$

3 Biharmonic B-General Helices with Bishop Frame In The Heisenberg Group Heis ^{3}

Let $\gamma:I\to Heis^3$ be a non geodesic curve on the Heisenberg group Heis ^{3} parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis ^{3} along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}&\nabla_{\mathbf{T}}\mathbf{T}=\kappa\mathbf{N},\\&\nabla_{\mathbf{T}}\mathbf{N}=-\kappa\mathbf{T}+\tau\mathbf{B},\\&\nabla_{\mathbf{T}}\mathbf{B}=-\tau\mathbf{N},\end{aligned}\tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned}&g(\mathbf{T},\mathbf{T})=1,g(\mathbf{N},\mathbf{N})=1,g(\mathbf{B},\mathbf{B})=1,\\&g(\mathbf{T},\mathbf{N})=g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.\end{aligned}\tag{3.2}$$

In the rest of the paper, we suppose everywhere $\kappa\neq 0$ and $\tau\neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &g(\mathbf{T},\mathbf{T}) = 1, g(\mathbf{M}_1,\mathbf{M}_1) = 1, g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ &g(\mathbf{T},\mathbf{M}_1) = g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures. where $\theta(s) = \arctan\dfrac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_2^2 + k_1^2}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\theta(s), \\ k_2 &= \kappa(s)\sin\theta(s). \end{aligned} \tag{3.5}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned} \tag{3.6}$$

Theorem 3.1. *$\gamma : I \to Heis^3$ is a biharmonic curve with Bishop frame if and only if*

$$\begin{aligned} &k_1^2 + k_2^2 = \text{constant} = C \neq 0, \\ &k_1'' - Ck_1 = k_1\left[\frac{1}{4} - \left(M_2^3\right)^2\right] - k_2M_1^3M_2^3, \\ &k_2'' - Ck_2 = k_1M_1^3M_2^3 + k_2\left[\frac{1}{4} - \left(M_1^3\right)^2\right]. \end{aligned} \tag{3.7}$$

To separate a general helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as B-general helix.

Theorem 3.2. *Let $\gamma_{\mathsf{B}} : I \to Heis^3$ be a unit speed biharmonic B-general helix with non-zero natural curvatures. Then the parametric equation of γ_{B} are*

$$x_{\mathrm{B}}(s)=\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2,$$

$$y_{\mathrm{B}}(s)=-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3,$$

(3.8)

$$z_{\mathrm{B}}(s)=(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}$$
$$-\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_4,$$

where $\zeta_0,\ \zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4$ *are constants of integration, [10].*

We can draw unit speed biharmonic B-general helices according to Bishop frame with helping the programme of Mathematica as follow:

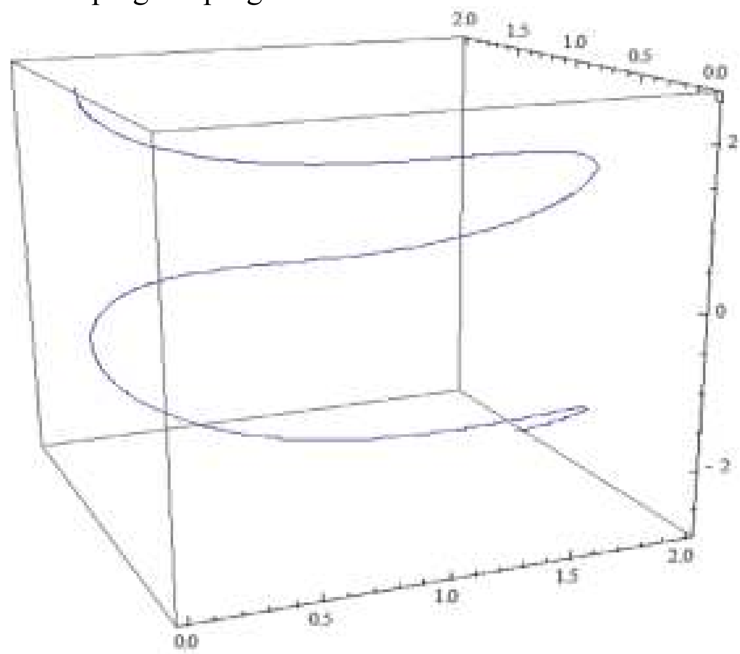

Fig.1.

4 Tangent Surface of Biharmonic $\mathbb{B}$-General Helices with Bishop Frame In The Heisenberg Group Heis^3

The purpose of this section is to study tangent developable of biharmonic $\mathbb{B}$-general helices with Bishop frame in the Heisenberg group Heis^3.

The tangent surface of $\gamma_{\mathbb{B}}$ is a ruled surface

$$\mathsf{R}(s,u)=\gamma_{\mathbb{B}}(s)+u\mathbf{T}(s). \tag{4.1}$$

Theorem 4.1. *(Main Theorem) Let $\gamma_{\mathbb{B}}: I\rightarrow Heis^3$ be a unit speed biharmonic $\mathbb{B}$-general helix with non-zero natural curvatures. Then the parametric equation of tangent surface of $\gamma_{\mathbb{B}}$ are*

$$\begin{aligned}
x_{\mathbb{B}}(s,u)&=\frac{\sin\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]\\
&+u\sin\theta\cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]+\zeta_2,\\
y_{\mathbb{B}}(s,u)&=-\frac{\sin\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]\\
&+u\sin\theta\sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]+\zeta_3,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
z_{\mathbb{B}}(s,u)&=(\cos\theta)s+\frac{\sin^2\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\left(\frac{s}{2}-\frac{\sin 2\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]}{4\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\right.\\
&-\frac{\zeta_1\sin\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]+u\cos\theta\\
&+\frac{u\sin^2\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\sin^2\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]
\end{aligned}$$

$$+u\zeta_1\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_4,$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 *are constants of integration.*

Proof. From orthonormal basis (2.2) and (3.8), we obtain

$$\mathbf{T}=(\sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0],\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0],$$
$$\cos\theta+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin^2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \tag{4.3}$$
$$+\zeta_1\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]),$$

where ζ_1 is constant of integration.

Using above equation, we have (4.2), the theorem is proved.

We need following lemma.

Lemma 4.2. *Let* $\gamma_{\mathrm{B}}:I\to Heis^3$ *be a unit speed biharmonic* B*-general helix with non-zero natural curvatures. Then the position vector of* γ_{B} *is*

$$\gamma_{\mathrm{B}}(s)=[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2]\mathbf{e}_1$$
$$+[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3]\mathbf{e}_2$$
$$+[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \tag{4.4}$$
$$[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3]$$
$$+(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}$$

$$-\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_4]\mathbf{e}_3,$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 *are constants of integration.*

Theorem 4.3. *Let* $\gamma_{\mathsf{B}}: I \to Heis^3$ *be a unit speed biharmonic* B*-general helix with non-zero natural curvatures. Then the equation of tangent surface of* γ_{B} *is*

$$\begin{aligned}
\mathsf{R}_{\mathsf{B}}(s,u) = {}&[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\\
&+u\sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2]\mathbf{e}_1\\
&+[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\\
&+u\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3]\mathbf{e}_2 \qquad (4.5)\\
&+[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2]\\
&[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3]\\
&+(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\\
&-\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+u\cos\theta+\zeta_4]\mathbf{e}_3,
\end{aligned}$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 *are constants of integration.*

Proof. From section 3, we immediately arrive at

$$\mathbf{T} = \sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_1+\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_2$$
$$+\cos\theta\mathbf{e}_3. \tag{4.6}$$

Using above equation we have (4.5). Thus, the proof is finished.

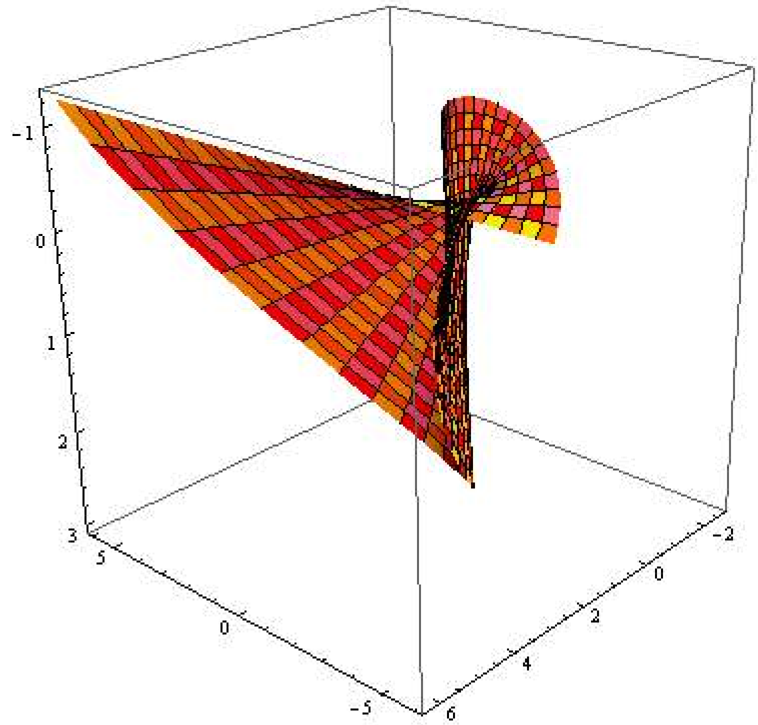

Fig.2.

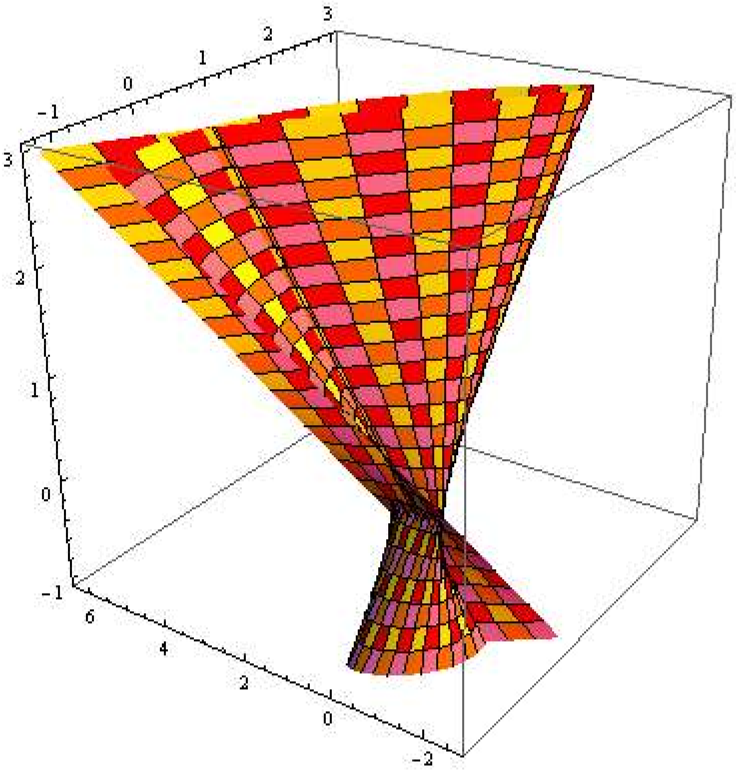

Fig.3.

5 Open Problem

The authors can be resarch minimal tangent surfaces of biharmonic $\mathbf{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3.

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