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Coderivations and $*$-Coderivations On Matrix Coalgebra

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Abstract

In this paper we study coderivations on $M_n^c(\mathbb{R})$, the coalgebra of $n \times n$ matrices over $\mathbb{R}$. Proving $M_n^c(\mathbb{R})$ is a coseparable coalgebra, we show that every coderivation on $M_n^c(\mathbb{R})$ is inner. Using this, we demonstrate that if f is a coderivation on $M_n^c(\mathbb{R})$ and $\{e_{ij}\}_{1\leq i,j\leq n}$ is the canonical $\mathbb{R}$-basis of $M_n(\mathbb{R})$ then $f(e_{ij}) = \sum_{k=1}^n (c_{ik}^{ij} e_{ik} + c_{kj}^{ij} e_{kj})$. We also define an involution on $M_n^c(\mathbb{R})$ which turns it into a $*$-coalgebra, and we inspect $*$-coderivations on $M_n^c(\mathbb{R})$.

Keywords: *Coalgebra, coderivation, inner coderivation, $*$-coderivation.*

1 Introduction

A *coalgebra* (C, Δ, ε) over a field κ is a κ-vector space C together with the κ-linear maps $\Delta : C \longrightarrow C \otimes C$, and $\varepsilon : C \longrightarrow \kappa$, called *coproduct* and *counit*, respectively, such that $(I_C \otimes \Delta)\Delta = (\Delta \otimes I_C)\Delta$, and $(I_C \otimes \varepsilon)\Delta = (\varepsilon \otimes I_C)\Delta$. The (n, n)*–matrix coalgebra over* $\mathbb{R}$, is $M_n(\mathbb{R})$ with the coproduct and the counit, which are defined, respectively, by the rules $e_{ij} \mapsto \sum_k e_{ik} \otimes e_{kj}$, and $e_{ij} \mapsto \delta_{ij}$, and it is denoted by $M_n^c(\mathbb{R})$. For an account on coalgebras the reader is referred to the book [4].

A κ-linear map $f : C \longrightarrow C$ on a coalgebra (C, Δ, ε) is called a *coderivation* if $\Delta f = (I_C \otimes f + f \otimes I_C)\Delta$. The coderivation f is called an *inner coderivation* provided that there exists a $\gamma \in C^*$ such that $f = (I_C \otimes \gamma - \gamma \otimes I_C)\Delta$ (see [5]). A coalgebra (C, Δ, ε) is called *coseparable* if there exists a κ-map $\tau : C \otimes C \to \kappa$ such that $(I_C \otimes \tau)(\Delta \otimes I_C) = (\tau \otimes I_C)(I_C \otimes \Delta)$ and $\tau\Delta = \varepsilon$. One can see a general definition of coderivations and coseparable coalgebras in the sense of comodules in [5].

A $*$-*coalgebra* is a coalgebra (C, Δ, ε), where C is equipped with an involution such that $\Delta(c^*) = \sum c^*_{(1)} \otimes c^*_{(2)}$, for all $c \in C$. For more examples of $*$-coalgebras see [9].

In this paper, we show that $M^c_n(\mathbb{R})$ is a coseparable coalgebra which implies that every coderivation on this space is inner. Using this, we demonstrate that if f is a coderivation on $M^c_n(\mathbb{R})$ and $\{e_{ij}\}_{1\leq i,j\leq n}$ is the canonical $\mathbb{R}$-basis of $M_n(\mathbb{R})$ then $f(e_{ij}) = \sum_{k=1}^{n} \left(c^{ij}_{ik} e_{ik} + c^{ij}_{kj} e_{kj}\right)$.

Then, defining a proper involution on $M^c_n(\mathbb{R})$, we make it a $*$-coalgebra and we inspect the form of a $*$-coderivation on this $*$-coalgebra. Giving an example, we show that a coderivation on $M^c_n(\mathbb{R})$ is not necessarily a $*$-coderivation in general.

Finally summary and conclusions will be presented , and we arise the open problem in the last section.

2 Preliminaries

Throughout the paper κ is a fixed field and I_C denotes the identity mapping on C. Moreover, $\{e_{ij}\}_{1\leq i,j\leq n}$ is the canonical $\mathbb{R}$-basis of $M_n(\mathbb{R})$, where $M_n(\mathbb{R})$ is the set of all $n \times n$ matrices over $\mathbb{R}$. We also denote the Kronecker delta by δ_{ij} which is defined to be 1 if $i = j$ and to be 0 elsewhere. Recall that a *coalgebra* (C, Δ, ε) over a field κ is a κ-vector space C together with the κ-linear maps

$$\begin{aligned} &\Delta : C \longrightarrow C \otimes C && \text{(coproduct)} \\ &\varepsilon : C \longrightarrow \kappa && \text{(counit)} \end{aligned}$$

such that

$$\begin{aligned} &(I_C \otimes \Delta)\Delta = (\Delta \otimes I_C)\Delta && \text{(coassociativity)} \\ &(I_C \otimes \varepsilon\ )\Delta = (\ \varepsilon \otimes I_C)\Delta && \text{(counitary).} \end{aligned}$$

When working with coalgebras, a certain notation for the coproduct simplifies the formulas considerably and has become quite popular. Given an element c of the coalgebra (C, Δ, ε), we know that there exist elements $c_{1,i}$ and $c_{2,i}$ in C such that $\Delta(c) = \sum_i c_{1,i} \otimes c_{2,i}$. In *Sweedlers notation*, this is abbreviated to

$$\Delta(c) = \sum_i c_{1,i} \otimes c_{2,i} =: \sum c_{(1)} \otimes c_{(2)}.$$

Here, the subscripts "(1)" and "(2)" indicate the order of the factors in the tensor product. Before moving any further we must have an understanding of tensor product of matrices. For the matrices $A = (a_{ij})_{1\leq i\leq m,1\leq j\leq n}$ and $B = (b_{\ell k})_{1\leq \ell\leq p,1\leq k\leq q}$, the tensor product is

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & & \vdots & \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}.$$

The following example has an essential role in our discussion.

Example 2.1. *[4, p. 6] Considering $M_n(\mathbb{R})$, we define the coproduct $\Delta : M_n(\mathbb{R}) \to M_n(\mathbb{R}) \otimes M_n(\mathbb{R})$ and the counit $\varepsilon : M_n(\mathbb{R}) \to \mathbb{R}$, respectively, by the rules $e_{ij} \mapsto \sum_k e_{ik} \otimes e_{kj}$ and $e_{ij} \mapsto \delta_{ij}$. The resulting coalgebra is called the (n, n)–matrix coalgebra over $\mathbb{R}$, and we denote it by $M_n^c(\mathbb{R})$.*

A κ-linear map $f : C \longrightarrow C$ on a coalgebra (C, Δ, ε) is called a *coderivation* if $\Delta f = (I_C \otimes f + f \otimes I_C)\Delta$. The coderivation f is called an *inner coderivation* provided that there exists a $\gamma \in C^*$ such that $f = (I_C \otimes \gamma - \gamma \otimes I_C)\Delta$. Now we make some preparations that serve a very useful purpose and we present, without proof, those results of [5] which are necessary for our work. A coalgebra (C, Δ, ε) is called *coseparable* if there exists a κ-map $\tau : C \otimes C \to \kappa$ such that $(I_C \otimes \tau)(\Delta \otimes I_C) = (\tau \otimes I_C)(I_C \otimes \Delta)$ and $\tau\Delta = \varepsilon$.
Regarding to the facts mentioned in [5, p. 41], we can deduce the following.

Proposition 2.2. *A coalgebra (C, Δ, ε) is coseparable if and only if there exist the linear maps $\rho^-, \rho^+ : C \otimes C \to C \otimes C \otimes C$ and $\pi : C \otimes C \to C$ satisfying the following conditions*

i. $(I_C \otimes \rho^-)\rho^- = (\Delta \otimes I_{C\otimes C})\rho^-,\ (\varepsilon \otimes I_{C\otimes C})\rho^- = I_{C\otimes C}$;

ii. $(\rho^+ \otimes I_C)\rho^+ = (I_{C\otimes C} \otimes \Delta)\rho^+,\ (I_{C\otimes C} \otimes \varepsilon)\rho^+ = I_{C\otimes C}$;

iii. $(I_C \otimes \rho^+)\rho^- = (\rho^- \otimes I_C)\rho^+$;

iv. $(I_C \otimes \pi)\rho^- = (\pi \otimes I_C)\rho^+ = \Delta\pi$;

v. $\pi\Delta = I_C$.

Theorem 2.3. *[5, Theorem 3] The following statements concerning a coalgebra (C, Δ, ε) are equivalent* *i. the coalgebra C is coseparable;*

ii. every coderivation on C is inner.

An *involution* on a real vector space A is a map $* : A \to A, a \mapsto a^*$, that is conjugate linear and involutive in the sense that $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \lambda a^*$, $(a^*)^* = a$, for all $a, b \in A$ and $\lambda \in \mathbb{R}$. A real vector space with a fixed involution is called also a $*$-*vector space*. A linear map $\phi : A \to B$ of $*$-vector spaces is $*$-*linear* if $\phi(a^*) = \phi(a)^*$, for all $a \in A$. A $*$-*coalgebra* is a coalgebra (C, Δ, ε), where C equipped with an involution such that $\Delta(c^*) = \sum c^*_{(1)} \otimes c^*_{(2)}$, for all $c \in C$.

3 Coderivations on Matrix Coalgebra

In this section we show that $M^c_n(\mathbb{R})$ is a coseparable coalgebra and so every coderivation on $M^c_n(\mathbb{R})$ is inner. Using this, we determine the form of a coderivation on $M^c_n(\mathbb{R})$. In what follows, f denotes a coderivation on $M^c_n(\mathbb{R})$. In order to characterize coderivations on $M^c_n(\mathbb{R})$, it is enough to find coefficients of the matrix units in the matrix representation of f. Finding the zero coefficients is the first and the most important step. For this purpose, we prove that every coderivation on $M^c_n(\mathbb{R})$ is inner and using the innerness property, we have the zero coefficients.

Theorem 3.1. *$M^c_n(\mathbb{R})$, is a coseparable coalgebra.*

Proof. Putting $C = M^c_n(\mathbb{R})$, it is enough to define the maps $\rho^-, \rho^+ : C \otimes C \to C \otimes C \otimes C$, $\pi : C \otimes C \to C$, as the following

$$\rho^-(e_{ij} \otimes e_{mp}) = \sum\nolimits_{k=1}^{n} e_{ik} \otimes e_{kj} \otimes e_{mp}, \rho^+(e_{ij} \otimes e_{mp}) = \sum\nolimits_{k=1}^{n} e_{ij} \otimes e_{mk} \otimes e_{kp},$$
$$\pi(e_{ij} \otimes e_{mp}) = \begin{cases} e_{ip} & j = m \\ 0 & j \neq m \end{cases}$$

for $1 \leq i, j, m, p \leq n$. Now Proposition 2.2 gives the result. $\square$

Attending Theorem 2.3, we have the following result.

Corollary 3.2. *Every coderivation on $M^c_n(\mathbb{R})$, is inner.*

Before stating the main theorem, we simplify the condition of a κ-linear map f being a coderivation on $M^c_n(\mathbb{R})$. We know that a κ-linear map f is a coderivation on $M_n^c(\mathbb{R})$ if and only if $\Delta f(e_{ij}) = \big(I_C \otimes f + f \otimes I_C\big)\Delta(e_{ij})$. Considering $f(e_{ij}) = \sum_{m,p=1}^{n} c_{mp}^{ij} e_{mp}$, for the first side of the equality we have

$$\begin{aligned}\Delta f(e_{ij}) &= \Delta(\sum_{m,p=1}^{n} c_{mp}^{ij} e_{mp}) \\ &= \sum_{m,p,k=1}^{n} c_{mp}^{ij} e_{mk} \otimes e_{kp} \\ &= \sum_{m,p,k=1}^{n} c_{mp}^{ij} e_{n(m-1)+k,n(k-1)+p},\end{aligned}$$

and for the other side we deduce

$$\begin{aligned}&\big(I_C \otimes f + f \otimes I_C\big)\Delta(e_{ij}) \\ =& \sum_{k=1}^{n} e_{ik} \otimes f(e_{kj}) + f(e_{ik}) \otimes e_{kj} \\ =& \sum_{m,p,k=1}^{n} c_{mp}^{kj} e_{ik} \otimes e_{mp} + c_{mp}^{ik} e_{mp} \otimes e_{kj} \\ =& \sum_{m,p,k=1}^{n} c_{mp}^{kj} e_{n(i-1)+m,n(k-1)+p} + c_{mp}^{ik} e_{n(m-1)+k,n(p-1)+j}.\end{aligned}$$

So a κ-linear map f is a coderivation on $M_n^c(\mathbb{R})$ if and only if

$$\begin{aligned}&\sum_{m,p,k=1}^{n} c_{mp}^{kj} e_{n(i-1)+m,n(k-1)+p} + c_{mp}^{ik} e_{n(m-1)+k,n(p-1)+j} \\ =& \sum_{m,p,k=1}^{n} c_{mp}^{ij} e_{n(m-1)+k,n(k-1)+p}. \qquad (*)\end{aligned}$$

Theorem 3.3. *Let f be a coderivation on $M_n^c(\mathbb{R})$. Then there exist real coefficients c_{ik}^{ij}, and c_{kj}^{ij}, $1 \leq k \leq n$, such that $f(e_{ij}) = \sum_{k=1}^{n}(c_{ik}^{ij} e_{ik} + c_{kj}^{ij} e_{kj})$. There is a regular relation between the coefficients given by the equalities i-vi, recursively, where $m_\ell, p_\ell, k_\ell, \ell = 1, 2$ and m_3, p_3 are the numbers that satisfy the equality $c_{m_1 p_1}^{k_1 j} + c_{m_2 p_2}^{i k_2} = c_{m_3 p_3}^{ij}$.*

i. $m_1 = k_1 = i$, $p_1 = n$; $\big(np_2, nm_2 + k_2\big) = \big(n(i+1) - j, (n+1)i\big)$; $\big(nm_3, p_3\big) = \big((n+1)i - k_3, n(i - k_3 + 1)\big)$;

ii. $m_2 = i,\ p_2 = n,\ k_2 = j;\ \big(m_1, nk_1 + p_1\big) = \big(j, n^2 + j\big);\ \big(nm_3, p_3\big) = \big(ni + j - k_3, n(n - k_3) + j\big);$

iii. $m_3 = i,\ p_3 = n;\ \big(m_1, nk_1 + p_1\big) = \big(k_3, n(k_3 + 1)\big);\ \big(nm_2 + k_2, np_2\big) = \big(ni + k_3, n(k_3 + 1) - j\big);$

iv. $m_1 = n,\ p_1 = j,\ k_1 = i;\ \big(p_2, nm_2 + k_2\big) = \big(i, n(i + 1)\big);\ \big(nm_3, p_3\big) = \big(n(i + 1) - k_3, n(i - k_3) + j\big);$

v. $m_2 = n,\ p_2 = k_2 = j;\ \big(m_1, nk_1 + p_1\big) = \big(n(n-i)+j, (n+1)j\big);\ \big(nm_3, p_3\big) = \big(n^2 + j - k_3, (n + 1)j - nk_3\big);$

vi. $m_3 = n,\ p_3 = j;\ \big(m_1, nk_1 + p_1\big) = \big(n(n-i)+k_3, nk_3+j\big);\ \big(p_2, nm_2+k_2\big) = \big(k_3, n^2 + k_3\big).$

$k_3 = 1, \ldots, n$ *and for each acceptable value of* k_3 *we have a distinct equality.*

Proof. By Corollary 3.2, for every coderivation f on $M_n^c(\mathbb{R})$, we may find $\gamma \in M_n^c(\mathbb{R})^*$ satisfying innerness condition of f. Using the fact $\kappa \otimes C = C \otimes \kappa \cong C$, for the coalgebra (C, Δ, ε) over the field κ, we can write

$$\begin{aligned} f(e_{ij}) &= \big(I_C \otimes \gamma - \gamma \otimes I_C\big)\Delta(e_{ij}) \\ &= \sum_{k=1}^{n} e_{ik} \otimes \gamma(e_{kj}) - \gamma(e_{ik}) \otimes e_{kj} \\ &= \sum_{k=1}^{n} \gamma(e_{kj})e_{ik} - \gamma(e_{ik})e_{kj}, \end{aligned}$$

for $\gamma \in M_n^c(\mathbb{R})^*$. Making this equal with $f(e_{ij}) = \sum_{m,p=1}^{n} c_{mp}^{ij} e_{mp}$, we deduce that all coefficients except c_{ik}^{ij} and c_{kj}^{ij}, $k = 1, \ldots, n$ are zero. We thus have $f(e_{ij}) = \sum_{k=1}^{n} (c_{ik}^{ij} e_{ik} + c_{kj}^{ij} e_{kj})$.

As we see, there exist some coefficients which may be nonzero. Now we show that there is a regular relation between them. For this, we search for terms containing c_{ik}^{ij} and $c_{kj}^{ij}, k = 1, \ldots, n$ in the equality $(*)$. Consider the values of m, p, k in the first and the second part of the first side of the equality $(*)$, respectively, as m_1, p_1, k_1, and m_2, p_2, k_2 and in the second side as m_3, p_3, k_3. Because of the form of $f(e_{ij})$, we may assume $c_{mp}^{ij}, m, p = 1, \ldots, n - 1$ are identified, when we compute the coefficients for the $n \times n$ matrix, corresponded to $f(e_{ij})$. Now we find the relation between c_{in}^{ij} and c_{nj}^{ij} for different values of i, j. Making equal c_{mp}^{kj}, the first coefficient in the equality $(*)$, with c_{in}^{ij}, we have $m_1 = i,\ p_1 = n,\ k_1 = i$ and then we have

$$e_{n(i-1)+m, n(k-1)+p} = e_{n(i-1)+i, n(i-1)+n} = e_{(n+1)i-n, ni}.$$

Now we look for all coefficients of $e_{(n+1)i-n,ni}$ in the two sides of the equality $(*)$. Putting

$$n(m_2-1)+k_2=(n+1)i-n, n(p_2-1)+j=ni,$$

and

$$n(m_3-1)+k_3=(n+1)i-n, n(k_3-1)+p_3=ni,$$

we can find all $m_\ell, p_\ell, k_\ell, \ell=1,2$ and m_3, p_3 satisfying

$$c^{k_1 j}_{m_1 p_1}+c^{ik_2}_{m_2 p_2}=c^{ij}_{m_3 p_3}.$$

Obviously the number of the summands in each side of the equality above depends on the number of the qualified m, p, k that has been found. So we have deduced the equalities stated in (i).
Repeating this method for c^{ik}_{mp} and then c^{ij}_{mp}, we have the relations of (ii) and (iii). Checking this equalities we have all the summands in two sides of the equality $(*)$ containing c^{ij}_{in}. Similarly, we can obtain that the relations stated in (iv),(v),(vi), give us all sentences containing c^{ij}_{nj}. □

Now let us examine Theorem 3.3 for $n=4$.

Example 3.4. *By Theorem 3.3, the only coefficients which may be nonzero are $c^{ij}_{ik}, c^{ij}_{kj}, k=1,\ldots,4$. Now we can obtain the relations between this coefficients with a little bit computation. For example, for finding equations containing c^{11}_{14}, it is enough to check the equalities i-iii with $i=j=1, n=4$. We have*

i. $m_1=k_1=1,\ p_1=4$*;* $(4p_2, 4m_2+k_2)=(7,5)$ *unacceptable;*
$(4m_3, p_3)=(5-k_3, 8-4k_3) \Rightarrow m_3=k_3=1,\ p_3=4,$ *and* $k_3=2,3,4$ *unacceptable* $\Rightarrow c^{11}_{14}=c^{11}_{14}$.

ii. $m_2=k_2=1,\ p_2=4$*;* $(m_1, 4k_1+p_1)=(1,17) \Rightarrow m_1=p_1=1,\ k_1=4;$
$(4m_3, p_3)=(5-k_3, 17-4k_3)$ *unacceptable* $\Rightarrow c^{41}_{11}+c^{11}_{14}=0$.

iii. $m_3=1,\ p_3=4$*;* $(4m_2+k_2, 4p_2)=(4+k_3, 3+4k_3)$ *unacceptable;*
$k_3=1 \Rightarrow (m_1, 4k_1+p_1)=(1,8) \Rightarrow m_1=k_1=1,\ p_1=4 \Rightarrow c^{11}_{14}=c^{11}_{14}$.
$k_3=2 \Rightarrow (m_1, 4k_1+p_1)=(2,12) \Rightarrow m_1=k_1=2,\ p_1=4 \Rightarrow c^{21}_{24}=c^{11}_{14}$.
$k_3=3 \Rightarrow (m_1, 4k_1+p_1)=(3,16) \Rightarrow m_1=k_1=3,\ p_1=4 \Rightarrow c^{31}_{34}=c^{11}_{14}$.
$k_3=4 \Rightarrow (m_1, 4k_1+p_1)=(4,20) \Rightarrow m_1=p_1=k_1=4 \Rightarrow c^{41}_{44}=c^{11}_{14}$.

We therefore have $c^{11}_{14}=-c^{41}_{11}=c^{21}_{24}=c^{31}_{34}=c^{41}_{44}$. *This is a short way to find the qualified relations in comparison with the direct way. We have computed* *coefficients for $n = 2, 3, 4$, directly, by checking the equality $(*)$ separately for each n. After a long calculation, we have deduced*

$$\begin{aligned}
a &:= c^{11}_{12} = -c^{21}_{11} = c^{21}_{22} = -c^{22}_{12} = c^{31}_{32} = -c^{23}_{13} = c^{41}_{42} = -c^{24}_{14},\\
b &:= -c^{12}_{11} = c^{11}_{21} = -c^{22}_{21} = c^{12}_{22} = -c^{32}_{31} = c^{13}_{23} = -c^{42}_{41} = c^{14}_{24},\\
c &:= c^{12}_{12} = -c^{21}_{21} = c^{32}_{32} = -c^{23}_{23} = c^{42}_{42} = -c^{24}_{24},\\
d &:= c^{11}_{13} = -c^{31}_{11} = c^{31}_{33} = -c^{33}_{13} = c^{21}_{23} = -c^{32}_{12} = c^{41}_{43} = -c^{34}_{14},\\
g &:= -c^{13}_{11} = c^{11}_{31} = -c^{33}_{31} = c^{13}_{33} = -c^{23}_{21} = c^{12}_{32} = -c^{43}_{41} = c^{14}_{34},\\
h &:= c^{13}_{12} = -c^{21}_{31} = c^{23}_{22} = -c^{22}_{32} = c^{33}_{32} = -c^{23}_{33} = c^{43}_{42} = -c^{24}_{34},\\
k &:= -c^{12}_{13} = c^{31}_{21} = -c^{22}_{23} = c^{32}_{22} = -c^{32}_{33} = c^{33}_{23} = -c^{42}_{43} = c^{34}_{24},\\
r &:= c^{11}_{14} = -c^{41}_{11} = c^{21}_{24} = -c^{42}_{12} = c^{31}_{34} = -c^{43}_{13} = c^{41}_{44} = -c^{44}_{14},\\
s &:= -c^{14}_{11} = c^{11}_{41} = -c^{24}_{21} = c^{12}_{42} = -c^{34}_{31} = c^{13}_{43} = -c^{44}_{41} = c^{14}_{44},\\
t &:= -c^{14}_{12} = c^{21}_{41} = -c^{24}_{22} = c^{22}_{42} = -c^{34}_{32} = c^{23}_{43} = -c^{44}_{42} = c^{24}_{44},\\
u &:= c^{12}_{14} = -c^{41}_{21} = c^{22}_{24} = -c^{42}_{22} = c^{32}_{34} = -c^{43}_{23} = c^{42}_{44} = -c^{44}_{24},\\
v &:= -c^{14}_{13} = c^{31}_{41} = -c^{24}_{23} = c^{32}_{42} = -c^{34}_{33} = c^{33}_{43} = -c^{44}_{43} = c^{34}_{44},\\
w &:= c^{13}_{14} = -c^{41}_{31} = c^{23}_{24} = -c^{42}_{32} = c^{33}_{34} = -c^{43}_{33} = c^{43}_{44} = -c^{44}_{34},
\end{aligned}$$

and other coefficients are zero. We see that the zero coefficients are exactly the ones obtained by Theorem 3.3.
By Corollary 3.2, we know that f is inner. Let us define $\gamma \in M_4^c(\mathbb{R})^*$ satisfying the innerness property. Using the equality $\sum_{m,p=1}^4 c^{ij}_{mp} e_{mp} = \sum_{k=1}^4 \gamma(e_{kj})e_{ik} - \gamma(e_{ik})e_{kj}$, and by determination of f we have the expected γ. It is enough for γ to satisfy the following conditions.

$$\begin{aligned}
&\gamma(e_{11}) = \gamma(e_{33}) = \gamma(e_{44}),\ \gamma(e_{12}) = -b,\ \gamma(e_{13}) = -g,\ \gamma(e_{14}) = -s,\\
&\gamma(e_{21}) = a,\ \gamma(e_{22}) - \gamma(e_{11}) = c,\ \gamma(e_{23}) = h,\ \gamma(e_{24}) = -t,\\
&\gamma(e_{31}) = d,\ \gamma(e_{32}) = -k,\ \gamma(e_{34}) = -v,\\
&\gamma(e_{41}) = r,\ \gamma(e_{42}) = u,\ \gamma(e_{43}) = w,
\end{aligned}$$

where the letters have been defined above.

4 ∗-Coderivations on Matrix Coalgebra

In this section, we define an involution on $M_n^c(\mathbb{R})$ that makes it a ∗-coalgebra, and we inspect the form of a ∗-coderivation on this ∗-coalgebra.

We equip the coalgebra $M_n^c(\mathbb{R})$ with the involution given by

$$e^*_{ij} = (-1)^{i+j} e_{n+1-i,n+1-j}.$$

We can see easily, this involution turns $M_n^c(\mathbb{R})$ into a $*$-coalgebra.
Let f be a coderivation on $M_n^c(\mathbb{R})$. We can simplify the condition of f being a $*$-coderivation with the mentioned involution. We have

$$\begin{aligned} f(e^*_{ij}) &= f\big((-1)^{i+j}e_{n+1-i,n+1-j}\big) \\ &= \sum_{m,p=1}^{n}(-1)^{i+j}c_{mp}^{n+1-i,n+1-j}e_{mp}, \end{aligned}$$

and

$$\begin{aligned} f(e_{ij})^* &= (\sum_{m,p=1}^{n} c_{mp}^{ij}e_{mp})^* \\ &= \sum_{m,p=1}^{n} c_{mp}^{ij}e^*_{mp} \\ &= \sum_{m,p=1}^{n}(-1)^{m+p}c_{mp}^{ij}e_{n+1-m,n+1-p} \\ &= \sum_{k,\ell=1}^{n}(-1)^{2(n+1)-k-\ell}c_{n+1-k,n+1-\ell}^{ij}e_{k\ell}. \end{aligned}$$

So for being f a $*$-coderivation on $M_n^c(\mathbb{R})$, we should have

$$(-1)^{i+j}c_{mp}^{n+1-i,n+1-j} = (-1)^{m+p}c_{n+1-m,n+1-p}^{ij},$$

for $m,p = 1,\ldots,n$. Now let f be a $*$-coderivation on $M_2^c(\mathbb{R})$. Checking the equality above, at first, for $i = j = m = 1$, $p = 2$, we get $a = b$ and then for $i = p = 1$, $j = m = 2$, we get $c = 0$. So we obtain the following simple form for f.

$$\begin{aligned} f(e_{11}) &= -f(e_{22}) = a(e_{12}+e_{21}), \\ f(e_{12}) &= f(e_{21}) = a(-e_{11}+e_{22}). \end{aligned}$$

Similarly, if f is a $*$-coderivation on $(M_3^c(\mathbb{R}),\Delta)$, we deduce $a = -h$, $b = -k$

and $d=-g$, therefore

$$\begin{array}{rcl}
f(e_{11}) &=& ae_{12}+de_{13}+be_{21}-de_{31}\\
f(e_{12}) &=& -be_{11}+ce_{12}+be_{13}+be_{22}-de_{32}\\
f(e_{13}) &=& de_{11}-ae_{12}+be_{23}-de_{33}\\
f(e_{21}) &=& -ae_{11}-ce_{21}+ae_{22}+de_{23}+ae_{31}\\
f(e_{22}) &=& -ae_{12}-be_{21}+be_{23}+ae_{32}\\
f(e_{23}) &=& -ae_{13}+de_{21}-ae_{22}-ce_{23}+ae_{33}\\
f(e_{31}) &=& -de_{11}-be_{21}+ae_{32}+de_{33}\\
f(e_{32}) &=& -de_{12}-be_{22}-be_{31}+ce_{32}+be_{33}\\
f(e_{33}) &=& -de_{13}-be_{23}+de_{31}-ae_{32},
\end{array}$$

and for f as a $*$-coderivation on $(M_4^c(\mathbb{R}),\Delta)$, deducing $a=v$, $b=w$, $c=0$, $d=-t$, $g=-u$, $h=k$ and $r=s$, we have

$$\begin{array}{rcl}
f(e_{11}) &=& ae_{12}+de_{13}+re_{14}+be_{21}+ge_{31}+re_{41}\\
f(e_{12}) &=& -be_{11}-he_{13}-ge_{14}+be_{22}+ge_{32}+re_{42}\\
f(e_{13}) &=& -ge_{11}+he_{12}+be_{14}+be_{23}+ge_{33}+re_{43}\\
f(e_{14}) &=& -re_{11}+de_{12}-ae_{13}+be_{24}+ge_{34}+re_{44}\\
f(e_{21}) &=& -ae_{11}+ae_{22}+de_{23}+re_{24}-he_{31}-de_{41}\\
f(e_{22}) &=& -ae_{12}-be_{21}-he_{23}-ge_{24}-he_{32}-de_{42}\\
f(e_{23}) &=& -ae_{13}-ge_{21}+he_{22}+be_{24}-he_{33}-de_{43}\\
f(e_{24}) &=& -ae_{14}-re_{21}+de_{22}-ae_{23}-he_{34}-de_{44}\\
f(e_{31}) &=& -de_{11}+he_{21}+ae_{32}+de_{33}+re_{34}+ae_{41}\\
f(e_{32}) &=& -de_{12}+he_{22}-be_{31}-he_{33}-ge_{34}+ae_{42}\\
f(e_{33}) &=& -de_{13}+he_{23}-ge_{31}+he_{32}+be_{34}+ae_{43}\\
f(e_{34}) &=& -de_{14}+he_{24}-re_{31}+de_{32}-ae_{33}+ae_{44}\\
f(e_{41}) &=& -re_{11}+ge_{21}-be_{31}+ae_{42}+de_{43}+re_{44}\\
f(e_{42}) &=& -re_{12}+ge_{22}-be_{32}-be_{41}-he_{43}-ge_{44}\\
f(e_{43}) &=& -re_{13}+ge_{23}-be_{33}-ge_{41}+he_{42}+be_{44}\\
f(e_{44}) &=& -re_{14}+ge_{24}-be_{34}-re_{41}+de_{42}-ae_{43}.
\end{array}$$

Finally, we give an example of a coderivation on $M_n^c(\mathbb{R})$, which is not a $*$-coderivation.

Example 4.1. *Consider f as a map on $M_n^c(\mathbb{R})$ by the rule*

$$f(e_{ij})=\sum_{k=1}^{n} e_{kj}-e_{ik},1\leq i,j\leq n.$$

We have

$$\begin{aligned}
&\big(I_C\otimes f+f\otimes I_C\big)\Delta(e_{ij})\\
=&\sum_{k=1}^{n}e_{ik}\otimes f(e_{kj})+f(e_{ik})\otimes e_{kj}\\
=&\sum_{k=1}^{n}e_{ik}\otimes(\sum_{\ell=1}^{n}e_{\ell j}-e_{k\ell})+(\sum_{\ell=1}^{n}e_{\ell k}-e_{i\ell})\otimes e_{kj}\\
=&\sum_{k,\ell=1}^{n}e_{\ell k}\otimes e_{kj}-e_{ik}\otimes e_{k\ell}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\Delta f(e_{ij}) &= \Delta(\sum_{\ell=1}^{n}e_{\ell j}-e_{i\ell})\\
&= \sum_{k,\ell=1}^{n}e_{\ell k}\otimes e_{kj}-e_{ik}\otimes e_{k\ell},
\end{aligned}$$

and so f is a coderivation on $M_n^c(\mathbb{R})$.
Now we show that f is not a $*$-coderivation. By definition of the mentioned involution, we have

$$\begin{aligned}
f(e_{ij}^*) &= f\big((-1)^{i+j}e_{n+1-i,n+1-j}\big)\\
&= (-1)^{i+j}\sum_{\ell=1}^{n}e_{\ell,n+1-j}-e_{n+1-i,\ell},
\end{aligned}$$

and for the other side of the equality $f(e_{ij}^*)=f(e_{ij})^*$, we deduce

$$\begin{aligned}
f(e_{ij})^* &= \sum_{k=1}^{n}e_{kj}^*-e_{ik}^*\\
&= \sum_{k=1}^{n}(-1)^{k+j}e_{n+1-k,n+1-j}-(-1)^{i+k}e_{n+1-i,n+1-k}\\
&= \sum_{\ell=1}^{n}(-1)^{n+1-\ell+j}e_{\ell,n+1-j}-(-1)^{n+1-\ell+i}e_{n+1-i,\ell}.
\end{aligned}$$

We see that the signs are not the same at two sides of the equality $f(e_{ij}^*)=f(e_{ij})^*$, for even and odd values of $i+j$, and f is not a $*$-coderivation.

5 Conclusion

In this paper, we have recognized the form of coderivations on $M_n^c(\mathbb{R})$. At the first step, we have shown that $M_n^c(\mathbb{R})$ is a coseparable coalgebra, which implies that every coderivation on this space is inner. Using this, we have proved that if f is a coderivation on $M_n^c(\mathbb{R})$, then $f(e_{ij}) = \sum_{k=1}^{n}(c_{ik}^{ij} e_{ik} + c_{kj}^{ij} e_{kj})$, and there exists a regular relation between the coefficients. Explaining an example, we have inspected the results for $n = 4$. Defining an involution on $M_n^c(\mathbb{R})$ to make it a $*$-coalgebra was the next stage which has been followed by stating a condition on f to be a $*$-coderivation. After that we have distinguished the form of $*$-coderivations on $M_n^c(\mathbb{R})$ for $n = 2, 3, 4$. Stating an example, at the last part, we have defined a coderivation on $M_n^c(\mathbb{R})$, which is not a $*$-coderivation.

6 Open Problem

If we replace $\mathbb{R}$ with κ in the definition of $M_n^c(\mathbb{R})$, we will have the definition of $M_n^c(\kappa)$, when κ is a field. For a coalgebra C over a field κ, the *matrix coalgebra of order n over C*, denoted by $M_n^c(C)$ is defined as $C \otimes M_n^c(\kappa)$. We finalize the paper with the following open question: Attending the obtained results, is there any way to recognize coderivations on $M_n^c(C)$.

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