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On Hermite-Hadamard Type Inequalities for (α, m)-Convex Functions

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Abstract

In the paper, some new inequalities of Hermite-Hadamard type for (α, m)-convex functions are obtained.

Keywords: *Hermite-Hadamard's integral inequality; (α, m)-convex function; Hölder's inequality.*

1 Introduction

Throughout this paper, we use b^* to represent a positive number and adopt the following notations:

$$\mathbb{R} = (-\infty, \infty), \qquad \mathbb{R}_0 = [0, +\infty), \qquad \mathbb{R}_+ = (0, \infty),$$
$$\bar{a} = \min\{a, mb\}, \qquad \bar{b} = \max\{a, mb\}, \qquad \|g\|_\infty = \sup_{t \in [\bar{a}, \bar{b}]} |g(t)|,$$

where $a, b \in \mathbb{R}$ and $m \in (0, 1]$.

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Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality is well known in the literature as Hermite-Hadamard's integral inequality for convex functions. See [9] and closely related references therein.

We now recite two definitions.

Definition 1.1 ([10])**.** *Let $f : [0, b^*] \to \mathbb{R}$ be a function and $m \in (0, 1]$. If*

$$f(\lambda x + m(1-\lambda)y) \le \lambda f(x) + m(1-\lambda)f(y) \tag{2}$$

holds for all $x, y \in [0, b^*]$ and $\lambda \in [0, 1]$, then we say that the function $f(x)$ is m-convex on $[0, b^*]$.

Definition 1.2 ([8])**.** *Let $f : [0, b^*] \to \mathbb{R}$ be a function and $(\alpha, m) \in (0, 1]^2$. If*

$$f(\lambda x + m(1-\lambda)y) \le \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y) \tag{3}$$

is valid for all $x, y \in [0, b^*]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b^*]$.

The following theorems are some known results obtained in recent years.

Theorem 1.1 ([6, Theorem 2.2])**.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)\big(|f'(a)|+|f'(b)|\big)}{8}. \tag{4}$$

Theorem 1.2 ([5, Theorem 2])**.** *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be m-convex and $m \in (0, 1]$. If $f \in L[a, b]$ for $0 \le a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}. \tag{5}$$

Theorem 1.3 ([3, Theorem 2.2])**.** *Let $I \supseteq \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function such that $f'(x) \in L[a, b]$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-convex on $[a, b]$ for some given numbers $m \in (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{4} \\ &\times \min\left\{\left(\frac{|f'(a)|^q + m|f'(b/m)|^q}{2}\right)^{1/q}, \left(\frac{m|f'(a/m)|^q + |f'(b)|^q}{2}\right)^{1/q}\right\}.\end{aligned} \tag{6}$$

Theorem 1.4 ([11, Theorem 1]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is a continuous function, then for all $x \in [a, b]$ we have*

$$\left| f(a) \int_a^x g(s)\,\mathrm{d}s + f(b) \int_x^b g(s)\,\mathrm{d}s - \int_a^b f(s)g(s)\,\mathrm{d}s \right| \le \|g\|_\infty \times \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]^{1-1/q} \left[\frac{(3b - x - 2a)(x-a)^2 + (b-x)^3}{6(b-a)} |f'(a)|^q + \frac{(x-a)^3 + (2b + x - 3a)(b-x)^2}{6(b-a)} |f'(b)|^q \right]^{1/q}. \quad (7)$$

Theorem 1.5 ([11, Theorem 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is a continuous function, then for all $x \in [a, b]$ we have*

$$\left| f(x) \int_a^b g(s)\,\mathrm{d}s - \int_a^b f(s)g(s)\,\mathrm{d}s \right| \le \|g\|_\infty \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]^{1-1/q} \times \left[\frac{(3b - 2x - a)(x-a)^2 + 2(b-x)^3}{6(b-a)} |f'(a)|^q + \frac{2(x-a)^3 + (b + 2x - 3a)(b-x)^2}{6(b-a)} |f'(b)|^q \right]^{1/q}. \quad (8)$$

For more information on this topic, please refer to [1, 2, 4, 7, 9, 12, 13, 14, 15, 16] and plenty of references cited therein.

Our goal of this paper is to establish some new Hermite-Hadamard type inequalities for (α, m)-convex functions.

2 Lemmas

For establishing new integral inequalities of Hermite-Hadamard type for (α, m)-convex functions, we need the following lemmas.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function, $a, b \in I$ with $a < b$, $m \in (0, 1]$, $a \neq mb$, and $g : [\bar{a}, \bar{b}] \to \mathbb{R}$. If $f', g \in L[\bar{a}, \bar{b}]$, then, for all $x \in [\bar{a}, \bar{b}]$, we have*

$$f(a) \int_a^x g(s)\,\mathrm{d}s + f(mb) \int_x^{mb} g(s)\,\mathrm{d}s - \int_a^{mb} f(s)g(s)\,\mathrm{d}s = \int_a^{mb} \int_x^t g(s) f'(t)\,\mathrm{d}s\,\mathrm{d}t. \quad (9)$$

Lemma 2.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function, $a, b \in I$ with $a < b$, $m \in (0,1]$, $a \neq mb$, and $g : [\bar{a}, \bar{b}] \to \mathbb{R}$. If $f', g \in L[\bar{a}, \bar{b}]$, then for all $x \in [\bar{a}, \bar{b}]$, we have*

$$f(x) \int_a^{mb} g(s)\,\mathrm{d}s - \int_a^{mb} f(s)g(s)\,\mathrm{d}s = \int_a^{mb} S_g(t) f'(t)\,\mathrm{d}t, \tag{10}$$

where

$$S_g(t) = \begin{cases} \displaystyle\int_{\bar{a}}^{t} g(s)\,\mathrm{d}s, & t \in [\bar{a}, x), \\ \displaystyle -\int_{t}^{\bar{b}} g(s)\,\mathrm{d}s, & t \in [x, \bar{b}] \end{cases} \quad \text{and} \quad S_1(t) = \begin{cases} t - \bar{a}, & t \in [\bar{a}, x), \\ \bar{b} - t, & t \in [x, \bar{b}]. \end{cases}$$

Proof of Lemmas 2.1 and 2.2. These lemmas can be deduced directly from integrating by part the right-hand sides of (9) and (10) respectively. □

3 Some new integral inequalities of Hermite-Hadamard type for (α, m)-convex functions

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type for functions whose derivatives are (α, m)-convex.

Theorem 3.1. *Let $f : [0, b^*] \to \mathbb{R}$ be a differentiable function, $a, b \in [0, b^*]$ with $a < b$, $(\alpha, m) \in (0,1]^2$, $a \neq mb$, and $f' \in L[\bar{a}, \bar{b}]$. If $|f'(x)|^q$ for $q \geq 1$ is (α, m)-convex on $[0, b]$, $g : [\bar{a}, \bar{b}] \to \mathbb{R}$ is a continuous function, and $x \in [\bar{a}, \bar{b}]$, then*

$$\begin{aligned} &\left| f(a) \int_a^x g(s)\,\mathrm{d}s + f(mb) \int_x^{mb} g(s)\,\mathrm{d}s - \int_a^{mb} f(s)g(s)\,\mathrm{d}s \right| \\ &\quad \leq \|g\|_\infty \left[\frac{(x-a)^2 + (mb-x)^2}{2} \right]^{1-1/q} \left\{ m \frac{(x-a)^2 + (mb-x)^2}{2} |f'(b)|^q \right. \\ &\qquad + \frac{|f'(a)|^q - m|f'(b)|^q}{(\alpha+1)(\alpha+2)} \left[2(mb-x)^2 \left(\frac{mb-x}{mb-a} \right)^\alpha \right. \\ &\qquad \left.\left. + (mb-a)[(\alpha+1)(x-a) - (mb-x)] \right] \right\}^{1/q}. \end{aligned} \tag{11}$$

Proof. Using Lemma 2.1, Hölder inequality, and the (α, m)-convexity of $|f'|^q$, it follows that

$$\left| f(a) \int_a^x g(s)\,\mathrm{d}s + f(mb) \int_x^{mb} g(s)\,\mathrm{d}s - \int_a^{mb} f(s)g(s)\,\mathrm{d}s \right|$$

$$
\begin{aligned}
&\le \int_{\bar a}^{\bar b}\left|\int_x^t g(s)\,\mathrm{d}s\right||f'(t)|\,\mathrm{d}t \le \left(\int_{\bar a}^{\bar b}\left|\int_x^t g(s)\,\mathrm{d}s\right|\mathrm{d}t\right)^{1-1/q}\\
&\quad\times\left(\int_{\bar a}^{\bar b}\left|\int_x^t g(s)\,\mathrm{d}s\right||f'(t)|^q\,\mathrm{d}t\right)^{1/q} \le \|g\|_\infty\left[\int_{\bar a}^{\bar b}|t-x|\,\mathrm{d}t\right]^{1-1/q}\\
&\quad\times\left[\int_{\bar a}^{\bar b}|t-x||f'(t)|^q\,\mathrm{d}t\right]^{1/q} = \|g\|_\infty\left[\frac{(x-a)^2+(mb-x)^2}{2}\right]^{1-1/q}\\
&\quad\times\left[\int_{\bar a}^{\bar b}|t-x|\left|f'\left(\frac{\bar b-t}{\bar b-\bar a}\bar a+\frac{t-\bar a}{\bar b-\bar a}\bar b\right)\right|^q\mathrm{d}t\right]^{1/q}\\
&\le \|g\|_\infty\left[\frac{(x-a)^2+(mb-x)^2}{2}\right]^{1-1/q}\left\{\int_{\bar a}^{\bar b}|t-x|\left[\left(\frac{mb-t}{mb-a}\right)^\alpha|f'(a)|^q\right.\right.\\
&\quad\left.\left.+\,m\left(1-\left(\frac{mb-t}{mb-a}\right)^\alpha\right)|f'(b)|^q\right]\mathrm{d}t\right\}^{1/q}\\
&= \|g\|_\infty\left[\frac{(x-a)^2+(mb-x)^2}{2}\right]^{1-1/q}\left\{m|f'(b)|^q\int_{\bar a}^{\bar b}|t-x|\,\mathrm{d}t\right.\\
&\quad\left.+\,(|f'(a)|^q-m|f'(b)|^q)\int_{\bar a}^{\bar b}|t-x|\left(\frac{mb-t}{mb-a}\right)^\alpha\mathrm{d}t\right\}^{1/q}
\end{aligned}
$$

for $x\in[\bar a,\bar b]$. Substituting $\int_{\bar a}^{\bar b}|t-x|\,\mathrm{d}t=\frac{1}{2}[(x-a)^2+(mb-x)^2]$ and

$$
\begin{aligned}
\int_{\bar a}^{\bar b}|t-x|\left(\frac{mb-t}{mb-a}\right)^\alpha\mathrm{d}t &= \frac{1}{(\alpha+1)(\alpha+2)}\\
&\quad\times\left\{2(mb-x)^2\left(\frac{mb-x}{mb-a}\right)^\alpha+(mb-a)[(\alpha+1)(x-a)-(mb-x)]\right\}
\end{aligned}
$$

into the above inequality leads to (11). Theorem 3.1 is thus proved. □

Remark 3.2. *The inequality* (7) *is a special case of* (11) *applied to* $m=\alpha=1$.

Corollary 3.1. *Under the conditions of Theorem 3.1, if* $q=1$, *we have*

$$
\begin{aligned}
&\left|f(a)\int_a^x g(s)\,\mathrm{d}s+f(mb)\int_x^{mb}g(s)\,\mathrm{d}s-\int_a^{mb}f(s)g(s)\,\mathrm{d}s\right|\le\|g\|_\infty\\
&\quad\times\left\{m\frac{(x-a)^2+(mb-x)^2}{2}|f'(b)|+\left[2(mb-x)^2\left(\frac{mb-x}{mb-a}\right)^\alpha\right.\right.\\
&\quad\left.\left.+\,(mb-a)[(\alpha+1)(x-a)-(mb-x)]\right]\frac{|f'(a)|-m|f'(b)|}{(\alpha+1)(\alpha+2)}\right\}. \qquad (12)
\end{aligned}
$$

Theorem 3.3. *Let* $f:[0,b^*]\to\mathbb{R}$ *be a differentiable function,* $a,b\in[0,b^*]$ *with* $a<b$, $(\alpha,m)\in(0,1]^2$, $a\ne mb$, *and* $f'\in L[\bar a,\bar b]$. *If* $|f'(x)|^q$ *for* $q>1$

is (α, m)*-convex on* $[0, b]$*,* $g : [\bar{a}, \bar{b}] \to \mathbb{R}$ *is a continuous function, then, for all* $x \in [\bar{a}, \bar{b}]$*, we have*

$$\begin{aligned}\left|f(a)\int_a^x g(s)ds + f(mb)\int_x^{mb} g(s)ds - \int_a^{mb} f(s)g(s)\,\mathrm{d}s\right| \\ \le \|g\|_\infty\left\{\frac{q-1}{2q-1}\left[|x-a|^{(2q-1)/(q-1)} + |mb-x|^{(2q-1)/(q-1)}\right]\right\}^{1-1/q} \\ \times\left\{\frac{|mb-a|}{\alpha+1}[|f'(a)|^q + \alpha m|f'(b)|^q]\right\}^{1/q}. \qquad (13)\end{aligned}$$

Proof. By Lemma 2.1, Hölder inequality, and the (α, m)-convexity of $|f'|^q$, for $x \in [\bar{a}, \bar{b}]$, it follows that

$$\begin{aligned}&\left|f(a)\int_a^x g(s)\,\mathrm{d}s + f(mb)\int_x^{mb} g(s)\,\mathrm{d}s - \int_a^{mb} f(s)g(s)\,\mathrm{d}s\right| \\ &\le \int_{\bar{a}}^{\bar{b}}\left|\int_x^t g(s)\,\mathrm{d}s\right||f'(t)|\mathrm{d}t \le \left(\int_{\bar{a}}^{\bar{b}}\left|\int_x^t g(s)\,\mathrm{d}s\right|^{q/(q-1)}\mathrm{d}t\right)^{1-1/q} \\ &\quad\times\left(\int_{\bar{a}}^{\bar{b}}|f'(t)|^q\,\mathrm{d}t\right)^{1/q} \le \|g\|_\infty\left[\int_{\bar{a}}^{\bar{b}}|t-x|^{q/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left[\int_{\bar{a}}^{\bar{b}}|f'(t)|^q\,\mathrm{d}t\right]^{1/q} \\ &= \|g\|_\infty\left(\frac{q-1}{2q-1}\left[|x-a|^{(2q-1)/(q-1)} + |mb-x|^{(2q-1)/(q-1)}\right]\right)^{1-1/q} \\ &\quad\times\left[\int_{\bar{a}}^{\bar{b}}\left|f'\left(\frac{\bar{b}-t}{\bar{b}-\bar{a}}\bar{a} + \frac{t-\bar{a}}{\bar{b}-\bar{a}}\bar{b}\right)\right|^q\mathrm{d}t\right]^{1/q} \\ &\le \|g\|_\infty\left(\frac{q-1}{2q-1}\left[|x-a|^{(2q-1)/(q-1)} + |mb-x|^{(2q-1)/(q-1)}\right]\right)^{1-1/q} \\ &\quad\times\left\{\int_{\bar{a}}^{\bar{b}}\left[\left(\frac{mb-t}{mb-a}\right)^\alpha|f'(a)|^q + m\left(1-\left(\frac{mb-t}{mb-a}\right)^\alpha\right)|f'(b)|^q\right]\mathrm{d}t\right\}^{1/q} \\ &= \|g\|_\infty\left\{\frac{q-1}{2q-1}\left[|x-a|^{(2q-1)/(q-1)} + |mb-x|^{(2q-1)/(q-1)}\right]\right\}^{1-1/q} \\ &\quad\times\left\{\frac{|mb-a|}{\alpha+1}[|f'(a)|^q + \alpha m|f'(b)|^q]\right\}^{1/q}.\end{aligned}$$

The proof of Theorem 3.3 is complete. □

Corollary 3.2. *Under the conditions of Theorem 3.3, if* $m = \alpha = 1$*, we have*

$$\left|f(a)\int_a^x g(s)\,\mathrm{d}s + f(b)\int_x^b g(s)\,\mathrm{d}s - \int_a^b f(s)g(s)\,\mathrm{d}s\right|$$

$$\leq \|g\|_\infty \left\{ \frac{q-1}{2q-1} \left[(x-a)^{(2q-1)/(q-1)} + (b-x)^{(2q-1)/(q-1)} \right] \right\}^{1-1/q} \times \left\{ \frac{b-a}{2} [|f'(a)|^q + |f'(b)|^q] \right\}^{1/q}. \quad (14)$$

Theorem 3.4. *Let $f : [0, b^*] \to \mathbb{R}$ be a differentiable function, $a, b \in [0, b^*]$ with $a < b$, $(\alpha, m) \in (0, 1]^2$, $a \neq mb$, and $f' \in L[\bar{a}, \bar{b}]$. If $|f'(x)|^q$ for $q \geq 1$ is (α, m)-convex on $[0, b]$, $g : [\bar{a}, \bar{b}] \to \mathbb{R}$ is a continuous function, and $x \in [\bar{a}, \bar{b}]$, then*

$$\left| f(x) \int_a^{mb} g(s)\,\mathrm{d}s - \int_a^{mb} f(s)g(s)\,\mathrm{d}s \right| \leq \|g\|_\infty \left[\frac{(x-a)^2 + (mb-x)^2}{2} \right]^{1-1/q} \times \left\{ m \left[\frac{(x-a)^2 + (mb-x)^2}{2} \right] |f'(b)|^q + \left[[\alpha(mb-x)^2 - (\alpha+2) \times (mb-x)(x-a)] \left(\frac{mb-x}{mb-a} \right)^\alpha + (mb-a)^2 \right] \frac{|f'(a)|^q - m|f'(b)|^q}{(\alpha+1)(\alpha+2)} \right\}^{1/q}. \quad (15)$$

Proof. Using Lemma 2.2 yields $|S_g(t)| \leq \|g\|_\infty S_1(t)$ for $t \in [\bar{a}, \bar{b}]$. By Hölder inequality and the (α, m)-convexity of $|f'|^q$, it follows that

$$\left| f(x) \int_a^{mb} g(s)\,\mathrm{d}s - \int_a^{mb} f(s)g(s)\,\mathrm{d}s \right| \leq \int_{\bar{a}}^{\bar{b}} |S_g(t)||f'(t)|\,\mathrm{d}t$$

$$\leq \|g\|_\infty \left[\int_{\bar{a}}^{\bar{b}} S_1(t)\,\mathrm{d}t \right]^{1-1/q} \left[\int_{\bar{a}}^{\bar{b}} S_1(t)|f'(t)|^q\,\mathrm{d}t \right]^{1/q}$$

$$\leq \|g\|_\infty \left[\frac{(x-a)^2 + (mb-x)^2}{2} \right]^{1-1/q} \left\{ m|f'(b)|^q \int_{\bar{a}}^{\bar{b}} S_1(t)\,\mathrm{d}t + (|f'(a)|^q - m|f'(b)|^q) \int_{\bar{a}}^{\bar{b}} S_1(t) \left(\frac{mb-t}{mb-a} \right)^\alpha \mathrm{d}t \right\}^{1/q}$$

$$\leq \|g\|_\infty \left[\frac{(x-a)^2 + (mb-x)^2}{2} \right]^{1-1/q} \left\{ m \left[\frac{(x-a)^2 + (mb-x)^2}{2} \right] |f'(b)|^q + \frac{1}{(\alpha+1)(\alpha+2)} \left[\left(\frac{mb-x}{mb-a} \right)^\alpha [\alpha(mb-x)^2 - (\alpha+2)(mb-x)(x-a)] + (mb-a)^2 \right] (|f'(a)|^q - m|f'(b)|^q) \right\}^{1/q}.$$

The proof of Theorem 3.4 is complete. □

Remark 3.5. *The inequality* (8) *is a special case of* (15) *applied to $m = \alpha = 1$.*

Corollary 3.3. *Under the conditions of Theorem 3.4, if $q = 1$, we have*

$$\begin{aligned}\bigg|f(x)\int_a^{mb} g(s)\,\mathrm{d}s-\int_a^{mb} f(s)g(s)\,\mathrm{d}s\bigg| &\le \|g\|_\infty\bigg\{m\bigg[\frac{(x-a)^2+(mb-x)^2}{2}\bigg]\\ &\quad\times|f'(b)|+\frac{1}{(\alpha+1)(\alpha+2)}\bigg[[\alpha(mb-x)^2-(\alpha+2)(mb-a)(x-a)]\\ &\quad\times\bigg(\frac{mb-x}{mb-a}\bigg)^\alpha+(mb-a)^2\bigg](|f'(a)|-m|f'(b)|)\bigg\}. \qquad (16)\end{aligned}$$

Theorem 3.6. *Let $f:[0,b^*]\to\mathbb{R}$ be a differentiable function, $a,b\in[0,b^*]$ with $a<b$, $(\alpha,m)\in(0,1]^2$, $a\ne mb$, and $f'\in L[\bar a,\bar b]$. If $|f'(x)|^q$ for $q>1$ is (α,m)-convex on $[0,b]$, $g:[\bar a,\bar b]\to\mathbb{R}$ is a continuous function, and $x\in[\bar a,\bar b]$, then*

$$\begin{aligned}\bigg|f(x)\int_a^{mb} g(s)\,\mathrm{d}s-\int_a^{mb} f(s)g(s)\,\mathrm{d}s\bigg| &\le \|g\|_\infty\bigg\{\frac{q-1}{2q-1}\big[|x-a|^{(2q-1)/(q-1)}\\ &\quad+|mb-x|^{(2q-1)/(q-1)}\big]\bigg\}^{1-1/q}\bigg\{\frac{|mb-a|}{\alpha+1}[|f'(a)|^q+\alpha m|f'(b)|^q]\bigg\}^{1/q}. \qquad (17)\end{aligned}$$

Proof. By Lemma 2.2, Hölder inequality, and the (α,m)-convexity of $|f'|^q$, for $x\in[\bar a,\bar b]$, it follows that

$$\begin{aligned}\bigg|f(x)\int_a^{mb} g(s)\,\mathrm{d}s-\int_a^{mb} f(s)g(s)\,\mathrm{d}s\bigg| &\le \|g\|_\infty\bigg\{\int_{\bar a}^{\bar b}[S_1(t)]^{q/(q-1)}\,\mathrm{d}t\bigg\}^{1-1/q}\\ &\quad\times\bigg\{\int_{\bar a}^{\bar b}|f'(t)|^q\,\mathrm{d}t\bigg\}^{1/q}\le\|g\|_\infty\bigg\{\frac{q-1}{2q-1}\big[|x-a|^{(2q-1)/(q-1)}\\ &\quad+|mb-x|^{(2q-1)/(q-1)}\big]\bigg\}^{1-1/q}\bigg[\int_{\bar a}^{\bar b}\bigg|f'\bigg(\frac{\bar b-t}{\bar b-\bar a}\bar a+\frac{t-\bar a}{\bar b-\bar a}\bar b\bigg)\bigg|^q\mathrm{d}t\bigg]^{1/q}\\ &=\|g\|_\infty\bigg\{\frac{q-1}{2q-1}\big[|x-a|^{(2q-1)/(q-1)}+|mb-x|^{(2q-1)/(q-1)}\big]\bigg\}^{1-1/q}\\ &\quad\times\bigg\{\frac{|mb-a|}{\alpha+1}[|f'(a)|^q+\alpha m|f'(b)|^q]\bigg\}^{1/q}.\end{aligned}$$

The proof of Theorem 3.6 is complete. □

Corollary 3.4. *Under the conditions of Theorem 3.6, if $m=\alpha=1$, we have*

$$\begin{aligned}\bigg|f(x)\int_a^{b} g(s)\,\mathrm{d}s-\int_a^{b} f(s)g(s)\,\mathrm{d}s\bigg| &\le \|g\|_\infty\bigg\{\frac{q-1}{2q-1}\big[(x-a)^{(2q-1)/(q-1)}\\ &\quad+(b-x)^{(2q-1)/(q-1)}\big]\bigg\}^{1-1/q}\bigg\{\frac{b-a}{2}[|f'(a)|^q+|f'(b)|^q]\bigg\}^{1/q}. \qquad (18)\end{aligned}$$

4 Two open problems

Finally, we would like to pose two open problems as follows.

1. When the function g is symmetric with respect to the midpoint $\frac{\bar{a}+\bar{b}}{2}$ of the interval $[\bar{a},\bar{b}]$, can one simplify the inequalities (11), (13), (15), and (17) in Theorems 3.1, 3.3, 3.4, and 3.6 respectively?

2. Can these inequalities established in this paper be applied to construct inequalities of special means of two positive numbers, as done in [6, 14] and other papers?

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