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On existence of nonlinear fractional differential equations for $0 < \alpha \le 2$

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Abstract

We investigate in this article the existence problem of a fractional nonlinear differential system with $\boldsymbol{\alpha} \in (\mathbf{0}, \mathbf{2}]$***. We obtain the results by using Banach fixed-point theorem.***

Keywords: *Fractional differential equations, Banach fixed point, nonlocal condition.*

1 Introduction

Fractional differential equations are considered as a new branch of applied mathematics by which many physical and engineering approaches can be modeled. The fact that fractional differential equations are considered as alternative models to nonlinear differential equations which induced extensive researches in various fields including the theoretical part. The existence and uniqueness problems of fractional nonlinear differential equations as a basic theoretical part are investigated by many authors (see [1]-[13] and references therein). In [1] and [10], the authors obtained sufficient conditions for the existence of solutions for a class of boundary value problem for fractional differential equations (in the case of $1 < \alpha \le 2$) involving the Caputo fractional derivative and nonlocal conditions using the Banach, Schaefer's, and Krasnoselkii fixed points theorems. The Cauchy problems for some fractional abstract differential equations (in the case of $0 < \alpha \le 1$) with nonlocal conditions are investigated by the authors in [2], [9] and [12] using the Banach and Krasnoselkii fixed point theorems. The Banach fixed point theorem is used in [4] and [8] to investigate the existence problem of fractional integrodifferential equations (in the case of $0 < \alpha \le 1$) on Banach spaces. Motivated by these works we study in this paper the existence of solution to fractional differential equations when $0 < \alpha \le 2$ on Banach spaces by using Banach fixed point theorem.

2 Preliminaries

We need some basic definitions and properties of fractional calculus (see [5], [11], and [13]) which will be used in this paper.

Definition 2.1 *A real function $f(t)$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0,\infty)$, and it is said to be in the space C_μ^n iff $f^{(n)} \in C_\mu, n \in \mathbb{N}$.*

Definition 2.2 *A function $f \in C_\mu, \mu \geq -1$ is said to be fractional integrable of order $\alpha \geq 0$ if*

$$(I^\alpha f)(t) = I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds < \infty,$$

where $\alpha > 0$, and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Next, we introduce the Caputo fractional derivative.

Definition 2.3 *The fractional derivative in the Caputo sense is defined as*

$$(D^\alpha f)(t) = D^\alpha f(t) = I^{n-\alpha}(f^{(n)}(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds$$

for $-1 < \alpha \leq n, n \in \mathbb{N}, t > 0, f \in C_\mu^n$. In particular,

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \qquad for\ 0 < \alpha \leq 1$$

and

$$D^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f''(s) ds, \qquad for\ 1 < \alpha \leq 2$$

where $f \in C_{-1}^2$ is a function with values in abstract space X.

Lemma 2.4 [1] *If $n-1 < \alpha \leq n$,* then

$$I^\alpha D^\alpha f(t) = f(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $t \in J = [0,T]$, $c_0, c_1, \ldots,$ and c_{n-1} are constants.

Let $Y = C(J,X)$ be a Banach space of all continuous functions $x(t)$ from a compact interval J into a Banach space X.

Consider the fractional nonlinear differential system

$$\begin{cases} D^\alpha x_i(t) = f_i(t, x_i(t)), \\ x_i(0) = y_i \in X, x_2{}'(T) = 0, \end{cases} \tag{1}$$

where $i-1<\alpha\leq i; i=1,2,$ $f_i: J\times Y\to Y$ is continuous which satisfies the following hypothesis;

(H1) *There exists a positive constant A_i such that*

$$\|f_i(t,x_i)-f_i(t,y_i)\|\leq A_i\|x_i-y_i\|, \qquad i=1,2,$$

for any $t\in J, x_i, y_i\in Y$. Moreover, let $B_i=sup_{t\in J}\|f_i(t,0)\|$.

Lemma 2.5 *The fractional differential system (1) is equivalent to*

$$x_i(t)=y_i+\int_0^T G_i(t,s)f_i(s,x_i(s))ds,\ \ t\in J, i-1\leq\alpha\leq i, i=1,2, \tag{2}$$

where

$$G_1(t,s)=\begin{cases}\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} & for\ 0\leq s\leq t\\ 0, & for\ t\leq s\leq T\end{cases}$$

$$G_2(t,s)=\begin{cases}\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\dfrac{t(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & for\ 0\leq s\leq t\\ -\dfrac{t(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & for\ t\leq s\leq T\end{cases},$$

Proof. The case $0<\alpha\leq 1$ is trivial. Let $1<\alpha\leq 2$, then by Lemma 2.4 and conditions in (1), we have

$$I^\alpha f_2(t,x_2(t))=I^\alpha D^\alpha x_2(t)=x_2(t)+c_0+c_1t.$$

For $t=0$, we get $0=y_2+c_0$ which implies that $c_0=-y_2$. On the other hand, for $t=T$, we get $I^{\alpha-1}f_2(T,x_2(T))=x_2{}'(T)+c_1$ which implies that

$$c_1=I^{\alpha-1}f_2(T,x_2(T))=\frac{1}{\Gamma(\alpha-1)}\int_0^T(T-s)^{\alpha-2}f_2(s,x_2(s))ds\,.$$

Hence

$$x_2(t)=y_2-\frac{t}{\Gamma(\alpha-1)}\int_0^T(T-s)^{\alpha-2}f_2(s,x_2(s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_2(s,x_2(s))ds.$$

Conversely, if we apply the fractional differential operator D^α to the integral equations in (2), it can easily get the system (1).

3 Existence of a solution

The existence problems of fractional nonlinear differential systems are investigated in this section by using the well-known Banach fixed point theorem. The first result is the existence of solution for the system (1). The following condition is essential to get the contraction property.

(H2) Let, for $i = 1,2$, and $i-1 < \alpha \le i$,

$$\begin{cases} C_i = \max\{A_i, B_i\} T^\alpha \left(\dfrac{1}{\Gamma(\alpha+1)} + \dfrac{i(i-1)}{i!\,\Gamma(\alpha)} \right) < 1 \\ r_i \ge \dfrac{\|y_i\| + C_i}{1 - C_i}. \end{cases}$$

Moreover, let $B_{r_i} = \{x_i \in Y : \|x_i\| \le r_i\}$.

Theorem 3.1 *If the hypotheses (H1), and (H2) are satisfied, then the fractional differential system (1) has a solution on J.*

Proof. We prove, by using the Banach fixed point, the operator $\Lambda_i : Y \to Y, i = 1,2$, given by

$$\Lambda_i x_i(t) = y_i + \int_0^T G_i(t,s) f_i\big(s, x_i(s)\big) ds, \ \ t \in J$$

has a fixed point on B_{r_i}. Firstly, we show that $\Lambda_i B_{r_i} \subset B_{r_i}$. Let $i = 1$, and $0 < \alpha \le 1$, then

$$\begin{aligned} \|\Lambda_1 x_1(t)\| &\le \|y_1\| + \int_0^T \left\| G_1(t,s) \left(f_1\big(s, x_1(s)\big) - f_1(s,0) + f_1(s,0) \right) \right\| ds \\ &\le \|y_1\| + \frac{A_1 \|x_1\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{B_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\le \|y_1\| + (A_1\|x_1\| + B_1) \frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Next, for $i = 2$, and $1 < \alpha \le 2$, we have

$$\|\Lambda_2 x_2(t)\| \le \|y_2\| + (A_2\|x_2\| + B_2) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{tT^{\alpha-1}}{\Gamma(\alpha)} \right)$$

The two cases can be combined to give that

$$\begin{aligned} &\|\Lambda_i x_i(t)\| \le \|y_i\| \\ &\quad + T^\alpha \left(\frac{1}{\Gamma(\alpha+1)} + \frac{i(i-1)}{i!\,\Gamma(\alpha)} \right) (A_i\|x_i\| + B_i). \end{aligned}$$

Therefore, if $x_i \in B_{r_i}$, we get $\|\Lambda_{\mathrm{i}} x_i\| \leq (1 - C_i) r_i + C_i r_i = r_i$. Hence, the operator Λ_{i} maps B_{r_i} into itself. Next, we prove that Λ_{i} is a contraction mapping on B_{r_i}. Let $x_i, z_i \in B_{r_i}$, then for $i = 1,2,$ and $i - 1 < \alpha \leq i$, we have

$$\|\Lambda_{\mathrm{i}} x_i(t) - \Lambda_{\mathrm{i}} z_i(t)\| \leq \int_0^T |G_i(t,s)| \left\| f_i\big(s, x_i(s)\big) - f_i\big(s, z_i(s)\big) \right\| ds$$

$$\leq A_i T^{\alpha} \left(\frac{1}{\Gamma(\alpha+1)} + \frac{i(i-1)}{i!\,\Gamma(\alpha)} \right) \|x_i - z_i\| \leq C_i \|x_i - z_i\|.$$

Hence, the operator Λ_{i} has a unique fixed point $x_i; i = 1,2$, by which one can formulate a solution $x = (x_1, x_2)$ to the system (1).

Next result in this section is the existence of solution to the system (1) in the pace $Y \times Y$.

Let $x = (x_1, x_2) \in Z = Y \times Y$ such that $\|x\|_Z = \|x_1\| + \|x_2\|$. Then $(Z, \|x\|_Z)$ becomes a complete normed space. Also, assume that

$$f(t,x) = \big(f_1(t,x_1), f_2(t,x_2)\big), G(t,s) = \big(G_1(t,s), G_2(t,s)\big)$$

and that

$$\int_0^T G(t,s) * f(s,x)\, ds = \left(\int_0^T G_1(t,s) f_1(t, x_1(s)) ds \,, \int_0^T G_2(t,s) f_2(t, x_2(s)) ds \right).$$

Therefore, the system (1) can be rewritten in the form

$$\begin{cases} D^{\alpha} x(t) = f\big(t, x(t)\big), & t \in J, 0 < \alpha \leq 2, x \in Z; \\ z(0) = \big(x_1(0), x_2(0)\big) = z_0 \in Z, x_2{}'(T) = 0, \end{cases} \tag{3}$$

which is equivalent to the integral form (see eq.(2))

$$x(t) = z_0 + \int_0^T G(t,s) * f(s, x(s))\, ds \tag{4}$$

(H3) *Let $D, C,$ and r be positive constants such that*

$$\begin{cases} D = max_{1 \leq i \leq 2}\{A_i, B_i\} \\ C = D T^{\alpha} \left(\dfrac{2}{\Gamma(\alpha+1)} + \dfrac{1}{\Gamma(\alpha)} \right) < 1 \\ r \geq \dfrac{\|z_0\| + C}{1 - C}. \end{cases}$$

Moreover, let $B_r = \{x \in Z : \|x\| \leq r\}$.

Let $x = (x_1, x_2), y = (y_1, y_2,) \in Z$, in view of (H1), it easy to get

$$\begin{aligned}\|f(t,x) - f(t,y)\| &= \left\|\left(f_1(t,x_1) - f_1(t,y_1), f_2(t,x_2) - f_2(t,y_2)\right)\right\| \\ &\leq A_1\|x_1 - y_1\| + A_2\|x_2 - y_2\| \leq D\|x - y\|,\end{aligned}$$

and $\sup_{t\in J}\|f(t,0)\| = B_1 + B_2$.

Theorem 3.2 *If the hypotheses (H1), and (H3) are satisfied, then the fractional differential system (3) has a solution on J.*

Proof. Let $x = (x_1, x_2) \in Z$. Define the operator Λ on Z given by

$$\Lambda x(t) = (\Lambda_1 x_1(t), \Lambda_2 x_2(t)) = z_0 + \int_0^T G(t,s) * f(s, x(s))\, ds,$$

where $\|\Lambda x\| = \|\Lambda_1 x_1\| + \|\Lambda_2 x_2\|$. Hence, following the proof of Theorem 3.1, we have

$$\begin{aligned}\|\Lambda x(t)\| &\leq \|z_0\| + D(\|x_1\| + 1)\frac{T^\alpha}{\Gamma(\alpha+1)} + D(\|x_2\| + 1)\left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha)}\right) \\ &\leq \|z_0\| + \frac{DT^\alpha}{\Gamma(\alpha+1)}\|x\| + DT^\alpha\left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right).\end{aligned}$$

Therefore if $x \in B_r$, so does Λx. On the other hand, if $x, y \in B_r$, then

$$\begin{aligned}\|\Lambda x(t) - \Lambda y(t)\| &= \left\|\int_0^T G_1(t,s)\left(f_1(s, x_1(s)) - f_1(s, y_1(s))\right) ds\right\| \\ &\quad + \left\|\int_0^T G_2(t,s)\left(f_2(s, x_2(s)) - f_2(s, y_2(s))\right) ds\right\| \\ &\leq \frac{DT^\alpha}{\Gamma(\alpha+1)}\|x_1 - y_1\| + DT^\alpha\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right)\|x_2 - y_2\| \leq C\|x - y\|,\end{aligned}$$

which ends the proof.

We close the article by considering the following nonlocal fractional differential system

$$\begin{cases} D^\alpha x_i(t) = f_i(t, x_i(t)), \\ x_i(0) = g_i(x_i) \in Y, x_2'(T) = a, i = 1,2 \end{cases} \tag{5}$$

where f_i as before satisfies the hypothesis (H1), a is a constant, and the nonlinear function g satisfies the following assumption.

(H4) The function g_i is defined on the Banach space *Y,* satisfying the Lipsitchitz condition, i.e., there exists a positive constant C_i *s*uch that

$$\|g_i(x_i) - g_i(y_i)\| \leq C_i\|x_i - y_i\|, \qquad i = 1,2,$$

for any $x_i, y_i \in Y$.

The system (5) is equivalent to

$$x(t) = g(x) + At + \int_0^T G(t,s) * f(s, x(s))\, ds$$

where $g(x) = g(x_1, x_2) = \big(g_1(x_1), g_2(x_2)\big)$, $A = (0, a)$, and the other term is defined as previous.

Before going on to the last result in the sequel, we need to modify the hypothesis (H3) to be suitable for the case.

(H5) *Let $D, C,$ and r be positive constants such that*

$$\begin{cases} D = max_{1\leq i\leq 2}\{A_i, B_i, C_i\} \\ C = D\, max\left\{ 1 + \dfrac{T^\alpha}{\Gamma(\alpha+1)}, T^\alpha\left(\dfrac{2}{\Gamma(\alpha+1)} + \dfrac{1}{\Gamma(\alpha)}\right)\right\} < 1 \\ r \geq \dfrac{\|g(0)\| + |a|T + C}{1 - C}. \end{cases}$$

Moreover, let $B_r = \{x \in Z: \|x\| \leq r\}$.

Now, we can state the next theorem whose proof is similar to that of Theorem 3.2 with some modifications.

Theorem 3.3 *If the hypotheses (H1), (H4), and (H5) are satisfied, then the fractional differential system (5) has a unique solution on J.*

4 Open Problem

Consider the following nonlinear fractional differential system

$$\begin{cases} D^\alpha x_i(t) = f_i\big(t, x_i(t)\big), i - 1 < \alpha \leq i; i = 1,2,\dots,n, \\ x_i(0) = u_i \in X, {x_{i+k}}^{(i)}(T) = v_{ik} \in X, 1 \leq k \leq n, \end{cases} \tag{6}$$

where $x_i \in Y$ and f_i satisfies some conditions. One can use a fixed point theorem to show that the system (6) has a solution $x = (x_1, x_2, \dots, x_n) \in Y \times Y \times \cdots \times Y$ under specific assumptions.

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