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On Timelike Biharmonic General Helices in the Lorentzian $\mathbb{E}(1,1)$

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Abstract

In this paper, we study timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We characterize the timelike biharmonic general helices in terms of their curvature and torsion in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

Keywords: *Bienergy, Biharmonic curve, General helix, Rigid motion*

1 Introduction

A curve of constant slope or general helix is defined by the property that the tangent lines make a constant angle with a fixed direction. A necessary and sufficient condition that a curve to be general helix is that ratio of curvature to torsion be constant. Indeed, a helix is a special case of the gerenal helix. If both curvature and torsion are non-zero constants, it is called a helix or only a W-curve.

On the other hand in Mechanics, selection of helical gears involves specifying whether the gear should be single or double helix. Often, turbomachinery operators make this decision based on past experience or the traditions followed in their engineering firm. The double helical gear eliminates internal thrust without introducing a thrust bearing, while maintaining a helical design for load sharing and smooth transfer of load from tooth to tooth. However, this style of gear is subject to a mismatch of the helices between the pinion and the gear. In the case of a single helix, the pinion can be corrected with respect to the gear whereas in the case of two helices a compromise must be settled for the left and right-hand helices. This is because mismatch is not necessarily symmetrical but accumulative, [10].

The notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by J. Eells and J.H. Sampson (see [4]).

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathrm{T}(\phi) \right|^2 dv_h,$$

where $\mathrm{T}(\phi) := \operatorname{tr} \nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_{\phi} \mathrm{T}(\phi) + \operatorname{tr} R(\mathrm{T}(\phi), d\phi) d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

There are a few results on biharmonic curves in arbitrary Riemannian manifolds. The biharmonic curves in the Heisenberg group Heis^3 are investigated in [7,8,11-13] by Körpınar and Turhan.

In this paper, we study timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathrm{E}(1,1)$. We characterize the timelike biharmonic general helices in terms of their curvature and torsion in the Lorentzian group of rigid motions $\mathrm{E}(1,1)$.

2 Preliminaries

Let $\mathrm{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathrm{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathrm{E}(1,1) \to \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \to (x, y, z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z}, \mathbf{e}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,[\mathbf{e}_2,\mathbf{e}_3]=0,[\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_2.$$

Put

$$x^1=x, x^2=\frac{1}{2}(y+z), x^3=\frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{e}_1=\frac{\partial}{\partial x^1},\mathbf{e}_2=\frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2}+e^{-x^1}\frac{\partial}{\partial x^3}\right),\mathbf{e}_3=\frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2}-e^{-x^1}\frac{\partial}{\partial x^3}\right). \quad (2.1)$$

The bracket relations are

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,[\mathbf{e}_2,\mathbf{e}_3]=0,[\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_2. \quad (2.2)$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3\}$. We consider left-invariant Lorentzian metric, given by

$$g=-(dx^1)^2+\left(e^{-x^1}dx^2+e^{x^1}dx^3\right)^2+\left(e^{-x^1}dx^2-e^{x^1}dx^3\right)^2, \quad (2.3)$$

where

$$g(\mathbf{e}_1,\mathbf{e}_1)=-1, g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1. \quad (2.4)$$

Let coframe of our frame be defined by

$$\boldsymbol{\theta}^1=dx^1,\boldsymbol{\theta}^2=e^{-x^1}dx^2+e^{x^1}dx^3,\boldsymbol{\theta}^3=e^{-x^1}dx^2-e^{x^1}dx^3.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla=\begin{pmatrix}0 & 0 & 0\\ -\mathbf{e}_3 & 0 & -\mathbf{e}_1\\ -\mathbf{e}_2 & -\mathbf{e}_1 & 0\end{pmatrix}, \quad (2.5)$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k=1,2,3\}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

3 Timelike Biharmonic General Helices in the Lorentzian Group of Rigid Motions $\mathrm{E}(1,1)$

Let $\gamma : I \to \mathrm{E}(1,1)$ be a non geodesics timelike curve in the group of rigid motions $\mathrm{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the group of rigid motions $\mathrm{E}(1,1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}
&\nabla_{\mathbf{T}(s)}\mathbf{T}(s) = \kappa(s)\mathbf{N}(s),\\
&\nabla_{\mathbf{T}(s)}\mathbf{N}(s) = \kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s),\\
&\nabla_{\mathbf{T}(s)}\mathbf{B}(s) = -\tau(s)\mathbf{N}(s),
\end{aligned} \tag{3.1}$$

where $\kappa(s)$ is the curvature of γ, $\tau(s)$ is its torsion and

$$\begin{aligned}
&g(\mathbf{T}(s),\mathbf{T}(s)) = -1,\ g(\mathbf{N}(s),\mathbf{N}(s)) = 1,\ g(\mathbf{B}(s),\mathbf{B}(s)) = 1,\\
&g(\mathbf{T}(s),\mathbf{N}(s)) = g(\mathbf{T}(s),\mathbf{B}(s)) = g(\mathbf{N}(s),\mathbf{B}(s)) = 0.
\end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned}
&\mathbf{T}(s) = T_1(s)\mathbf{e}_1 + T_2(s)\mathbf{e}_2 + T_3(s)\mathbf{e}_3,\\
&\mathbf{N}(s) = N_1(s)\mathbf{e}_1 + N_2(s)\mathbf{e}_2 + N_3(s)\mathbf{e}_3,\\
&\mathbf{B}(s) = \mathbf{T}(s)\times\mathbf{N}(s) = B_1(s)\mathbf{e}_1 + B_2(s)\mathbf{e}_2 + B_3(s)\mathbf{e}_3.
\end{aligned} \tag{3.3}$$

Theorem 3.1. $\gamma : I \to \mathrm{E}(1,1)$ *is a non geodesic timelike biharmonic curve in the Lorentzian group of rigid motions* $\mathrm{E}(1,1)$ *if and only if*

$$\begin{aligned}
&\kappa(s) = \text{constant} \neq 0,\\
&\kappa^2(s) - \tau^2(s) = 1 + 2B_1^2(s),\\
&\tau'(s) = -2N_1(s)B_1(s).
\end{aligned} \tag{3.4}$$

Proof. Using (3.1), we have (3.4). This complete the proof of the theorem.

If we write this curve in the another parametric representation $\gamma = \gamma(\theta)$, where $\theta = \int_0^s \kappa(s)ds.$ We have new Frenet equations as follows:

$$\begin{aligned}
&\nabla_{\mathbf{T}(\theta)}\mathbf{T}(\theta) = \mathbf{N}(\theta),\\
&\nabla_{\mathbf{T}(\theta)}\mathbf{N}(\theta) = \mathbf{T}(\theta) + f(\theta)\mathbf{B}(\theta),
\end{aligned} \tag{3.5}$$

$$\nabla_{\mathbf{T}(\theta)}\mathbf{B}(\theta) = -f(\theta)\mathbf{N}(\theta),$$

where $f(\theta) = \dfrac{\tau(\theta)}{\kappa(\theta)}$.

If we write $\{\mathbf{T}(\theta), \mathbf{N}(\theta), \mathbf{B}(\theta)\}$ with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as following:

$$\begin{aligned}
\mathbf{T}(\theta) &= T_1(\theta)\mathbf{e}_1 + T_2(\theta)\mathbf{e}_2 + T_3(\theta)\mathbf{e}_3,\\
\mathbf{N}(\theta) &= N_1(\theta)\mathbf{e}_1 + N_2(\theta)\mathbf{e}_2 + N_3(\theta)\mathbf{e}_3,\\
\mathbf{B}(\theta) &= \mathbf{T}(\theta)\times\mathbf{N}(\theta) = B_1(\theta)\mathbf{e}_1 + B_2(\theta)\mathbf{e}_2 + B_3(\theta)\mathbf{e}_3.
\end{aligned} \tag{3.6}$$

Theorem 3.2. *Let $\gamma : I \to \mathsf{E}(1,1)$ is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions $\mathsf{E}(1,1)$. Then, the parametric equations of γ are*

$$x^1(\theta) = \cosh \ell\theta + \wp_3,$$

$$\begin{aligned}
x^2(\theta) = &\frac{\sinh \ell e^{\cosh \ell\theta + \wp_3}}{2(\wp_1^2 + \cosh^2 \ell)}\{(\cosh \ell - \wp_1)\cos(\wp_1\theta + \wp_2)\\
&+ (\cosh \ell + \wp_1)\sin(\wp_1\theta + \wp_2)\} + \wp_4,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
x^3(\theta) = &\frac{\sinh \ell e^{-\cosh ö\theta - \wp_3}}{2(\wp_1^2 + \cosh^2 \ell)}\{-(\cosh \ell - \wp_1)\cos(\wp_1\theta + \wp_2)\\
&+ (\cosh \ell + \wp_1)\sin(\wp_1\theta + \wp_2)\} + \wp_5,
\end{aligned}$$

where $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$, $\wp_5$ are constants of integration.

Proof. Since the curve $\gamma(\theta)$ is a timelike general helix, i.e. the tangent vector $\mathbf{T}(\theta)$ makes a constant angle ℓ, with the constant timelike vector called the axis of the general helix. So, without loss of generality, we take the axis of a general helix as being parallel to the timelike vector $\mathbf{X}_1$. Then, using first equation of (3.6), we get

$$T_1(\theta) = g(\mathbf{T}(\theta), \mathbf{X}_1) = \cosh \ell. \tag{3.8}$$

So, substituting the components $T_1(\theta)$, $T_2(\theta)$ and $T_3(\theta)$ in the first equation of (3.6), we have the following equation

$$\mathbf{T} = \cosh \ell\mathbf{e}_1 + \sinh \ell \cos \eta(\theta)\mathbf{e}_2 + \sinh \ell \sin \eta(\theta)\mathbf{e}_3. \tag{3.9}$$

If we substitute (3.5) in (3.9), we have

$$\eta(\theta) = \wp_1\theta + \wp_2, \tag{3.10}$$

where $\wp_1$, $\wp_2$ are constants of integration.

Since (3.9) and (3.10), imply

$$\mathbf{T} = \cosh \ell \mathbf{e}_1 + \sinh \ell \cos(\wp_1\theta + \wp_2)\mathbf{e}_2 + \sinh \ell \sin(\wp_1\theta + \wp_2)\mathbf{e}_3. \tag{3.11}$$

Using (2.1) in (3.11), we obtain

$$\mathbf{T} = (\cosh \ell, \frac{1}{2} e^{\cosh \ell \theta + \wp_3} \sinh \ell \cos(\wp_1\theta + \wp_2) + \sin(\wp_1\theta + \wp_2)],$$
$$\frac{1}{2} e^{-\cosh \ell \theta - \wp_3} \sinh \ell \cos(\wp_1\theta + \wp_2) - \sin(\wp_1\theta + \wp_2)]),$$

where $\wp_3$ is constant of integration.

Also, we have

$$\frac{dx^1}{d\theta} = \cosh \ell,$$
$$\frac{dx^2}{d\theta} = \frac{1}{2} e^{\cosh \ell \theta + \wp_3} \sinh \ell \cos(\wp_1\theta + \wp_2) + \sin(\wp_1\theta + \wp_2)], \tag{3.12}$$
$$\frac{dx^3}{d\theta} = \frac{1}{2} e^{-\cosh \ell \theta - \wp_3} \sinh \ell \cos(\wp_1\theta + \wp_2) - \sin(\wp_1\theta + \wp_2)].$$

If we take the integral of (3.12), we get (3.7). Thus, the proof is completed.

Theorem 3.3. *Let $\gamma : I \to \mathsf{E}(1,1)$ is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions $\mathsf{E}(1,1)$. Then, the parametric equations of γ are*

$$x^1(s) = \cosh \ell \kappa s + \wp_3,$$

$$x^2(s) = \frac{\sinh \ell e^{\cosh \ell \kappa s + \wp_3}}{2(\wp_1^2 + \cosh^2 \ell)} \{(\cosh \ell - \wp_1)\cos(\wp_1 \kappa s + \wp_2)$$
$$+ (\cosh \ell + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_4, \tag{3.13}$$

$$x^3(s) = \frac{\sinh \ell e^{-\cosh \ell \kappa s - \wp_3}}{2(\wp_1^2 + \sinh^2 \ell)} \{-(\cosh \ell - \wp_1)\cos(\wp_1 \kappa s + \wp_2)$$
$$+ (\cosh \ell + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_5,$$

where $\wp_1,\ \wp_2,\ \wp_3,\ \wp_4,\ \wp_5$ *are constants of integration.*

Proof. From first equation of (3.4) and the definition of θ, we have

$$\theta = \kappa s. \tag{3.14}$$

So, substituting (3.14) in the system (3.7), we have (3.13) and the assertion is proved.

We can use Mathematica in Theorem 3.3, yields

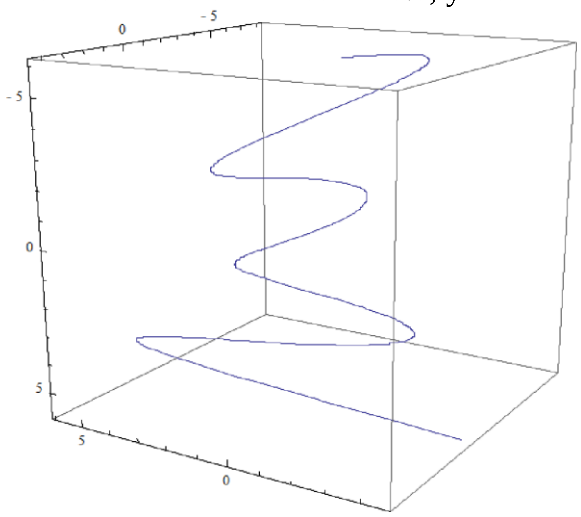

Figure1.

5 Open Problem

In this work, we study timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathrm{E}(1,1)$. We have given some explicit characterizations of biharmonic curves. Additionally, problems such as; investigation timelike biharmonic curves or extending such kind curves to higher dimensional Heisenberg group can be presented as further researches.

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