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A Brief Note on an Exponential Recursive Sequence

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Abstract

Recursive sequence has been used to help solve and characterize some implicit functions. In this note, we consider an exponential recursive sequence $\beta_k = 1 - e^{-\lambda_k \beta_{k-1}}$ ***with*** $\beta_k \in (0,1)$ ***and*** $\lambda_k > 1$***. Under some mild conditions, we show a concise and beautiful property that*** $\prod_{k=1}^{\infty} \beta_k > 0$ ***if and only if*** $\sum_{k=1}^{\infty} e^{-\lambda_k} < \infty$***.***

Keywords: *implicit function, recursive sequence, series.*

1 Three Introductory Examples

For $\lambda > 1$, the equation

$$\beta = 1 - e^{-\lambda\beta} \tag{1}$$

determines a unique $\beta \in (0,1)$ (see e.g. [2] Chap. 5). The form of implicit function (1) appears in various fields in mathematics.

The following are three examples.

Example 1.1 (Galton-Watson branching process) *Let* $X \in \mathbb{N}$ *be a random variable. The Galton-Watson branching process defined by* X *starts with a single particle, which produces* Z_1 *other particles, where the number* Z_1 *of first-generation particles has the same distribution as* X*. Each of the offspring particles produces, in turn, its own children, whose number has distribution* X*, independently for each particle, and so on. If we denote the number of offspring in the* i*-th generation by* Z_i*, then* $Z_0 \equiv 1$*, while for* $i \geq 1$ *the* *variable Z_i is the sum of Z_{i-1} independent copies of X. The probability ρ of extinction of the branching process is defined as $\rho = \lim_{n\to\infty} P(Z_n = 0)$.*

Let $X \sim Poi(\lambda)$. Then the probability generating function of X is

$$f(x) = \sum_{i=0}^{\infty} x^i P(X = i) = \sum_{i=0}^{\infty} \frac{\lambda^i x^i}{i!} e^{-\lambda} = e^{\lambda(x-1)}. \tag{2}$$

It is known that [1], if $EX > 1$ and $P(X = 0) > 0$, the probability ρ that the branching process defined by X dies out is equal to the unique solution of the equation $f(x) = x$ which belongs to the interval $(0, 1)$. Hence, if $\lambda > 1$, $\rho = 1 - \beta$, where $\beta \in (0, 1)$ is uniquely determined by the equation (1).

Example 1.2 (Erdős-Rényi random graph) *Given a real number $p \in [0, 1]$, the Erdős-Rényi random graph, denoted by $G(n, p)$, is defined by taking as Ω the set of all graphs on vertex set $[n] = \{1, 2, \cdots, n\}$ and setting*

$$P(G) = p^e (1-p)^{\binom{n}{2}-e}, \tag{3}$$

where $e = |E(G)|$ stands for the number of edges of G. It can be viewed as a result of $\binom{n}{2}$ independent coin flippings, one for each pair of vertices, with the probability of success (i.e., drawing an edge) equal to p.

Let $p = \lambda/n$ with $\lambda > 1$. Then $G(n, p)$ contains a giant component of $(1 + o(1))\beta n$ vertices, where $\beta \in (0, 1)$ is defined as in (1); see [3, 5].

Example 1.3 (SIR model for epidemic process) *The fully mixed SIR (susceptible-infected-removed) model consists of n people. Suppose that people meet and make contacts sufficient to result in the spread of disease entirely at random with a per-individual rate β, meaning that each individual has, on average, β contacts with randomly chosen others per unit time. Infected individuals remove at some constant average rate γ. Assume the disease starts with a small number c of individuals and everyone else in the susceptible state.*

Let r be the fraction of removed individuals at $t \to \infty$, which is also the total number of individuals who ever catch the disease during the entire course of the epidemic—the total size of the outbreak. It can be shown (see e.g. [4]) that r satisfies the following equation

$$r = 1 - \left(1 - \frac{c}{n}\right) e^{-\frac{\beta r}{\gamma}}. \tag{4}$$

In the limit of large population size $n \to \infty$, by setting $\lambda = \beta/\gamma$, the final value of r satisfies

$$r = 1 - e^{-\lambda r}, \tag{5}$$

which again follows the form of (1).

Now, the implicit function (1) naturally induces the following recursive relation

$$\beta_k = 1 - e^{-\lambda_k \beta_{k-1}}, \tag{6}$$

where $\lambda_k > 1$ and $\beta_k \in (0,1)$ for all $k \geq 1$. We make the following basic observations:

- If $\lim_{k\to\infty} \lambda_k = \lambda < \infty$, then $\lim_{k\to\infty} \beta_k = \beta \in (0,1)$.
- If $\lim_{k\to\infty} \lambda_k = \infty$, then $\lim_{k\to\infty} \beta_k = 1$.

In the sequel, we will focus on a general sequence $\{\lambda_k\}_{k\geq 1}$ and derive an interesting property for $\{\beta_k\}_{k\geq 1}$.

2 A Property for the Recursive Sequence

For each $k \geq 1$, let $\beta_k \in (0,1)$ satisfy the recursive relation (6), where $\beta_0 = 1$. We establish the following result.

Theorem 2.1 *Assume that λ_k is non-decreasing, $\lambda_1 > 1$ and $\lambda_2 > 1/(1 - e^{-1})$. Then $\beta_k > 1 - e^{-1}$ for all $k \geq 1$, and*

$$\sum_{k=1}^{\infty} e^{-\lambda_k(1-e^{-1})} < \infty \implies \prod_{k=1}^{\infty} \beta_k > 0 \implies \sum_{k=1}^{\infty} e^{-\lambda_k} < \infty. \tag{7}$$

Moreover, if $\lambda_k/\lambda_{k-1} \leq C$ for some constant $C > 0$, then

$$\prod_{k=1}^{\infty} \beta_k > 0 \iff \sum_{k=1}^{\infty} e^{-\lambda_k} < \infty. \tag{8}$$

Proof. (6) with $k = 1$ and $\lambda_1 > 1$ imply $\beta_1 > 1 - e^{-1}$. Since $\lambda_2\beta_1 > (1 - e^{-1})/(1 - e^{-1}) = 1$, from (6) we have $\beta_2 > 1 - e^{-1}$. Likewise, we obtain for all $k \geq 1$, $\lambda_k\beta_{k-1} > 1$ and $\beta_k > 1 - e^{-1}$.

We next turn to the proof of (7). It follows from (6) that

$$\begin{aligned} 1 - \beta_k &= e^{-\lambda_k \beta_{k-1}} \\ &= e^{-\lambda_k\left(1-e^{-\lambda_{k-1}\beta_{k-2}}\right)} \\ &= e^{-\lambda_k} e^{\lambda_k e^{-\lambda_{k-1}\beta_{k-2}}} \\ &\leq e^{-\lambda_k} e^{\lambda_k e^{-1}} \\ &= e^{-\lambda_k(1-e^{-1})}, \end{aligned} \tag{9}$$

where the last inequality follows from the fact that $\lambda_k\beta_{k-1} > 1$. Since $\beta_k > 1-e^{-1}$ and there exists α large enough such that $\alpha x > e^x$ for all $x \in [1-e^{-1}, 1]$, we obtain

$$\prod_{k=1}^{\infty} \beta_k \geq \frac{e}{\alpha} \prod_{k=1}^{\infty} e^{-(1-\beta_k)} = \frac{e}{\alpha} e^{-\sum_{k=1}^{\infty}(1-\beta_k)}. \tag{10}$$

Combining (9) and (10), we derive that

$$\sum_{k=1}^{\infty} e^{-\lambda_k(1-e^{-1})} < \infty \implies \prod_{k=1}^{\infty} \beta_k > 0. \tag{11}$$

On the other hand, since $ex \leq e^x$ for all $x \in \mathbb{R}$, and $\beta_k < 1$, we have

$$\begin{aligned} \prod_{k=1}^{\infty} \beta_k &\leq \prod_{k=1}^{\infty} e^{-(1-\beta_k)} \\ &= e^{-\sum_{k=1}^{\infty}(1-\beta_k)} \\ &= e^{-\sum_{k=1}^{\infty}(e^{-\lambda_k\beta_{k-1}})} \\ &< e^{-\sum_{k=1}^{\infty} e^{-\lambda_k}}. \end{aligned} \tag{12}$$

Accordingly,

$$\prod_{k=1}^{\infty} \beta_k > 0 \implies \sum_{k=1}^{\infty} e^{-\lambda_k} < \infty, \tag{13}$$

which together with (11) concludes the proof of (7).

To prove (8), note that

$$\begin{aligned} 1-\beta_k &= e^{-\lambda_k\beta_{k-1}} \\ &= e^{-\lambda_k\left(1-e^{-\lambda_{k-1}\beta_{k-2}}\right)} \\ &= e^{-\lambda_k} e^{\lambda_k e^{-\lambda_{k-1}\beta_{k-2}}} \\ &= e^{-\lambda_k} e^{\frac{\lambda_k\lambda_{k-1}\beta_{k-2}}{\lambda_{k-1}\beta_{k-2}} e^{-\lambda_{k-1}\beta_{k-2}}} \\ &\leq e^{-\lambda_k} e^{\frac{\lambda_k}{\lambda_{k-1}\beta_{k-2} e}} \\ &< e^{-\lambda_k} e^{\frac{C}{e-1}}, \end{aligned} \tag{14}$$

where the last but two inequality follows from the inequality $xe^{-x} \leq e^{-1}$, and the last inequality follows from the fact $\beta_k > 1 - e^{-1}$ and the assumption $\lambda_k/\lambda_{k-1} \leq C$. Hence, (14) and (10) imply that

$$\sum_{k=1}^{\infty} e^{-\lambda_k} < \infty \implies \prod_{k=1}^{\infty} \beta_k > 0. \tag{15}$$

The proof of (8) is thus completed. □

We remark that the cycle in (7) can not be closed, since in general $\sum_{k=1}^{\infty} e^{-\lambda_k} < \infty \not\Longrightarrow \sum_{k=1}^{\infty} e^{-\lambda_k(1-e^{-1})} < \infty$.

3 Open Problems

In this section, we mention some open problems that deserve further investigation. In order to show the equivalence relation (8), we assumed that the growth rate of λ_k is upper bounded. This condition is presumably not optimal and it would be interesting to examine how far it can be relaxed. Indeed, some condition on the rate of change involving λ_k is presumably necessary. Moreover, an alternative generating function approach [6] may be use to help further understand the recursive sequence (6). Potential applications within and beyond each setting of the aforementioned examples would be more than desirable.

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