Int. J. Open Problems Compt. Math., Vol. 6, No. 1, March 2013
ISSN 1998-6262;
www.i-csrs.org

New Existence Results for Fractional Evolution Equations With Nonlocal Boundary Conditions

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Abstract

In this paper, we establish sufficient conditions for the existence and uniqueness of solutions for impulsive fractional differential equations of order α, $(n-1<\alpha\leq n, n\in\mathbb{N}^*)$ in Banach spaces. These results are obtained using Banach contraction fixed point theorem. Other existence results are also presented using Krasnoselskii theorem.

Keywords: *Caputo derivative, Fixed point theorem.*

1 Introduction

The fractional differential equations theory is a new branch of applied mathematics by which many physical phenomena in various fields of science and engineering can be modeled. Significant development in this area has been achieved for the last two decades. For details, we refer to [3, 4]. Moreover, the study of impulsive fractional differential equations is also of great importance. The study of such equations is linked to their utility in processes which experience a sudden change of their state at certain moments. These processes arise in phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics [1, 2]. Many researchers have discussed existence of solutions for impulsive systems in Banach space. For more details, we refer the reader to [7, 8, 9, 10]. Motivated by the works [5], the main aim of this paper is to establish some existence results for evolution fractional differential equations in Banach spaces by using the fractional calculus and fixed point theorems. So, let us consider the following α^{th}-order evolution fractional differential equations with integral boundary conditions:

$$D^{\alpha}x\left(t\right)+f\left(t,x\left(t\right)\right)=0,\ t_i\neq t,\ t\in J:=[0,1],\ n-1<\alpha\leq n,\ n\in\mathbb{N}^{*}$$

$$-\Delta^{(n-1)}|_{t=t_i},\ i=1,2,...,m, \tag{1}$$

$$x(0)=x^{'}(0)=...=x^{(n-2)}(0)=0,\ x(1)=\int_0^1 h(t)x(t)dt,$$

where D^{α} denotes the Caputo derivative, $f\in C(J\times X,X)$, $I_i\in C(X,X)$, X is a real space, $(t)_{i=1,...,m}$ are fixed points, with $0<t_1<t_2<...<t_i<...<t_m$, m fixed in $\mathbb{N}$, and $\Delta x^{(n-1)}|_{t=t_i}=x^{(n-1)}(t_i^{+})-x^{(n-1)}(t_i^{-})$, such that $x^{(n-1)}(t_i^{+})$ and $x^{(n-1)}(t_i^{-})$ represent the right-hand limit and left-hand limit of $x^{(n-1)}(t)$ at $t=t_i$, respectively and $h\in L^1(J)$ is nonnegative.

2 Preliminaries

We give the necessary notation and basic definitions which will be used in this paper [6, 11].

Definition 1: A real valued function$f(t)$, $t>0$ is said to be in the space $C_{\mu},\mu\in\mathbb{R}$, if there exists a real number $p>\mu$ such that $f(t)=t^{p}f_1(t)$, where $f_1(t)\in C([0,\infty))$.
Definition 2: A function $f(t),t>0$ is said to be in the space $C_{\mu}^{n},n\in\mathbb{N}$, if $f^{(n)}\in C_{\mu}$.
Definition 3: The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for a function $f\in C_{\mu},\mu\geq -1$, is defined as

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau;\quad \alpha>0,t>0, \tag{2}$$

$$J^{0}f(t)=f(t),$$

where $\Gamma(\alpha):=\int_0^{\infty}e^{-u}u^{\alpha-1}du$.

The fractional derivative of $f\in C_{-1}^{n}([0,\infty[)$ in the Caputo's sense is defined as

$$D^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, & n-1<\alpha<n,n\in N^{*},\\ \frac{d^n}{dt^n}f(t), & \alpha=n.\end{cases} \tag{3}$$

In order to define the solutions of problem (1), we will consider the following sets:
Let $J^{'} = J - \{t_1, t_2, ..., t_n\}$, and

$$PC^{n-1}(J,X) = \{x : J \to X : x \in C(J,X), x^{(n-1)}|_{(t_i,t_{i+1})} \in C(t_i, t_{i+1}), \\ x^{(n-1)}|_{(t_i^-)} = x^{(n-1)}|_{(t_i)}, \exists x^{(n-1)}|_{t_i^+}, i = 1, 2, ..., m\}. \tag{4}$$

Then $PC^{n-1}(J,X)$ is a real Banach space with norm

$$||x|| = max\{||x||_\infty, ||x^{'}||_\infty, ..., ||x^{(n-1)}||_\infty\}, \tag{5}$$

where $||x^{(n-1)}||_\infty = sup_{t\in J}|x^{(n-1)}|, n = 1, 2, ...$
A function $PC^{n-1}(J,X)$ is called a solution of (1) if it satisfies (1).

We need the following lemma:

Lemma 2.1 *[12] For $\alpha > 0$, the general solution of equation $D^\alpha x = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ...c_{n-1}t^{n-1}, \tag{6}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, ...n-1, n = [\alpha] + 1$.

Also, we have the following properties

$$D^\alpha J^\alpha x = x$$

and

$$J^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} x^{(i)}(0)\frac{t^i}{i!}. \tag{7}$$

We need also:

Lemma 2.2 *A solution of the problem (1) is given by:*

$$x(t) = -J^\alpha f(t,x(t)) - \sum_{t_i<t} \frac{I_i(x(t_i))(t-t_i)^{n-1}}{(n-1)!} + \frac{t^{n-1}}{(n-1)!}\Big[(n-1)! \int_0^1 h(t)x(t)dt \\ + \tfrac{(n-1)!}{\Gamma(\alpha)} \textstyle\int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau + \sum_{t_i<1} I_i(x(t_i))(t-t_i)^{n-1}\Big]. \tag{8}$$

Proof: We have:

$$D^\alpha x(t) = -f(t,x(t)). \tag{9}$$

Applying the fractional integral operator J^α to both sides of (9), and using Lemma 1, we get

$$J^\alpha D^\alpha x(t) = -J^\alpha f(t,x(t)) - \sum_{t_i<t} \frac{I_i(x(t_i))(t-t_i)^{n-1}}{(n-1)!} \qquad (10)$$
$$- \textstyle\sum_{i=0}^{n-1} x^{(i)}(0)\frac{t_i}{i!}, \quad n-1<\alpha<n,$$

that is

$$x(t) = -J^\alpha f(t,x(t)) - \sum_{t_i<t} \frac{I_i(x(t_i))(t-t_i)^{n-1}}{(n-1)!} \qquad (11)$$
$$-x^{(n-1)}(0)\tfrac{t^{n-1}}{(n-1)!}, \quad n-1<\alpha<n.$$

If $t=1$, then we have

$$\int_0^1 h(t)x(t)dt = -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,x(\tau))d\tau \qquad (12)$$
$$- \textstyle\sum_{t_i<t} \frac{I_i(x)(t_i)(1-t_i)^{n-1}}{(n-1)!} - \frac{x^{(n-1)}(0)}{(n-1)!}.$$

Consequently,

$$x^{(n-1)}(0) = -\frac{(n-1)!}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,x(\tau))d\tau - \sum_{t_i<t} I_i(x(t_i))(t-t_i)^{n-1}$$
$$-(n-1)!\textstyle\int_0^1 h(t)x(t)dt. \qquad (13)$$

Therefore

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,x(\tau))d\tau - \sum_{t_i<t} \frac{I_i(x(t_i))(t-t_i)^{n-1}}{(n-1)!}$$
$$+\tfrac{t^{n-1}}{(n-1)!}\Big[\tfrac{(n-1)!}{\Gamma(\alpha)}\textstyle\int_0^1 (1-\tau)^{\alpha-1} f(\tau,x(\tau))d\tau + \sum_{t_i<t} I_i(x(t_i))(1-t_i)^{n-1} \qquad (14)$$
$$+(n-1)!\textstyle\int_0^1 h(t)x(t)dt\Big].$$

Lemma 2 is thus proved.

To establish the existence of solutions in $PC^{n-1}(J,X)$ of problem (1), let us list the following assumptions:

(H_1) : The nonlinear function $f : J \times X \to X$ is continuous and there exist constants $\beta > 0, ß > 0$, such that

$$||f(t, x(t)) - f(t, y(t))|| \leq \beta||x - y||; x, t \in X, t \in J \tag{15}$$

and

$$ß = max_{t\in J}||f(t, 0)||.$$

(H_2) : $I_i : X \to X$ is continuous and there exist constants ϖ_i such that

$$||I_i(x) - I_i(y) \leq \varpi_i||x - y||, i = 1, 2, ..., m, x, y \in X$$

and

$$\omega = max_{t\in J}||I_i(0)||.$$

(H_3) : For $t \in J$, the function $h : J \to X$ is continuous and there exists a constant $M > 0$; such that

$$||h(t)|| \leq M, fort \in J.$$

(H_4) : There exists a positive constant $r > 0$ such that

$$\gamma(2\beta r + 2ß) + Mr + \frac{1}{(n-1)!}\Big(2\sum_{i=1}^{m}\varpi_i r + 2m\omega\Big) \leq r,$$

where

$$\gamma = \frac{1}{\Gamma(\alpha + 1)}.$$

3 Main Results

3.1 Existence Results Using Banach Fixed Point Theorem

We prove the existence and the uniqueness of a solution for the impulsive fractional differential equation (1) by using Banach fixed point theorem. We have

Theorem 3.1 *If the hypotheses* $(H_j)_{j=\overline{1,4}}$ *and*

$$0 \leq \Lambda := 2\beta\gamma + \frac{2\sum_{i=1}^{m}\varpi_i}{(n-1)!} + M < 1 \tag{16}$$

are satisfied, then the impulsive fractional differential system (1) has a unique solution on $PC^{n-1}(J, X)$.

Proof: Let us take $B_r = \{x \in PC^{n-1}(J,X) : ||x|| \leq r\}$. Based on (1), we define an operator $\Phi : PC^{n-1}(J,B_r) \to PC^{n-1}(J,B_r)$ by

$$\Phi(x(t)) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,x(\tau))d\tau - \sum_{t_i<t} \frac{I_i(x)t_i))(t-t_i)^{n-1}}{(n-1)!}$$

$$+\tfrac{t^{n-1}}{(n-1)!}\Big[(n-1)!\int_0^1 h(t)x(t)dt + \tfrac{(n-1)!}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,x(\tau))d\tau \tag{17}$$

$$+\textstyle\sum_{t_i<t} I_i(x(t_i))(1-t_i)^{n-1}\Big].$$

(1*) We show that $\Phi B_r \subset B_r$. We have:

$$||\Phi(x(t))|| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}||f(\tau,x(\tau))||d\tau + \sum_{t_i<t}\frac{||I_i(x(t_i))||}{(n-1)!}$$

$$+\tfrac{1}{(n-1)!}\Big[(n-1)!\int_0^1 ||h(t)x(t)||dt + \tfrac{(n-1)!}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}||f(\tau,x(\tau))||d\tau \tag{18}$$

$$+\textstyle\sum_{t_i<1}||I_i(x(t_i))||\Big].$$

Consequently,

$$||\Phi(x(t))|| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}||f(\tau,x(\tau))-f(\tau,0)||d\tau$$

$$+\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}||f(\tau,0)d\tau + \textstyle\sum_{t_i<t}\frac{||I_i(x(t_i))-I_i(0)||}{(n-1)!}$$

$$+\textstyle\sum_{t_i<t}\frac{||I_i(0)||}{(n-1)!} + M||x|| + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}||f(\tau,x(\tau))-f(\tau,0)||d\tau \tag{19}$$

$$+\tfrac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}||f(\tau,0)||d\tau + \textstyle\sum_{t_i<1}\frac{||I_i(x(t_i))-I_i(0)||}{(n-1)!}$$

$$+\textstyle\sum_{t_i<1}\frac{||I_i(0)||}{(n-1)!}.$$

Using (H_1) and (H_2), we can write

$$||\Phi(x(t))|| \leq \frac{2\beta}{\Gamma(\alpha+1)}||x|| + \frac{2ß}{\Gamma(\alpha+1)}$$

$$\tfrac{2\sum_{i=1}^m \varpi_i||x||}{(n-1)!} + \tfrac{2m\omega}{(n-1)!} + M||x||. \tag{20}$$

This implies that

$$||\Phi(x(t))|| \leq \gamma(2\beta + 2ß) + Mr + \frac{1}{(n-1)!}\Big(2\sum_{i=1}^m \varpi_i r + 2m\omega\Big). \tag{21}$$

Thanks to (H_4), we obtain

$$||\Phi(x(t))|| \leq r.$$

Then $\Phi B_r \subset B_r$; which implies that the operator Φ maps B_r into itself.
(2^*) Now we prove that Φ is a contraction mapping on B_r :
Let x and $y \in B_r$, then we can write:

$$\begin{aligned} ||\Phi(x(t)) - \Phi(y(t))|| \leq || - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau - \sum_{t_i<t} \frac{I_i(x(t_i))(t-t_i)^{n-1}}{(n-1)!} \\ + \tfrac{t^{n-1}}{(n-1)!}[(n-1)! \textstyle\int_0^1 h(t)y(t)dt + \tfrac{(n-1)!}{\Gamma(\alpha)} \textstyle\int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \\ + \textstyle\sum_{t_i<t} I_i(x(t_i))(1-t_i)^{n-1} + \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \\ + \textstyle\sum_{t_i<t} \tfrac{I_i(y(t_i))(t-t_i)^{n-1}}{(n-1)!} - \tfrac{t^{n-1}}{(n-1)!}[(n-1)! \textstyle\int_0^1 h(t)y(t)dt \\ + \tfrac{(n-1)!}{\Gamma(\alpha)} \textstyle\int_0^1 (1-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau + \textstyle\sum_{t_i<t} I_i(y(t_i))(1-t_i)^{n-1}]||. \end{aligned} \tag{22}$$

Consequently,

$$\begin{aligned} ||\Phi(x(t)) - \Phi(y(t))|| \leq || - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (f(\tau, x(\tau)) - f(\tau, y(\tau))) d\tau \\ - \textstyle\sum_{t_i<t} \tfrac{(I_i(x(t_i)) - I_i(y(t_i)))(t-t_i)^{n-1}}{(n-1)!} \\ + \tfrac{t^{n-1}}{(n-1)!}[(n-1)! \textstyle\int_0^1 h(t)[x(t) - y(t)]dt \\ + \tfrac{(n-1)!}{\Gamma(\alpha)} \textstyle\int_0^1 (1-\tau)^{\alpha-1} (f(\tau, x(\tau)) - f(\tau, y(\tau))) d\tau \\ + \textstyle\sum_{t_i<t} (I_i(x(t_i)) - I_i(y(t_i)))(1-t_i)^{n-1}]||. \end{aligned} \tag{23}$$

By a simple calculate we obtain

$$||\Phi(x(t)) - \Phi(y(t))|| \leq \Big(2\gamma\beta + \frac{2\sum_{i=1}^m \varpi_i}{(n-1)!} + M\Big)||x-y||.$$

Then by (16), we get

$$||\Phi(x(t)) - \Phi(y(t))|| \leq \Lambda ||x-y||.$$

Since $0 \leq \Lambda < 1$, then Φ is a contraction and so by Banach fixed point theorem, there exists a unique fixed point $x \in PC^{n-1}(J, B_r)$ such that $(\Phi x)(t) = x(t)$.

3.2 Existence of Solution Using Krasnoselskii's Fixed Point Theorem

Our second result is based on the existence of solution using the following Krasnoselskii's fixed point theorem:

Theorem 3.2 *[8] Let S be a closed convex and nonempty subset of a Banach space X. Let P, Q be the operators such that*
(i) $Px + Qy \in S, x, y \in S$,
(ii) P is compact and continuous, (iii) Q is a contraction mapping.
Then there exists x^* such that $x^* = Px^* + Qx^*$.

We have:

Theorem 3.3 *Suppose that the hypotheses $(H_j)_{j=\overline{1,4}}$ are satisfied. If there exists a constant Υ such that*

$$\Upsilon := 2\gamma\beta + M < 1,$$

then the system (1) has a solution on $PC^{n-1}(J, X)$.

Proof: On B_r, we define the operators R and S as:

$$\begin{aligned} Rx(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \\ +\tfrac{t^{n-1}}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \\ +t^{n-1} \int_0^1 h(t)x(t)dt \end{aligned} \tag{24}$$

and

$$Sx(t) = -\sum_{t_i<t} \frac{I_i(x(t_i))(t-t_i)^{n-1}}{(n-1)!} + \sum_{t_i<1} \frac{I_i(x(t_i))t^{n-1}(1-t_i)^{n-1}}{(n-1)!}. \tag{25}$$

For $x, y \in B_r$, we have

$$||Rx(t) + Sy(t)|| \leq ||Rx(t)|| + ||Sy(t)||.$$

Then we can write

$$\begin{aligned} ||Rx(t) + Sy(t)|| \leq ||\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau|| + ||t^{n-1} \int_0^1 h(t)x(t)dt|| \\ +||\tfrac{t^{n-1}}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau|| \\ +|| \textstyle\sum_{t_i<t} \frac{I_i(x(t_i))(t-t_i)^{n-1}}{(n-1)!}|| + || \sum_{t_i<1} \frac{I_i(x(t_i))t^{n-1}(1-t_i)^{n-1}}{(n-1)!}||. \end{aligned} \tag{26}$$

By $(H_1), (H_2)$ and (H_3), it follows that

$$||Rx(t) + Sy(t)|| \leq \gamma(2\beta r + 2ß) + Mr + \frac{1}{(n-1)!}\Big(2\sum_{i=1}^{m} \varpi_i r + 2m\omega\Big). \quad (27)$$

Thanks to (H_4), we get

$$||Rx(t) + Sy(t)|| \leq r.$$

Hence $Rx + Sy \in B_R$.
On other hand, we have

$$||Rx(t) - Ry(t)|| = || - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))d\tau$$

$$+\tfrac{t^{n-1}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau - \tfrac{t^{n-1}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau, y(\tau))d\tau$$

$$+t^{n-1}\int_0^1 h(t)x(t)dt - t^{n-1}\int_0^1 h(t)y(t)dt||. \quad (28)$$

Hence,

$$||Rx(t) - Ry(t)|| \leq (2\gamma\beta + M)||x-y|| \leq \Upsilon||x-y||. \quad (29)$$

Since $\Upsilon < 1$, then the operator R is a contraction.
Now, we shall prove that the operator S is completely continuous from B_r to B_r. Since $I_i \in C(X, X)$, then S is continuous on B_r.
To prove the compactness of S, we shall prove that $S(B_r) \subseteq PC^{n-1}(J, X)$ is equicontinuous and $S(B_r)$ is precompact for any $r > 0, t \in J$.
Let $x \in B_r$ and $t + h \in J$. Then we can write

$$||Sx(t+h) - Sx(t)|| \leq || \sum_{t_i < t+h} \frac{I_i(x(t_i))((t+h) - t_i)^{n-1}}{(n-1)!} - \sum_{t_i<t} \tfrac{I_i(x(t_i))t^{n-1}(1-t_i)^{n-1}}{(n-1)!}||. \quad (30)$$

The inequality (30) is independent of x; thus S is equicontinous, and as $h \to 0$ the right hand side of the above inequality tends to zero. So $S(B_r)$ is relatively compact, and S is compact. Finally by Krasnosellkii theorem, there exists a fixed point $x(.)$ in B_r such that $(\Phi x)(t) = x(t)$ and this point $x(.)$ is a solution of (1).

4 Open Problems

We pose the following problems:
Open Problem 1: Using fractional differential operator of order α in the sense of Riemann-Liouville for a function $f \in C(J \times X, X)$, under what conditions do Theorem 3.1 and Theorem 3.3 hold for $1 < \alpha < 2$?
Open Problem 2: Is it possible to generalize Theorem 3.1 and Theorem 3.3 for $\alpha, n < \alpha < n+1, n \in \mathbb{N}$, using the Riemann-Liouville derivative approach?

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