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Numerical Solution of Poisson equation with Dirichlet Boundary Conditions

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Abstract

In this paper, we want to establish the existence, uniqueness and regularity of the solution is obtained with Dirichlet boundary conditions, for this aim we use Galerkin method which allow us to used the fixed point theorem of Brouwer for establish the existence of the solution, the variational inequalities are used to establish the uniqueness of solution, and establish the regularity of the solution. we introduce a finite difference scheme approximating Poisson equation in one, two, three-dimensional domain with Dirichlet boundary conditions, Gaussian elimination is undoubtedly the most widely used method for solving linear equations, Matlab is proposed for obtaining solutions for this problem

Keywords: *Poisson equation, Galerkin Method, Numerical solution, Dirichlet Boundary conditions, Matlab, Fixed Point theorem, Existence, Uniqueness, Regularity..*

1 Introduction

The fundamental partial differential equations that govern the equilibrium mechanics of multi-dimensional media are the Laplace equation and its inhomogeneous counterparts,the Poisson equation. In this paper, we concentrate on the Poisson equations in a n-dimensional domain. Their status as equilibrium equations implies that the solutions are determined by their values on the boundary of the domain. In particular, the introduction of Dirichlet conditions can improve the qualitative and quantitative characteristics of the problem which lead to good results concerning existence,uniqueness and regularity of the solution. In recent years much attention has been paid to this problem in many directions. The existence result will be a consequence of Brouwer fixed point theorem in[5], The uniqueness result will be a consequence of variational inequalities of [5] and Regularity Result of the solution will be a consequence of [5] and [12] and [6] and [7] and [10]. Many authors have studied the Laplace and Poisson equations For example in [4] and [3] and [1]. Many physical phenomena can be modeled by Poisson equation with Dirichlet conditions, In varies applications arise For example in fluid flow, flow in porous media, and electrostatics. Motivated by the above applications we study here the Poisson equation

$$-\Delta u(x) = f(x), x \in \Omega, \tag{1}$$

under the Dirichlet boundary conditions

$$u \mid_{\partial\Omega} = 0, \tag{2}$$

The Poisson equations arise as the basic equilibrium equations in a remarkable variety of physical systems. For example, we may u as the displacement of a membrane, the inhomogeneity f in the Poisson equation represent an external forcing. Suppose $\Omega = (]0,1[)^n$ is an open, bounded and connected set in $\mathbb{R}^n$ the boundary of Ω will be denoted $\partial\Omega$, we denoted by $D(\Omega)$ the space of real C^∞ function on Ω with a compact support in Ω. we seek a solution $u \in V =$ the closure of $D(\Omega)$ in $H^1(\Omega)$, such that $H^1(\Omega)$ is the Sobolev space

$$H^1(\Omega) = \{u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, 2, \ldots, n\}$$

then $V = H_0^1(\Omega)$ and $f \in \grave{V} = H^{-1}(\Omega)$, $\grave{V}$ the dual space of $V = H_0^1(\Omega)$, we use $\|\cdot\|_*$ to indicated the norm in $\grave{V} = H^{-1}(\Omega)$ defined by

$$\|\ell\|_* = \sup \frac{|\langle \ell, v\rangle|}{\|v\|_{H_0^1(\Omega)}}$$

such that $v \in V$, $v \neq 0$, $\ell \in \grave{V}$. In the application to Poisson equation, the space $H = L^2(\Omega)$ is provided such that, $V \subset H$, $L^2(\Omega)$ is a Hilbert space for the scalar product and the corresponding norm

$$\|u\|_{L^2(\Omega)} = \{(u, u)\}^{\frac{1}{2}} = (\int_\Omega u^2(x)\, dx)^{\frac{1}{2}}$$

we multiplying (1) by $v \in V = H_0^1(\Omega)$ and integrate in Ω by using integration by parts and the Dirichlet boundary conditions, we obtain V be a Hilbert space for the scalar product and the corresponding norm

$$\|u\|_{H_0^1(\Omega)} = (a(u,u))^{\frac{1}{2}} = (\int_{\Omega} (\nabla u)^2 \, dx)^{\frac{1}{2}}$$

we can associate a bilinear continuous form a on $V = H_0^1(\Omega)$ by setting

$$a(u,v) = \langle -\Delta u, v \rangle, \forall u, v \in V$$

where $\langle ., . \rangle$ is the scalar product between $V = H_0^1(\Omega)$ and $\grave{V} = H^{-1}(\Omega)$, Let the application $r \longmapsto \phi(r)$ defined from $\mathbb{R}^+$ into $\mathbb{R}^+$ is continuous,monotonous,and strictly increase such that $\phi(0) = 0$, $\phi(r) \longrightarrow \infty$ as $r \longrightarrow \infty$, the operator $A = -\Delta$ defined from $V = H_0^1(\Omega)$ into $\grave{V} = H^{-1}(\Omega)$ is called $\ll$ the dual operator$\gg$ is related by ϕ if satisfies the following conditions:

H1. $\langle -\Delta u, u \rangle = \| -\Delta u\|_* \|u\|_{H_0^1(\Omega)}, \forall u \in H_0^1(\Omega)$

H2. $\| -\Delta u\|_* = \phi(\|u\|_{H_0^1(\Omega)}), \forall u \in H_0^1(\Omega)$

and it satisfies the following properties:

B3. $A = -\Delta$ is monotonous, that is $\langle -\Delta u - (-\Delta v), u - v \rangle \geq 0, \forall u, v \in H_0^1(\Omega)$;

B4. $A = -\Delta$ is hemicontinuous, that is, for each $u, v, w \in H_0^1(\Omega)$ the function $\lambda \longmapsto \langle -\Delta(u + \lambda v), w \rangle$ is continuous from $\mathbb{R}$ into $\mathbb{R}$.

Since the scalar product a is a bilinear coercive form on $V = H_0^1(\Omega)$ then $-\Delta$ is an isomorphism from $V = H_0^1(\Omega)$ into $\grave{V} = H^{-1}(\Omega)$.
The basic idea of almost any numerical method for solving Poisson equation is to approximate the Poisson equation by a system of algebraic equations. In mathematics, Science and engineering linear systems has very important place. Many applications involve solving linear systems. Therefore, there have been great efforts to solve or approximate solutions to such systems. Several enhancements can be done in Algorithms that solve such linear systems. The objective of this paper is to provide a numerical method to solve a linear system of equations with Matlab.
the Poisson equation is solved numerically by the finite difference approximation in one, two, three-dimensional domain with Dirichlet boundary conditions. The numerical examples of the present study show that numerical solution of linear systems is obtained in generally, by Matlab.

2 Existence and uniqueness of solutions

Existence theorem is proved by using the Brouwer fixed point theorem, in this case we have the next Lemma plays a central role in the proof of the existence theorem.

Lemma 2.1 *Suppose that $\xi \longrightarrow P(\xi)$ is a continuous from $\mathbb{R}^m$ into $\mathbb{R}^m$, satisfy the following conditions for any ρ real-positive, there exist :*

$$(P(\xi), \xi) \geq 0, \forall \xi, \quad \textit{such that} \quad \|\xi\| = \rho$$

where if $\xi = \{\xi_i\}, \eta = \{\eta_i\} \in \mathbb{R}^m$:

$$(\xi, \eta) = \Sigma_{i=1}^{m} \xi_i \eta_i, \quad \|\xi\| = (\xi, \xi)^{\frac{1}{2}}$$

then there is $\xi, \|\xi\| \leq \rho, \quad$ such that $\quad P(\xi) = 0$.

Proof 2.2 *Let us argue by contradiction we assume that $P(\xi) \neq 0$ in the ball $K = \{\xi \mid \|\xi\| \leq \rho\}$ we consider that function $\xi \longrightarrow -p(\xi).\frac{\rho}{\|p(\xi)\|}$ is continuous from K into K by using the Brouwer fixed point theorem,there exist a ξ such that*

$$\xi = -p(\xi).\frac{\rho}{\|p(\xi)\|}$$

then we have $\|\xi\| = \rho$ and multiplying ξ by ξ,we obtain

$$(\xi, \xi) = \|\xi\|^2 = \rho^2 = -\frac{\rho}{\|p(\xi)\|}(p(\xi), \xi)$$

Then

$$(p(\xi), \xi) = -\rho\|p(\xi)\| < 0$$

then we obtain a contradiction with the fact that $(p(\xi), \xi) \geq 0$ for all ξ, then we obtain $p(\xi) = 0$.

by using the preceding Lemma, we obtain the following existence theorem.

Theorem 2.3 *Let $f \in H^{-1}(\Omega)$, be given then the problem* (1)-(2) *admit solution.*

Proof 2.4 *The solutions of Poisson equation with Dirichlet boundary conditions can be obtained as limits of approximate solutions calculated by Galerkin method.*

Approximate Solutions: *Let* $w_1, \dots, w_m, \dots$ *is a basis in* $H_0^1(\Omega)$ *for example that* $w_1, \dots, w_m, \dots$ *functions of the space* $D(\Omega)$ *we seek an approximate solution* $u_m \in [w_1, \dots, w_m]$ *where* $[w_1, \dots, w_m]$ *is the subspace of all linear Combinations of elements* $w_1, \dots, w_m$ *such that*

$$\langle -\Delta u_m, w_i \rangle = \langle f, w_i \rangle, i = 1, \dots, m. \tag{3}$$

we can prove that (3) *admit solution. by using the preceding Lemma, Since* $\xi = \{\xi_i\}, i = 1, 2, \dots, m$ *we can associate*

$$u_m = \Sigma_{i=1}^m \xi_i w_i, \, and \quad \eta_i = \langle -\Delta u_m, w_i \rangle - \langle f, w_i$$

putting

$$P(\xi) = \{\eta_i\}, \quad i = 1, 2, \cdots, m,$$

Then

$$(P(\xi), \xi) = \Sigma_{i=1}^m \xi_i \eta_i = \langle -\Delta u_m, u_m \rangle - \langle f, u_m \rangle$$

By using integration by parts and the dirichlet boundary conditions, we obtain

$$(P(\xi), \xi) = \|\nabla u_m\|_{L^2(\Omega)}^2 - \langle f, u_m \rangle \tag{4}$$

By using the Cauchy-Schwartz inequality and Poincar inequality,

$$|\langle f, u_m \rangle| \leq \|f\|_* \|\nabla u_m\|_{H_0^1(\Omega)} \leq \|f\|_* \|\nabla u_m\|_{L^2(\Omega)}$$

Then

$$(P(\xi), \xi) \geq \|\nabla u_m\|_{L^2(\Omega)}^2 - c_1 \|\nabla u_m\|_{L^2(\Omega)}$$

then $(P(\xi), \xi) \geq 0$, *if* $\|\nabla u_m\|_{L^2(\Omega)} \geq c_1$, *condition satisfying if* $\|\xi\| = \rho$, ρ *sufficiently large. Another, if* u_m *is a solution, then we have* $P(\xi) = 0$ *then from* (4), *we obtain*

$$\|\nabla u_m\|_{L^2(\Omega)}^2 = \langle f, u_m \rangle \leq c_1 \|\nabla u_m\|_{L^2(\Omega)}$$

and

$$\|\nabla u_m\|_{L^2(\Omega)} \leq c_1 \quad whereas \quad c_1 > 0 \tag{5}$$

Pass to the limit: *we deduce from* (5) *that* u_m *still in a bounded set of* $H_0^1(\Omega)$, *then we have there is a subsequence* u_μ *of* u_m *such that* u_m *tending to* u *weakly in* $H_0^1(\Omega)$, *and the injection of* $H_0^1(\Omega)$ *in* $L^2(\Omega)$ *is compact, then* u_μ *tending to* u *strongly in* $L^2(\Omega)$, *we obtain*
Let i *is fixed,* $\mu > i$, *we have:*

$$\langle -\Delta u_\mu, w_i \rangle = \langle f, w_i \rangle; \tag{6}$$

passing to the limit in (6), *we obtain*

$$\langle -\Delta u, w_i \rangle = \langle f, w_i \rangle \tag{7}$$

and

$$\langle -\Delta u - f, w_i \rangle = 0, for \ all \quad u \in H_0^1(\Omega) \tag{8}$$

Then u *satisfying* (1), *and the existence theorem is provide.*

Naturally: the equation (3) admit a unique solution if

$$\langle -\Delta u - (-\Delta v), u - v\rangle \geq 0, \forall u, v \in H_0^1(\Omega), u \neq v$$

There is in the sense the main result, Uniqueness theorem

Theorem 2.5 *Let $V = H_0^1(\Omega)$ is a banach space reflexive separable, let $A = -\Delta$ is a operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ satisfies the following conditions:*

D1. *$A = -\Delta$ is bounded and satisfies $B3$ and $B4$*

D2. *Let $\frac{\langle -\Delta v, v\rangle}{\|v\|_{H_0^1(\Omega)}} \longrightarrow +\infty$ as $\|v\|_{H_0^1(\Omega)} \longrightarrow +\infty$*

D3. *the norm $\|v\|_{H_0^1(\Omega)}$ is strictly convex on the unit sphere in $V = H_0^1(\Omega)$*

D4. *$-\Delta u = -\Delta v$ this implies that $\|u\|_{H_0^1(\Omega)} = \|v\|_{H_0^1(\Omega)}$, $\forall u, v \in H_0^1(\Omega)$*

then the equation (1) admit a unique solution.

Proof 2.6 *Let $w_1, \cdots, w_m, \cdots$ is a basis in $V = H_0^1(\Omega)$ since, $V = H_0^1(\Omega)$ is a separable space then that basis existing we seek a solution $u \in [w_1, \cdots, w_m]$ satisfies, $u = \Sigma_{i=1}^m \xi_i w_i$ and*

$$\langle -\Delta u, w_i\rangle = \langle f, w_i\rangle, i = 1, 2, \cdots, m \tag{9}$$

The existence of solutions u can be obtained by using the preceding Lemma, multiplying (9) by ξ_i add these relations for $i = 1, 2, \cdots, m$ which gives

$$\langle -\Delta u, u\rangle = \langle f, u\rangle \tag{10}$$

by using the Cauchy-Schwartz inequality, we obtain

$$\langle -\Delta u, u\rangle = \langle f, u\rangle \leq \|f\|_* \|u\|_{H_0^1(\Omega)}$$

Then

$$\|u\|_{H_0^1(\Omega)} \leq c_1 \tag{11}$$

and the function $u \longmapsto \langle -\Delta u, u\rangle$ is continuous from $[w_1, \cdots, w_m]$ into $\mathbb{R}$. First we can prove the next main result plays a central role in the study of the variational inequalities,

1. *u is solution to the equation (1) if and only if*

$$\langle -\Delta v - f, v - u\rangle \geq 0, \forall u, v \in H_0^1(\Omega)$$

we assume u is a solution of (1), then since

$$\langle -\Delta v - f, v - u\rangle = \langle -\Delta u - f, v - u\rangle + \langle -\Delta v - (-\Delta u), v - u\rangle$$

we obtain

$$\langle -\Delta v - f, v-u\rangle = \langle -\Delta v - (-\Delta u), v-u\rangle \geq 0$$

conversely, if we have

$$\langle -\Delta v - f, v-u\rangle \geq 0$$

then by taking $v = u + \lambda w$, $\lambda > 0$, $w \in V = H_0^1(\Omega)$ *we have from Lebesgue theorem:*

$$\langle -\Delta(u+\lambda w) - f, w\rangle \geq 0 \tag{12}$$

taking $\lambda \longrightarrow 0$ *in* (12), *we obtain*

$$\langle -\Delta u - f, w\rangle \geq 0, \textit{for all} \quad u, w \in H_0^1(\Omega) \tag{13}$$

since that passing to the limit $\lambda \longrightarrow 0$ *is satisfied in the estimate* (12) *if the monotonicity and hemicontinuous conditions are satisfied with the element of minimal of estimate* (13) *then we obtain,* (1)

2. *Consider a set* $E = \{u \in V \mid -\Delta u = f\}$ *is a closed convex, then for all* $v \in V$ *define the set*

$$S_v = \{u \in V \mid \langle -\Delta v - f, v-u\rangle \geq 0\}$$

from $H2$:

$$E = \bigcap_{v\in V} S_v$$

and since S_v *is a closed half space of* V, *we have the following result*

3. *by the condition* $D4$ *the set* $E \subset \{u \mid \|u\|_{H_0^1(\Omega)} = \rho\}$, ρ *convenient; from 2. the set* E *is a closed and convex and, since that the norm* $v \longrightarrow \|v\|$ *is strictly convex, then to formulate that main result the set* E *is coincide to a point. then the equation* (1) *admit a unique solution. we can prove that* $D1$, *we note that if* $V = H_0^1(\Omega)$ *and if* $v \in L^2(\Omega)$ *then*

$$\| - \Delta u\|_* \leq \|v\|_{L^2(\Omega)}$$

where we identify $L^2(\Omega) \subset H^{-1}(\Omega)$ *as indicated. then that* $A = -\Delta$ *is a bounded operator. we can also prove that* $D2$ *by using integration by parts and the Dirichlet boundary conditions, we obtain* $\langle -\Delta v, v\rangle = \|v\|^2_{H_0^1(\Omega)}$,*then to divide with* $\|v\|_{H_0^1(\Omega)}$ *is equal to* $\|v\|_{H_0^1(\Omega)}$ *tending to* $+\infty$ *as* $\|v\|_{H_0^1(\Omega)}$ *tending to* $+\infty$. *To prove* $D3$, *we define the norm on the unit sphere* K *in* $V = H_0^1(\Omega)$ *such that*

$$K = \{v \in V \mid \|v\|_{H_0^1(\Omega)} \leq 1\}$$

the set $K \subset V$ *is nomempty and convex and the norm on* K *satisfying*

$$\|\theta u + (1-\theta)v\|_{H_0^1(\Omega)} < \theta u\|_{H_0^1(\Omega)} + (1-\theta)\|v\|_{H_0^1(\Omega)}$$

for all $u, v \in K$, *for all* $\theta \in [0,1]$ *then* $\|.\|_{H_0^1(\Omega)}$ *is strictly convex on* $K \subset V$. *To prove D4. we assume that* $-\Delta u = -\Delta v$ *this implies that* $\Delta(u-v) = 0$, *multiplying* $-\Delta(u-v)$ *by* $u - v$, *we obtain*

$$\langle -\Delta(u-v), u-v\rangle = 0, u, v \in H_0^1(\Omega), u \neq v$$

by using integration by parts and the Dirichlet boundary conditions, we obtain

$$\int_\Omega (\nabla(u-v))^2\, dx = 0$$

using the poincar inequality,

$$\int_\Omega (u-v)^2\, dx \leq \int_\Omega (\nabla(u-v))^2\, dx$$

we obtain that $u - v = 0$ *then* $u = v$ *this gives us* $\|u\|_{H_0^1(\Omega)} = \|v\|_{H_0^1(\Omega)}$. *Then the equation* (1) *admit a unique solution.*

3 Regularity Result

we represent the sobolev space of order m,

$$H^m(\Omega) = \{v \in L^2(\Omega) \mid D^\alpha v \in L^2(\Omega), |\alpha| \leq m\}$$

and this is a hilbert space for the scalar product and the corresponding norm

$$\|u\|_{H^m(\Omega)} = \{((u,u))\}^{\frac{1}{2}} = (\Sigma_{\alpha \leq m} \int_\Omega (D^\alpha u)^2\, dx)^{\frac{1}{2}},$$

we indicate one regularity result for the solution u. The weak form of the problem (1)-(2) is : For f given in $H = L^2(\Omega)$, find $u \in V = H_0^1(\Omega)$ such that

$$a(u,v) = \langle -\Delta u, v\rangle, \forall v \in H_0^1(\Omega)$$

we conclude that the domain of $A = -\Delta$ in $L^2(\Omega)$ is $D(A) = D(-\Delta) = H^2(\Omega) \bigcap H_0^1(\Omega)$ we observe that

$$\langle f, v\rangle = (f, v), \forall f \in L^2(\Omega), \forall v \in H_0^1(\Omega).$$

Theorem 3.1 *Let* $f \in L^2(\Omega)$, *then there exists a unique solution* u *of* (1)-(2) *such that* $u \in H^2(\Omega) \bigcap H_0^1(\Omega)$.

Proof 3.2 *The solution of boundary value problem for Poisson equation can be obtained as limits of approximate solutions calculated by Galerkin method*

Existence theorem *we define an approximate solution u_m such that*

$$a(u_m, w_i) = \langle -\Delta u_m, w_i \rangle, i = 1, 2, \cdots, m$$

where $w_i, i = 1, 2, \cdots, m$ is a basis of the space $H^2(\Omega) \bigcap H_0^1(\Omega)$ because $H^2(\Omega) \bigcap H_0^1(\Omega)$ is a separable space, we can find an a priori estimate for u_m is to prove that existence theorem and we can prove $u \in H^2(\Omega) \bigcap H_0^1(\Omega)$ by using the equation (1). *we will treat this by steps:*

Step1. *we seek an approximate solution u_m to the problem* (1)-(2) *as follows:*

$$u_m = \Sigma_{i=1}^m \alpha_i w_i, \tag{14}$$

where $\alpha_i, i = 1, 2, \cdots, m$ are determined by the conditions:

$$a(u_m, w_i) = (-\Delta u_m, w_i), i = 1, 2, \cdots, m \tag{15}$$

where

$$a(u, v) = \Sigma_{i=1}^m \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx$$

then we have the equations (14)-(15) *are equivalent to value problem for a linear finite m-dimensional ordinary differential equation for the α_i,these equations guarantees the existence of solution of* (14)-(15)*, Then a priori estimate is obtained as follows: multiplying* (15) *by α_i, add these relations for $i = 1, 2, \cdots, m$ which gives*

$$a(u_m, u_m) = (-\Delta u_m, u_m)$$

by using Cauchy Schwartz inequality, we obtain

$$\|u_m\|_{H^2(\Omega) \bigcap H_0^1(\Omega)} \leq c_1 \tag{16}$$

it is easy to see that $D(A) = H^2(\Omega) \bigcap H_0^1(\Omega)$ is Hilbert space for norm, and that $A = -\Delta$ is an isomorphism of $D(A)$ into $L^2(\Omega)$.
Letting $m \longrightarrow +\infty$,

$$\textit{show that}\, u_m \textit{ still in a bounded set of}\, H^2(\Omega) \bigcap H_0^1(\Omega) \tag{17}$$

Pass to the limit *From Dunford-Pettis theorem see [6], show that the space $H^2(\Omega) \bigcap H_0^1(\Omega)$ be a given with dual $H^{-2}(\Omega) + H^{-1}(\Omega)$. by a consequence there is a subsequence u_μ of u_m such that*

$$u_\mu \longrightarrow u, \textit{ weak star in}\, H^2(\Omega) \bigcap H_0^1(\Omega) \tag{18}$$

on the other hand, in particular from (17) *show that* u_m *is a bounded in* $H_0^1(\Omega)$ *Then in particular show that* u_m *still in a bounded set of* $H^1(\Omega)$, *but from Rellich-Kondrachoff theorem, see [6] we have*

$$\textit{the injection of}\, H^1(\Omega)\ \textit{in}\, L^2(\Omega)\ \textit{is compact}$$

Then we assume that subsequence u_μ *of* u_m *satisfying,* (18)

$$u_\mu \longrightarrow u\ \textit{strongly in}\, L^2(\Omega)\ \textit{and almost everywhere}$$

Pass to the limit in (15) *to used for* $m = \mu$, *Let* i *is a fixed and* $\mu > i$, *then from* (15)*:*

$$a(u_\mu, w_i) = (f, w_i) \tag{19}$$

but from (18)

$$a(u_\mu, w_i) \longrightarrow a(u, w_i)\ \textit{weak star in}\, H_0^1(\Omega)$$

then from (19) *we conclude*

$$a(u, w_i) = (f, w_i) \tag{20}$$

this for i *is a fixed arbitrary, multiplying* (20) *by* α_i, *add these relations for* $i = 1, 2, \cdots, m$ *and since that* $w_1, \cdots, w_m$ *is a basis in* $H^2(\Omega) \bigcap H_0^1(\Omega)$ *we deduce that*

$$a(u, v) = (-\Delta u, v),\ \textit{for all}\, u, v \in H^2(\Omega) \bigcap H_0^1(\Omega)$$

Then u *satisfying* (1) *and, we have the existence theorem. In order to prove that* $u \in H^2(\Omega)$ *for verifying* $u \in H^2(\Omega) \bigcap H_0^1(\Omega)$

Step2. *we can prove that* $u \in H^2(\Omega)$ *we conclude from* (1) *that*

$$\Delta u = -f \tag{21}$$

but $f \in L^2(\Omega)$ *we deduce from* (21) *that* $\Delta u \in L^2(\Omega)$, *putting*

$$\Delta u = h$$

since $\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ *is an isomorphism with continuous inverse* $G = \Delta^{-1}$, *then we have :*

$$u = Gh \quad \textit{almost everywhere} \tag{22}$$

but from the regularity theory of solutions of linear elliptic boundary value problems, see [7] and [6], we obtain

$$G \in \mathscr{L}(L^2(\Omega), H^2(\Omega)) \tag{23}$$

where $\mathscr{L}(L^2(\Omega), H^2(\Omega))$ *the space of linear continuous operator* G *from* $L^2(\Omega)$ *into* $H^2(\Omega)$, *then from* (22) *and* (23), *we obtain that* $u \in H^2(\Omega)$.

Uniqueness Theorem

Theorem 3.3 *Assume that* $u, v \in H^2(\Omega) \bigcap H_0^1(\Omega)$ *are two solutions of the problem* (1)-(2), *then* $u = v$.

Proof 3.4 *Let* $w = u - v$. *Then*

$$\Delta w = 0, w \in H^2(\Omega) \bigcap H_0^1(\Omega)$$

w is harmonic. furthermore, since $u = v = 0$, on $\partial\Omega$, it follows that $w = 0$, on $\partial\Omega$, then we derive from corollary 6.3 *in A Maximum Principles for Harmonic Functions, see[11], that $w = 0$ in Ω. Then we obtain Uniqueness theorem.*

4 The Discretisation of the Problem

In this section, discrete the problem (1)-(2) we consider that a finite difference approximation in a one,two, three-dimensional domain of Poisson equation with Dirichlet boundary conditions. we let v_i denote an approximation of $u(x_i)$ and $v_{i,j}$ denote an approximation of $u(x_i, y_j)$ and $v_{i,j,k}$ denote an approximation of $u(x_i, y_j, z_k)$ we have the following approximations

$$u_{xx} = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

$$u_{xx} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + O(h^2)$$

$$\vdots$$

$$u_{zz} = \frac{u(x,y,z+h) - 2u(x,y,z) + u(x,y,z-h)}{h^2} + O(h^2)$$

These approximations motivate the following schemes:

In one-dimensional domain

$$\begin{aligned} L_h v = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} &= f(x_i) \quad \text{for} \quad i = 1, 2, \cdots, n \\ \text{and} \quad v_0 = v_{n+1} = 0 & \end{aligned} \tag{24}$$

In two-dimensional domain

$$\begin{aligned} L_h v = &-\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2} - \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h^2} \\ &= f(x_i, y_j), \text{ for } \quad i, j = 1, \cdots, n \\ &\text{and} \quad v_{0,j} = v_{n+1,j} = 0, \; v_{i,0} = v_{i,n+1} = 0 \end{aligned} \tag{25}$$

In three-dimensional domain

$$L_h v = -\frac{v_{i+1,j,k} - 2v_{i,j,k} + v_{i-1,j,k}}{h^2} - \frac{v_{i,j+1,k} - 2v_{i,j,k} + v_{i,j-1,k}}{h^2}$$

$$\begin{aligned} -\frac{v_{i,j,k+1} - 2v_{i,j,k} + v_{i,j,k-1}}{h^2} &= f(x_i, y_j, z_k) \\ \text{for} \quad i,j,k = 1,\cdots,n \quad \text{and } v_{0,j,k} &= v_{n+1,j,k} = 0 \\ v_{i,0,k} &= v_{i,n+1,k} = 0 \\ v_{i,j,0} &= v_{i,j,n+1} = 0 \end{aligned} \tag{26}$$

Definition 4.1 *Let $C(\Omega)$ denote the set of continuous on the open Ω For an integer $m \geq 0$, $C^m(\Omega)$ denotes the set of m-times continuously differentiable functions on Ω Similarly, we define*

$$C_0^2(\Omega) = \{u \in C^2(\Omega) \bigcap C(\overline{\Omega}) \mid u \mid_{\partial\Omega} = 0\}$$

5 Numerical examples

Our goal in this section is to show that the discrete solution v will indeed converge to the continuous solution u when the spacing h approaches Zero. we present numerical examples for the Poisson equation for a positive integer $n = N$fixed $\in \mathbb{N}$ where our abstract results apply.

In one-dimensional domain we consider the discretisation

$$[x_i, x_{i+1}],\ x_i = ih,\ h = \frac{1}{N+1},\ i = 0,1,\cdots,N+1 \quad \text{of the interval} \quad [0,1]$$

we define the approximation $\{v_i\}_{i=0}^{N+1}$ by requiring (24), the system (24) can be rewritten as a system of equations in the form

$$Av = b \tag{27}$$

where A is a $N \times N$ matrix v and b are both N-dimensional vectors, we define the following linear operator

$$A : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \text{defined by} \quad A(v_1,\cdots,v_N) = (2v_1 - v_2,\cdots,-v_{N+1} + 2v_N)$$

In fact that linear operator A is a bijection then the system (27) has a unique solution that can be computed by solving (27) by using the method of compensation, then the solution v of the discrete problem is

$$\begin{aligned}
v_1 &= \frac{2N}{(N+1)^3}f(\frac{1}{N+1}) + \frac{5N-1}{(N+1)^3}f(\frac{2}{N+1}) + \cdots + \frac{1}{(N+1)^3}f(\frac{N}{N+1}),\\
&\vdots\\
v_{N-p+1} &= \frac{p}{(N+1)^3}f(\frac{1}{N+1}) + \frac{2p}{(N+1)^3}f(\frac{2}{N+1}) + \cdots + \frac{N-p+1}{(N+1)^3}f(\frac{N}{N+1})\\
&\vdots\\
v_N &= \frac{1}{(N+1)^3}f(\frac{1}{N+1}) + \frac{2}{(N+1)^3}f(\frac{2}{N+1}) + \cdots + \frac{N}{(N+1)})f(\frac{N}{N+1})
\end{aligned}$$

The domain Ω is given by $\Omega =]0,1[$, we let D_h be a collection of discrete functions defined at the grid points x_i for $i = 0, 1, \cdots, n+1$ Next, we let $D_{h,0}$ be the subset of D_h containing discrete functions that are defined in each grid point, but with the special property that are Zero at the boundary.we consider a finite difference approximation of Poisson equation on Ω with homogeneous Dirichlet boundary conditions:

$$-\frac{\partial^2 u}{\partial x^2} = f, \text{ in } \Omega \tag{28}$$

$$u = 0, \text{ on } \partial\Omega \tag{29}$$

we can formulate the discrete problem (24) as follows:Find a discrete function $v \in D_{h,0}$ such that

$$L_h v(x_i) = f(x_i), \text{ for all } \quad i = 1, \cdots, n \tag{30}$$

and for two discrete functions u,v in $D_{0,h}$, we define the scalar product to be

$$\langle u, v\rangle_h = h\Sigma_{i=1}^n u_i v_i$$

we shall give a bound for the error between the solution u of the continuous Poisson problem (28)-(29) and the solution v of the discrete Poisson problem (30). For different values of h we compute the error

$$e_h = \max |u(x_i) - v_i|_{i=0,1,\cdots,n+1}$$

Furthermore, these values are used to estimate the rate of convergence.then to show that the discrete solution v will indeed converge to the continuous solution u when the spacing h approaches zero,then we want to introduce the concepts of truncation error and consistency. for any discrete function $v \in D_h$ we define this norm by

$$\|v\|_{h,\infty} = \max |v(x_i)|_{i=0,1,\cdots,n+1}$$

Definition 5.1 *Let $f \in C(]0,1[)$ and, let $u \in C_0^2(]0,1[)$ be the solution of* (28)-(29), *then we define the discrete vector τ_h called the truncation error defined by*

$$\tau_h(x_i) = (L_h u)(x_i) - f(x_i), \text{ for all } \quad i = 1, 2, \cdots, n$$

we say that the finite difference scheme (30) *is consistent with the differential equation* (28)-(29) *if*

$$\lim_{h \longrightarrow 0} \|\tau_h\|_{h,\infty} = 0$$

then that the truncation error is defined by applying the difference operator L_h to the exact solution u, thus a scheme is consistent if the exact solution almost solves the discrete problem. we introduce a norm on the set $C([0,1])$ for any function $f \in C([0,1])$, let

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

Proposition 5.2 *The solution $v \in D_{h,0}$ of* (30) *satisfies*

$$\|v\|_{h,\infty} \leq \frac{1}{8} \|f\|_{h,\infty}$$

its proof can be found in [11].

Lemma 5.3 *Suppose $f \in C^2([0,1])$, then the truncation error defined above satisfies*

$$\|\tau_h\|_{h,\infty} \leq \frac{\|f''\|_\infty}{12} h^2$$

Proof 5.4 *By using the fact that $-\frac{\partial^2 u(x)}{\partial x^2} = f$ and $-\frac{\partial^4 u}{\partial x^4} = f''$, we derive from the Taylor series expansion.Assume that $u(x)$ is a four-times continuously differentiable function. for any $h > 0$ we have*

$$u(x+h) = u(x) + h\frac{\partial u(x)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u(x+h_1)}{\partial x^4},$$

where h_1 is some number between 0 *and h. similarly,*

$$u(x-h) = u(x) - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u(x-h_2)}{\partial x^4},$$

for $0 \leq h_2 \leq h$. In particular, this implies that

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = \frac{\partial^2 u(x)}{\partial x^2} + E_h(x)$$

where the error term E_h satisfies

$$|E_h(x)| \leq \frac{M_u h^2}{12}$$

Here the constant M_u is given by

$$M_u = \sup_{x\in[0,1]} \left| \frac{\partial^4 u(x)}{\partial x^4} \right|$$

we observe that for a fixed function u, the error term E_h tends to zero as h tends to zero. Then we derive from the Taylor series expansion and the error estimate above that

$$|\tau_h(x_i)| = \left| \frac{u(x_{i-1} - 2u(x_i) + u(x_{i+1})}{h^2} + f(x_i) \right| \leq \left| \frac{\partial^2 u(x_i)}{\partial x^2} + f(x_i) \right| + \| \frac{\partial^4 u}{\partial x^4} \|_\infty \frac{h^2}{12}$$

since

$$\left| \frac{\partial^2 u(x_i)}{\partial x^2} + f(x_i) \right| + \| \frac{\partial^4 u}{\partial x^4} \|_\infty \frac{h^2}{12} = \frac{\|f''\|_\infty h^2}{12}$$

Then

$$|\tau_h(x_i)| \leq \frac{\|f''\|_\infty h^2}{12}$$

and

$$\|\tau\|_{h,\infty} \leq \frac{\|f''\|_\infty h^2}{12}$$

by using this bound on the truncation error, we can prove that the numerical solution converges towards the exact solution as the grid size tends to zero

Theorem 5.5 *Assume that $f \in C^2([0,1])$ is given, let u and v be the corresponding solutions of (28)-(29) and (30), then*

$$\|u - v\|_{h,\infty} \leq \frac{\|f''\|_\infty}{96} h^2$$

and

$$e_h \leq \frac{\|f''\|_\infty}{96} h^2$$

Proof 5.6 *Define the discrete error function $e \in D_{h,0}$ by $e(x_i) = u(x_i) - v(x_i)$ for $i = 1, \cdots, n$. Observe that*

$$L_h e = L_h u - L_h v = L_h u - f_h = \tau_h,$$

where f_h denotes the discrete function with elements $(f(x_1), \cdots, f(x_n))$. Then it follow from Proposition above that

$$\|e\|_{h,\infty} \leq (\frac{1}{8}) \|\tau_{h,\infty} \leq \frac{\|f''\|_\infty h^2}{96}$$

we say the sequence $\{e_h\}$ is of order $\{h^2\}$, and we write

$$e_h = o(h^2)$$

this theorem guarantees that the error measured in each grid point tends to zero as the mesh parameter h tends to zero Moreover, the rate of convergence is 2.

In two-dimensional domain The system (25) can be rewritten as a system of equations in the form

$$Av = b, \tag{31}$$

where A is a $N^2 \times N^2$ matrix v and b are both N^2-dimensional vectors, our goal is to solve the system $Av = b$, the key idea on which most direct method are based is that if A is an upper-triangular matrix, then that system (31) has a unique solution, indeed say A is an upper-triangular matrix, thus what we need is a method for transforming a matrix to an equivalent in upper-triangular form, this can be done by Elimination. we describe the method of Gaussian Elimination applied to a linear system $Av = b$, and we give some numerical examples.

Example 5.7 *we consider the discretisation*

$$[x_i, x_{i+1}], x_i = ih, h = \frac{1}{3}, i = 0, 1, 2 \quad \textit{of the interval,} \ [0, 1]$$

$$[y_j, y_{j+1}], y_j = jh, h = \frac{1}{3}, j = 0, 1, 2 \quad \textit{of the interval,} \ [0, 1]$$

we define the approximation $\{v_{i,j}\}_{i,j=0}^{2}$ *by requiring* (25), *we obtain the solution* v *of the discrete problem is*

$$v_{1,1} = \frac{1}{36} f(\frac{1}{2}, \frac{1}{2})$$

Example 5.8 *we consider the discretisation*

$$[x_i, x_{i+1}], x_i = ih, h = \frac{1}{4}, i = 0, 1, 2, 3 \quad \textit{of the interval,} \ [0, 1]$$

$$[y_j, y_{j+1}], y_j = jh, h = \frac{1}{4}, j = 0, 1, 2, 3 \quad \textit{of the interval,} \ [0, 1]$$

we define the approximation $\{v_{i,j}\}_{i,j=0}^{3}$ *by requiring* (25), *the system* (25) *can be rewritten as a system of equations in the form*

$$Av = b, \tag{32}$$

where A *is a* 4×4 *matrix* v *and* b *are both* 4*-dimensional vectors. Gaussian Elimination is a numerical method for solving linear system* (32)*, there are several operations that one perform on a system of equation* (32) *without changing its solution, In the matrix language the operations we apply on augmented matrix* $[A \mid b]$ *then* (32) *which being an upper triangular system, and can be rewritten as a system of equations in the form* $AX = b$ *by taking* $X = v$

is readily solved by back substitution given in Algorithm below, in [8] proposed the algorithm as follows

$$\begin{aligned} x_4 &= \frac{b_4}{a_{4,4}} \\ For \quad i &= 3,2,1 \\ x_i &= \frac{1}{a_{i,i}}[b_i - \Sigma_{j=i+1}^4 a_{i,j}x_j] \end{aligned}$$

then the solution v of the discrete problem is

$$\begin{aligned} v_{1,1} &= \frac{1}{4}[\frac{7}{54}f(\frac{1}{3},\frac{1}{3}) + \frac{1}{27}f(\frac{1}{3},\frac{2}{3}) + \frac{1}{27}f(\frac{2}{3},\frac{1}{3}) + \frac{1}{72}f(\frac{2}{3},\frac{2}{3})], \\ v_{1,2} &= \frac{1}{15}[\frac{35}{72}f(\frac{1}{3},\frac{2}{3}) + \frac{5}{36}f(\frac{1}{3},\frac{1}{3}) + \frac{5}{72}f(\frac{2}{3},\frac{1}{3}) + \frac{5}{36}f(\frac{2}{3},\frac{2}{3})], \\ v_{2,1} &= \frac{1}{224}[\frac{196}{27}f(\frac{2}{3},\frac{1}{3}) + \frac{56}{27}f(\frac{1}{3},\frac{1}{3}) + \frac{28}{27}f(\frac{1}{3},\frac{2}{3}) + \frac{56}{27}f(\frac{2}{3},\frac{2}{3})], \\ v_{2,2} &= \frac{1}{720}[\frac{70}{3}f(\frac{2}{3},\frac{2}{3}) + \frac{20}{3}f(\frac{1}{3},\frac{2}{3}) + \frac{10}{3}f(\frac{1}{3},\frac{1}{3}) + \frac{20}{3}f(\frac{2}{3},\frac{1}{3})] \end{aligned}$$

The complexity of the linear systems varies with the size of matrix A and b we can find A large enough matrix our method can be difficulty obtained, but can be easily reproduced by the interested reader with the aid of the Matlab, in the case $n = N$ is an integer fixed $\in \mathbb{N}$ we can find A very large matrix then our method does not hold, but can be easily reproduced by the interested reader with the aid of the Matlab.

Example 5.9 *we consider the discretisation*

$$[x_i, x_{i+1}], x_i = ih, h = \frac{1}{N+1}, i = 0,1,2,\cdots, N+1 \quad \textit{of the interval}, \ [0,1]$$

$$[y_j, y_{j+1}], y_j = jh, h = \frac{1}{N+1}, \ j = 0,1,2,\cdots, N+1 \quad \textit{of the interval}, \ [0,1]$$

the system (25) *can be rewritten as a system of equations in the form*

$$Av = b, \tag{33}$$

where A is $N^2 \times N^2$ matrix v and b are both N^2-dimensional vectors by using Matlab, Form augmented matrix $M = [A \mid b]$ and place the augmented matrix in upper- triangular form with the command $triu(M)$ which being an upper-triangular system is readily solved. the command $\gg B = triu(A)$ builds a matrix B with the matrix A, we can select to exact the upper-triangular part of the A matrix, but assign to all the lower triangle elements the value zero. and since (x_i, y_j)fixed $\in \mathbb{R}$ this implies $f(x_i, y_j)$fixed $\in \mathbb{R}$, $i, j = 0, 1, \cdots, N+1$

then the command $\gg c = triu(b)$ *builds a matrix* c *with the matrix* b*, we can select to extract the upper-triangular part of the* b *matrix, but assign to all the lower triangle elements the value zero:* $c = triu(b)$ *produces*

$$c = \begin{bmatrix} c_1 = \frac{1}{(N+1)^2} f(\frac{1}{N+1}, \frac{1}{N+1}), \\ c_2 = 0 \\ . \\ . \\ c_{N^2} = 0 \end{bmatrix}$$

by taking $X = v$ *the system* (33) *can be written in the form* $Bx = c$ *that can be computed by back substitution given in Algorithm below, In [8] proposed the algorithm as follows:*

$$\begin{aligned} x_{N^2} &= \frac{c_{N^2}}{b_{N^2,N^2}} \\ For \quad i &= N^2 - 1, \cdots, 1 \\ x_i &= \frac{1}{b_{i,i}}[c_i - \Sigma_{j=i+1}^{N^2} b_{i,j} x_j] \end{aligned}$$

then the solution v *of the discrete problem is*

$$\begin{aligned} v_{1,1} &= \frac{1}{4(N+1)^2} f(\frac{1}{N+1}, \frac{1}{N+1}), \\ v_{1,2} &= 0, \\ . \\ . \\ v_{N,N} &= 0 \end{aligned}$$

The domain Ω *is given by* $\Omega = (]0,1[)^2$*, we let* D_h *denote the set of all grid functions defined on* $\overline{\Omega}$*,*

$$D_h = \{v \mid v : \overline{\Omega_h} \longrightarrow \mathbb{R}\}$$

while $D_{h,0}$ *is the subset*

$$D_{h,0} = \{v \in D_h \mid v \mid_{\partial\Omega_h} = 0\}$$

we consider a finite difference approximation of Poisson equation on Ω *with homogeneous Dirichlet boundary conditions*

$$Lu = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f, \ in \quad \Omega \tag{34}$$

$$u = 0, \ on \quad \Omega \tag{35}$$

we recall that the finite difference approximation of the problem (34)-(35) *can be formulated as follows: Find* $v \in D_{h,0}$ *such that*

$$L_h v(x_i, y_j) = f(x_i, y_j) \quad \text{for all} \quad (x_i, y_j) \in \Omega_h, v \in D_{h,0} \tag{36}$$

and for two discrete functions u *and* v *in* $D_{h,0}$, *we define the scalar product to be*

$$\langle u, v\rangle_h = h^2 \Sigma_{i,j=1}^{n} u_{i,j} v_{i,j}$$

and for different values of h *we compute the error*

$$e_h = \max_{(x,y)\in\overline{\Omega_h}} |u(x, y) - v(x, y)|$$

Furthermore,these values are used to estimated the rate of convergence, we shall give a bound for the error between the solution u *of the continuous Poisson problem* (34)-(35) *and the solution* v *of the discrete Poisson problem* (36), *then to show the discrete solution* v *will indeed converge to the continuous solution* u *when the spacing* h *approaches zero. then we want to introduce the concepts of truncation error and consistency as in one-dimensional domain. we will assume that the solution* u *of Poisson problem* (34)-(35) *is four-times differentiable* $u \in C^4(\overline{\Omega})$. *Let:* α *be the finite constant given by*

$$\alpha = \max_{0\le i+j\le 4} \|\frac{\partial^{i+j} u}{\partial x^i \partial y^j}\|_\infty \tag{37}$$

where, as usual

$$\|u\|_\infty = \sup_{(x,y)\in\overline{\Omega}} |u(x, y)|$$

and for any discrete function $v \in D_h$ *we define this norm by*

$$\|v\|_{h,\infty} = \sup_{(x,y)\in\overline{\Omega_h}} |v(x, y)|$$

as in one-dimensional domain we introduce the truncation error

$$\tau_h(x_i, y_j) = (L_h u - f)(x_i, y_j) \quad \text{for all} \quad (x_i, y_j) \in \Omega_h$$

the following result is a generalization of Lemma 5.2.

Lemma 5.10 *Assume that* $u \in C^4(\overline{\Omega})$ *the truncation error* τ_h *satisfies*

$$\|\tau_h\|_{h,\infty} \le \frac{\alpha h^2}{6}$$

where α *is given by* (37).
by using its bound on the truncation error, we can prove that the numerical solution converges towards the exact solution as the grid size tends to zero.
the following error estimate for the finite difference method (36) *is a generalization of theorem* 5.4.

Theorem 5.11 *Let u and v be corresponding solutions of* (34)-(35) *and* (36) *respectively. if $u \in C^4(\overline{\Omega})$, then*

$$\|u-v\|_{h,\infty} \leq \frac{\alpha h^2}{48}, \text{ then } \quad e_h \leq \frac{\alpha h^2}{48}$$

where α is given by (37).
we say that the sequence $\{E_h\}$ is of order $\{h^2\}$, and we write

$$e_h = o(h^2)$$

this theorem guarantees that the error measured in each grid point tends to zero as the mesh parameter h tends to zero. Moreover, the rate of convergence is 2.

In three-Dimensional Domain The system (26) can be rewritten as a system of equations in the form

$$Av = b, \tag{38}$$

where A is a $N^3 \times N^3$ matrix v and b are both N^3-dimensional vectors, then Matlab is proposed for obtaining solutions for this system (38) and we give some numerical examples.

Example 5.12 *we consider the discretisation*

$$[x_i, x_{i+1}], x_i = ih, h = \frac{1}{2}, \ i = 0,1,2 \quad \textit{of the interval} \quad [0,1]$$

$$[y_j, y_{j+1}], y_j = jh, h = \frac{1}{2}, \ j = 0,1,2 \quad \textit{of the interval} \quad [0,1]$$

$$[z_k, z_{k+1}], z_k = kh, h = \frac{1}{2}, \ k = 0,1,2 \quad \textit{of the interval} \quad [0,1]$$

we define the approximation $\{v_{i,j,k}\}_{i,j,k=0}^{2}$ by requiring (26), *we obtain the solution v of the discrete problem* (26) *is*

$$v_{1,1,1} = \frac{1}{24} f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$$

Example 5.13 *we consider the discretisation*

$$[x_i, x_{i+1}], \ x_i = ih, \ h = \frac{1}{3}, \ i = 0,1,2,3 \quad \textit{of the interval} \quad [0,1]$$

$$[y_j, y_{j+1}], \ y_j = jh, \ h = \frac{1}{3}, \ j = 0,1,2,3 \quad \textit{of the interval} \quad [0,1]$$

$$[z_k, z_{k+1}],\ z_k = kh,\ h = \frac{1}{3},\ k = 0,1,2,3 \quad \textit{of the interval} \quad [0,1]$$

we define the approximation $\{v_{i,j,k}\}_{i,j,k=0}^{3}$ *by requiring* (26), *the system* (26) *can be rewritten as a system of equations in the form*

$$Av = b, \tag{39}$$

where A *is a* 8×8 *matrix* v *and* b *are both* 8*-dimensional vectors, by using Matlab above-mentioned on* (39) *which being an upper triangular system is readily solved, by taking* $X = v$ *then the system* (39) *can be written in the form*

$$BX = c, \tag{40}$$

and write a procedure based on algorithm above that solve the system (40), *then the solution* v *of the discrete problem is*

$$\begin{aligned}
v_{1,1,1} &= \frac{1}{54} f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \\
v_{1,1,2} &= 0, \\
&. \\
&. \\
v_{2,2,2} &= 0,
\end{aligned}$$

Example 5.14 *we consider the discretisation*

$$[x_i, x_{i+1}],\ x_i = ih,\ h = \frac{1}{N+1},\ i = 0,1,2,\cdots,N+1 \quad \textit{of the interval} \quad [0,1]$$

$$[y_j, y_{j+1}],\ y_j = jh,\ h = \frac{1}{N+1},\ j = 0,1,2,\cdots,N+1 \quad \textit{of the interval} \quad [0,1]$$

$$[z_k, z_{k+1}],\ z_k = kh,\ h = \frac{1}{N+1},\ k = 0,1,2,\cdots,N+1 \quad \textit{of the interval} \quad [0,1]$$

we define the approximation $\{v_{i,j,k}\}_{i,j,k=0}^{N+1}$ *by requiring* (26), *the system* (26) *can be rewritten as a system of equations in the form*

$$Av = b, \tag{41}$$

where A *is a* $N^3 \times N^3$ *matrix* v *and* b *are both* N^3*-dimensional vectors, by using Matlab above-mentioned on* (41) *which being an upper triangular system is readily solved, by taking* $X = v$ *then the system* (41) *can be written in the form*

$$BX = c, \tag{42}$$

and write a procedure based on a algorithm above that solve the system (42)*, then the solution v of the discrete problem is*

$$\begin{aligned} v_{1,1,1} &= \frac{1}{6(N+1)^2} f(\frac{1}{N+1}, \frac{1}{N+1}, \frac{1}{N+1}), \\ v_{1,1,2} &= 0, \\ &. \\ &. \\ v_{N,N,N} &= 0 \end{aligned}$$

the domain Ω is given by $\Omega = (]0,1[)^3$, we let D_h denote the set of all grid functions defined on $\overline{\Omega}$,

$$D_h = \{v \mid : \overline{\Omega_h} \longrightarrow \mathbb{R}\},$$

while $D_{h,0}$ is the subset

$$D_{h,0} = \{v \in D_h \mid v \mid_{\partial\Omega_h} = 0\}$$

we consider a finite difference approximation of Poisson equation on Ω with homogeneous Dirichlet boundary conditions

$$Lu = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = f, \ \ in \quad \Omega \tag{43}$$

$$u = 0, \ \ on \quad \Omega \tag{44}$$

we recall that the finite difference approximation (43)-(44) *can be formulated as follows: Find $v \in D_{h,0}$ such that*

$$(L_h v)(x_i, y_j, z_k) = f(x_i, y_j, z_k), \ \ for\ all \quad (x_i, y_j, z_k) \in \Omega_h, v \in D_{h,0} \tag{45}$$

and for two discrete functions u and v in $D_{0,h}$, we define the scalar product to be

$$\langle u, v\rangle_h = h^2 \Sigma_{i,j,k=1}^n u_{i,j,k} v_{i,j,k}$$

and for different values of h we compute the error

$$e_h = \max_{(x,y,z)\in\overline{\Omega_h}} |u(x,y,z) - v(x,y,z)|$$

Furthermore,these values are used to estimated the rate of convergence, we shall give a bound for the error between the solution u of the continuous Poisson problem (43)-(44) *and the solution v of the discrete Poisson problem* (45)*, then to show the discrete solution v will indeed converge to the continuous solution u when the spacing h approaches zero,then we want to introduce the*

concepts of truncation error and consistency as in one-dimensional domain. we will assume that the solution u of Poisson problem (43)-(44) *is four-times differentiable* $u \in C^4(\Omega)$.
Let α *be the finite constant given by*

$$\alpha = \max_{0 \le i+j+k \le 4} \| \frac{\partial^{i+j+k} u}{\partial x^i \partial y^j \partial z^k} \|_\infty \tag{46}$$

where, as usual

$$\|u\|_\infty = \sup_{(x,y,z)\in\overline{\Omega_h}} |u(x,y,z)|$$

and for any discrete function $v \in D_h$ *we define this norm by*

$$\|v\|_{h,\infty} = \sup_{(x,y,z)\in\overline{\Omega_h}} |v(x,y,z)|$$

as in one-dimensional domain, we introduce the truncation error

$$\tau_h(x_i, y_j, z_k) = (L_h u - f)(x_i, y_j, z_k) \quad \text{for all, } (x_i, y_j, z_k) \in \Omega_h$$

the following result is a generalization of Lemma 5.2.

Lemma 5.15 *Assume that* $u \in C^4(\overline{\Omega})$, *the truncation error* τ_h *satisfies*

$$\|\tau_h\|_{h,\infty} \le \frac{\alpha h^2}{4}$$

where α *is given by* (46).
by using this bound on the truncation error, we can prove the numerical solution converges towards the exact solution as the grid size tends to zero.
the following error estimate for the finite difference method (45) *is a generalization of theorem* 5.4.

Theorem 5.16 *Let* u *and* v *be corresponding solutions of* (43)-(44) *and* (45) *respectively, if* $u \in C^4(\overline{\Omega})$, *then*

$$\|u - v\|_{h,\infty} \le \frac{\alpha h^2}{32}, \text{ then } \quad e_h \le \frac{\alpha h^2}{32}$$

where α *is given by* (46).
we say that the sequence $\{e_h\}$ *is of order* $\{h^2\}$, *and we write*

$$e_h = o(h^2)$$

this theorem guarantees that the error measured in each grid point tends to zero as the mesh parameter h tends to zero, Moreover the rate of convergence is 2

6 Conclusion and Open Problem

In this paper, we have studied a Poisson equation in n-dimensional domain with Dirichlet boundary conditions we establish the existence, uniqueness and regularity of the solution. we were interested in the numerical solution of the Poisson equation by finite differences schemes in one, two, three- dimensional domain.Numerical examples are presented to see the performance of the method. In generally, Matlab is proposed for obtaining solutions for this problem for a positive integer $n = N\ fixed \in \mathbb{N}$ the numerical results shows that Matlab can be easily realized and is quite effective. For the future works, we propose to study numerical solution based on finite element analysis for the Poisson equation in $n = N$-dimensional domain with Dirichlet boundary conditions.

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