Int. J. Open Problems Compt. Math., Vol. 5, No. 4, December 2012
ISSN 1998-6262; Copyright ©ICSRS Publication, 2012
www.i-csrs.org

A weighted target- following algorithm for linearly constrained convex optimization

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Abstract

In recent years, convex optimization solved very large practical engineering problems reliably and efficiently. In this paper, we present an extension of an algorithm for convex quadratic programming using a new technique for finding a class of search directions and the strategy of the central path, for convex optimization under linear constraints. To solve the initialization problem, we have introduced a weighted vector with the property that starting from an initial feasible centred point, it generates iterates that simultaneously, gets closer to optimality and closer to centrality. Finally, the favorable polynomial complexity bound for the algorithm is deserved namely, O** $(\sqrt{n}\log(\frac{x^tz}{\epsilon}))$ **iterations.

Keywords: *interior points methods, linearly constrained convex optimization, primal-dual target following algorithm, equivalent algebraic transformation, polynomial complexity.*

1 Introduction

Interior point methods ($IPMs$) are among the most effective methods for solving wide classes of optimization problems because of their polynomial complexity and their numerical efficiency. Since the seminal work of Karmarkar [7] in 1984, many researchers have proposed and analyzed various $IPMs$ for linear optimization (LO) and a large amount of results have been reported. The search directions play an important role in finding new algorithms. Peng, Roos and Terlaky [8] have defined the notion of self regular functions and, using this concept, they have introduced a new class of search directions for LO. They have extended their results also to complementarity problems (CP), semidefinite optimization (SDO) and second order cone optimization ($SOCO$), and they have proved polynomial complexity of different large-update algorithms, which use self-regular functions to obtain new directions. An alternative method has been introduced in [3, 5, 6] by applying algebraically equivalent transformations to the nonlinear centering equation of the system, which defines the central path, this method has been applied with success to LO. Recently, the new technique for LO has been extended also to convex quadratic optimization (CQP) by Achache [1] and to monotone mixed linear complementarity problems ($LCPs$) by Wang, Cai and Yue [9]. The method of algebraically equivalent transformation has been generalized also to weighted path following algorithms. The first results for (LO) have been given in [3]. Later on, Achache [2] generalized this algorithm to standard $LCPs$. The above mentioned algebraic transformations, followed by a Newton step, resulted in small-update feasible algorithms, and for all of them the best known iteration bounds were obtained.

In this paper we extend the weighted path following algorithms to linearly constrained convex optimization ($LCCO$).

The paper is organized as follows: in Section 2 the statement of the problem is presented. In Section 3, we deal with the new search directions and the description of the algorithm. In Section 4 we state its complexity analysis. Finally, we present some conclusions in Section 5.

The notations used in this paper are the following: $\Re^n$ is the set of n dimentional vectors and $\Re^{m\times n}$ is the set of $m \times n$ matrices. Moreover, $\Re_{++}$ is the set of strictly positive real numbers.

2 Statement of the problem

Let us consider the following problem

$$(P)\begin{cases} min\ f(x) \\ Ax = b \\ x \geqslant 0 \end{cases}$$

and its dual

$$(D)\begin{cases} max\ b^t y + f(x) - (\nabla f(x))^t x \\ A^t y + z - \nabla f(x) = 0 \\ z \geq 0, \text{y} \in \Re^m \end{cases}$$

Where $A \in \Re^{m\times n}$, $rank(A) = m$, $b \in \Re^m$, $c \in \Re^n$ and $f : \Re^n \to \Re$ is a convex and twice continuously differentiable function.

we impose the following assumptions:

(H1): $K_{int} = \{x \in \Re^n / \ Ax = b, \ x > 0\}$ the set of strictly feasible points of (P) is non-empty,

(H2): $T_{int} = \{y \in \Re^m, z \in \Re^n / A^t y + z - \nabla f(x) = 0, \ z > 0\}$ the set of strictly feasible points of (D) is non-empty.

In order to introduce an interior point method to solve (P), we associate the following barrier minimization problem

$$(P_\mu) \begin{cases} min \ f(x) - \mu \sum_{i=1}^{n} r_i ln \ x_i = f_\mu(x) \\ Ax = b \\ x \ > \ 0 \end{cases}$$

where $\mu > 0$ be the barrier parameter and $r = (r_1, r_2, ..., r_n) \in \Re^n_{++}$ is a weighted vector introduced to ensure that the initial point (x^0, z^0, μ^0) verified $\delta(x^0 z^0, \mu^0) = 0 < 1$ (proximity measure which will be defined bellow), if $r_i = 1, \forall i$ then the weighted central path coincides with the classical one. Hence, this approach can be seen as a generalization of central path methods.

The resolution of (P_μ) is equivalent at that of (P) with that if $x^*(\mu)$ is an optimal solution of (P_μ) then $x^* = \lim\limits_{\mu \to 0} x^*(\mu)$ is an optimal solution of (P).

The problem (P_μ) is a convex optimization problem and then its first order optimality conditions are:

$$(1) \begin{cases} A^t y + z - \nabla f(x) = 0, \ x > 0, \ z > 0 \\ Ax = b \\ xz = \mu r \end{cases}$$

where xs denotes the coordinatewise product of the vectors x and s, hence $xs = (x_1 z_{1,} x_2 z_2, ..., x_n z_n)^T$.

Under the assumptions H1, H2 and A has full rank the system (1) has a unique solution. [10]

3 New search directions

The basic idea behind this approach is to replace the nonlinear equation:

$xz = \mu r$ in (1) by an equivalent equation: $\psi(xz) = \psi(\mu r\ )$

where ψ is a real valued function on $[0, +\infty)$ and differentiable on $(0, +\infty)$ such that $\psi(t)$ and $\psi\prime(t) > 0$, for all $t > 0$. Then the system (1) can be written as the following equivalent form:

$$(2)\ \begin{cases} A^t y + z - \nabla f(x) = 0,\ x > 0,\ z > 0 \\ Ax = b \\ \psi(xz) = \psi(\mu r) \end{cases}$$

Applying Newton's method for the system (2) we get

$$(3)\ \begin{cases} A^t \Delta y + \Delta z - \nabla^2 f(x)\Delta x\ = 0 \\ A\Delta x\ = 0 \\ z\psi\prime(xz)\Delta x + x\psi\prime(xz)\Delta z = \psi(\mu r) - \psi(xz) \end{cases}$$

Now, the following notations are useful for studying the complexity of the proposed algorithm.

Let $(x,\, z)$ be a pair of primal-dual interior feasible solutions, we introduce the scaled vectors v and d as follows:

$v = \sqrt{xz}$, $d = \sqrt{\frac{x}{z}}$

Using d we can rescale both x and z to the same vector:

$d^{-1}x = dz = v$

we also use d to rescale Δx and Δz :$p_x = d^{-1}\Delta x$, $p_z = d\Delta z$ and $p_y = \Delta y$

Now we may write

$x\Delta z + z\Delta x = xd^{-1}d\Delta z + zdd^{-1}\Delta x = v(\ p_x +\ p_z)$

Hence, Newton's direction is determined by the following linear system:

$$(4)\begin{cases} -\bar{H}p_x + \bar{A}^t p_y + p_z = 0 \\ \bar{A}p_x\ = 0 \\ p_x + p_z = p_v \end{cases}$$

where $D = diag(d)$, $\bar{H} = D\nabla^2 f(x)D$ is symmetric and positive semidefinite matrix, $\bar{A} = AD$ and $p_v = \frac{\psi(\mu r) - \psi(v^2)}{v\psi\prime(v^2)}$,

As in [3], we shall consider the following function:

$$\psi(t) = \sqrt{t}$$

with $\psi\prime(t) = \frac{1}{2\sqrt{t}} > 0$ for all $t > 0$.

We have from (4):

$$(5)\begin{cases} -\bar{H}p_x + \bar{A}^t p_y + p_z = 0 \\ \bar{A}p_x\ = 0 \\ p_x + p_z = 2(\sqrt{\mu r} - v) \end{cases}$$

We define for all vector v the following proximity measure by:
$\delta(xz,\mu) = \delta(v,\mu) = \frac{\|p_v\|}{2\min(\sqrt{\mu r})} = \frac{\|\sqrt{\mu r}-v\|}{\min(\sqrt{\mu r})}$
were $\|\|$ is the Euclidean norm (l_2 norm) and $\min(x) = \min\{x_1,\ x_2, ...,\ x_n\}$.
We introduce another measure $\sigma_c(r) = \frac{\max(r)}{\min(r)}$
Now, we get the short-step primal-dual algorithm to solve $(LCCO)$:

Algorithm for linearly constrained convex optimization

- **Input:** $(x^{(0)}, y^{(0)}, z^{(0)})$ where $x^{(0)}$ is a strictly feasible solution of (P), $(y^{(0)}, z^{(0)})$ is a strictly feasible solution of (D), $\mu^{(0)} > 0$ an initial barrier parameter, $0 < \theta < 1$ and ε is the accuracy parameter.

- compute: $r = \frac{x^{(0)}z^{(0)}}{\mu^0}$

- **begin:**

  - $x = x^{(0)}, z = z^{(0)}, v = \sqrt{xz}$,d=$\sqrt{\frac{x}{z}}$, $\mu = \mu^{(0)}$
  - **while** $x^t z > \varepsilon$ **do**
  - Solve the Newton system of equations in (5)
  - compute $\Delta x = dp_x, \Delta z = d^{-1}p_z$ and $\Delta y = p_y$
  - compute $x = x + \Delta x,\ \ y = y + \Delta y,\ z = z + \Delta z$ and $\mu = (1-\theta)\mu$.

- **end.**

Remark 1 *By construction, to guarantee that the next Newton iterate $\hat{x} = x + \alpha_x \Delta x > 0$ and $\hat{z} = z + \alpha_z \Delta z > 0$ for any $\alpha \in IR$, it suffices to set*

$$\alpha_x = \begin{cases} min\,(-x_i/\Delta x_i) & si\ \Delta x_i\ <\ 0 \\ \ 1 & si\ \Delta x_i \geq 0 \end{cases}$$
$$\alpha_z = \begin{cases} min\,(-z_i/\Delta z_i) & si\ \Delta z_i\ <\ 0 \\ \ 1 & si\ \Delta z_i \geq 0 \end{cases}$$

4 Complexity analysis

Let

$q_v = p_x - p_z$

we have

$p_x = \frac{1}{2}(p_v + q_v), \quad p_z = \frac{1}{2}(p_v - q_v)$

$p_x p_z = \frac{1}{4}(p_v^2 - q_v^2)$ and $\|q_v\| \leq \|p_v\|$

This last result follows directly from the equality

$\|p_v\|^2 = \|q_v\|^2 + 4p_x^T p_z$ We have $\delta(v,\mu) \geq \frac{\|q_v\|}{2\min(\sqrt{\mu r})}$

since

$p_x^T p_z = p_x^T \bar{H} p_x \geq 0$

because The function f is convex, thus the matrices $\nabla^2 f(x)$ and $\bar{H}$ are symmetric and positive semidefinite.

In the following Lemma, we state a condition which ensures the feasibility of the full Newton step. Let $\hat{x} = x + \Delta x$ and $\hat{z} = z + \Delta z$, be the new iterate after a full Newton step.

Lemma 1 *: Let $\delta = \delta(v,\mu) < 1$.Then the full Newton step is strictly feasible, hence: $\hat{x} > 0$ and $\hat{z} > 0$.see* [3]

In the next lemma we show that $\delta < 1$ is sufficient for the quadratic convergence of the Newton process.

Lemma 2 *: Let $\hat{x} = x + \Delta x$ and $\hat{z} = z + \Delta z$ be the iteration obtained after a full Newton step with $v = \sqrt{xz}$ and $\hat{v} = \sqrt{\hat{x}\hat{z}}$*

Suppose $\delta = \delta(v,\mu) < 1$.Then $\delta(\hat{v},\mu) \leq \frac{\delta^2}{1+\sqrt{1-\delta^2}}$

thus $\delta(\hat{v},\mu) < \delta^2(v,\mu)$, which means quadratic convergence of the Newton step.

Proof:

We have:

$$\begin{aligned}(\hat{v})^2 &= \hat{x}\hat{z} \\ &= (x+\Delta x)(z+\Delta z) \\ &= v^2 + vp_v + \frac{p_v^2}{4} - \frac{q_v^2}{4} \\ &= \mu r - \frac{p_v^2}{4} + \frac{p_v^2}{4} - \frac{q_v^2}{4} \\ &= \mu r - \frac{q_v^2}{4}\end{aligned}$$

we obtain

$\min(\hat{v})^2 \geq \min(\mu r) - \frac{\|q_v^2\|_\infty}{4} \geq \min(\mu r) - \frac{\|q_v\|^2}{4} \geq \min(\mu r)(1-\delta^2)$

and this relation yields:

$\min(\hat{v}) \geq \min(\sqrt{\mu r})(\sqrt{1-\delta^2})$

Furthermore

$$\begin{aligned}\delta(\hat{v},\mu) &= \frac{1}{\min\sqrt{\mu r}}\left\|\frac{\mu r-\hat{v}}{\sqrt{\mu r}+\hat{v}}\right\| \\ &\leq \frac{\left\|\mu r-v^{+}\right\|}{\min\sqrt{\mu r}(\min(\sqrt{\mu r}+\hat{v}))} \\ &\leq \frac{\left\|\mu r-v^{+}\right\|}{(\min\sqrt{\mu r})^2(1+\sqrt{1-\delta^2})} \\ &\leq \frac{\left\|q_v^2\right\|}{(\min\sqrt{\mu r})^2(1+\sqrt{1-\delta^2})} \\ &\leq \frac{\delta^2}{1+\sqrt{1-\delta^2}}\end{aligned}$$

In the next lemma we state an upper bound for the duality gap obtained after a full Newton step.

Lemma 3 : *Let* $\hat{x} = x + \Delta x$ *and* $\hat{z} = z + \Delta z$. *Then the duality gap is:* $(\hat{x})^T\hat{z} = \mu\left\|\sqrt{r}\right\|^2 - \frac{\|q_v\|^2}{4}$,
hence

$$(\hat{x})^T\hat{z} \leq \mu\left\|\sqrt{\frac{x^0z^0}{\mu^0}}\right\|^2.$$

Proof:
From

$(\hat{v})^2 = \mu r - \frac{q_v^2}{4}$

we have $\hat{x}\hat{z} = \mu r - \frac{q_v^2}{4}$

we obtain $(\hat{x})^T\hat{z} = e^T(\hat{x}\hat{z}) = \mu e^T r - \frac{e^T q_v^2}{4} = \mu\left\|\sqrt{r}\right\|^2 - \frac{\|q_v\|^2}{4}$

this relation yields

$$(\hat{x})^T\hat{z} \leq \mu\left\|\sqrt{r}\right\|^2 = \mu\left\|\sqrt{\frac{x^0z^0}{\mu^0}}\right\|^2$$

The next lemma discusses the influence on the proximity measure of the Newton process followed by a step along from the central path. We assume the parameter μ will be reduced by a constant factor $(1-\theta)$.

Lemma 4 : *Let* $\delta = \delta(xz,\mu) < 1$ *and* $\mu^{+} = (1-\theta)\mu$, *where* $0 < \theta < 1$. *Then*

$$\delta(\hat{v},\hat{\mu}) \leq \frac{\theta}{1-\theta}\sqrt{\sigma_c(r)} + \frac{1}{1-\theta}\,\delta(\hat{v},\mu).$$

Furthermore, if $\delta \leq \frac{1}{2}$, $\theta = \frac{2}{5\sqrt{\sigma_c(r)n}}$ and $n \geq 4$ then we get $\delta(\hat{v},\hat{\mu}) \leq \frac{1}{2}$.

Proof:

$$\begin{aligned}\delta(\hat{v},\hat{\mu}) &= \frac{\left\|\sqrt{\hat{\mu}r}-\hat{v}\right\|}{\min\sqrt{\mu^{+}r}} = \frac{\left\|\sqrt{\hat{\mu}r}-\sqrt{\mu r}+\sqrt{\mu r}-\hat{v}\right\|}{\min\sqrt{\hat{\mu}r}} \\ &\leq \frac{\left\|\sqrt{\hat{\mu}r}-\sqrt{\mu r}\right\|}{\min\sqrt{\mu^{+}r}} + \frac{\left\|\sqrt{\mu r}-\hat{v}\right\|}{\min\sqrt{\mu^{+}r}} \\ &= \frac{1-\sqrt{1-\theta}}{\sqrt{1-\theta}}\left(\frac{\|\mu r\|}{\min(\sqrt{\mu r})}\right) + \frac{1}{\sqrt{1-\theta}}\,\delta(\hat{v},\mu) \\ &\leq \frac{1-\sqrt{1-\theta}}{\sqrt{1-\theta}}\sqrt{n\sigma_c(r)} + \frac{1}{\sqrt{1-\theta}}\,\delta(\hat{v},\mu)\end{aligned}$$

Now let $\theta = \frac{2}{5\sqrt{n\sigma_c(r)}}$, observe that $\sigma_c(r) \geq 1$ and for $n \geq 4$ we obtain $\theta \leq \frac{2}{10}$ if $\delta(v,\mu) \leq \frac{1}{2}$ then from lemma 2 we deduce $\delta(\hat{v},\mu) \leq \frac{1}{4}$. finally, the above relation yields: $\delta(\hat{v},\hat{\mu}) \leq \frac{1}{2}$

In the next lemma we calculate an upper bound for the total number of iterations performed by the algorithm.

Lemma 5 *:Assume that x^0 and z^0 are strictly feasible, $\mu^0 = \frac{(x^0)^T z^0}{n} > 0, r = \frac{x^0 z^0}{\mu^0}$. Moreover, let x^k and z^k be the vectors obtained after k iterations. Then the inequality $(x^k)^T z^k \leq \varepsilon$ is satisfied for $k \geq \left[\frac{1}{\theta} \log \frac{(x^0)^T z^0}{\epsilon}\right]$*

Proof:
after k ierations, we get $\mu^k = (1-\theta)^k \mu^0$. using lemma 3 we find that

$$(x^k)^T z^k \leq \mu^k \left\|\sqrt{r}\right\|^2 = (1-\theta)^k \mu^0 \left\|\sqrt{r}\right\|^2 = (1-\theta)^k \left\|\sqrt{\mu^0 r}\right\|^2$$
$$= (1-\theta)^k \left\|\sqrt{x^0 z^0}\right\|^2 = (1-\theta)^k (x^0)^T z^0$$

hence $(x^k)^T z^k \leq \varepsilon$ hold if $(1-\theta)^k (x^0)^T z^0 \leq \varepsilon$
taking logarithms, we obtain
$k \log(1-\theta) + \log(x^0)^T z^0 \leq \log \epsilon$
Using the inequality $-log(1-\theta) \geq \theta$ we deduce that the above relation holds if
$k\theta \geq \log \frac{(x^0)^T z^0}{\varepsilon} \Rightarrow k \geq \frac{1}{\theta} \log \frac{(x^0)^T z^0}{\varepsilon}$
For the default $\theta = \frac{2}{5\sqrt{\sigma_c(r) n}}$, we obtain the following Corollary.

Corollary 1 *: Suppose that $x^0 \in K_{int}$, $z^0 \in T_{int}$, and let $\mu^0 = \frac{(x^0)^T z^0}{n}$. If $\theta = \frac{2}{5\sqrt{\sigma_c(r) n}}$, then the algorithm requires at most $\left[\frac{5}{2}\sqrt{\sigma_c(r) n} \log \frac{(x^0)^T z^0}{\epsilon}\right]$ iterations. For the resulting vectors we have $(x^k)^T z^k \leq \varepsilon$.*

5 Conclusion:

We have introduced a new weighted algorithm for solving linearly constrained convex optimization. The method of finding, an initial point close to the central path is based on the introduction of the weighted vector and a new search direction is based on an equivalent algebraic transformation of the centering equation from the system, which defines the central path. Polynomial complexity is proved, and the best known iteration bound is obtained.

6 Open problem

This method deserve some supplementary efforts to calculate, the initial point close to the central path and the search directions. This, until now, is the object of researchers aiming to reduce the iteration cost.

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