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# Ultra Linear Continuous Functionals and Ultra Generalized Complex Numbers In The Ultra Generalized Function Spaces

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**Abstract**

To study mathematical models in the Ultra Generalized Function Space $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ constructed in [6]it is important to define some tools like Ultra Generalized Complex Numbers and Ultra Generalized Functionals . In this paper , the Ultra Generalized Complex Numbers $C_\alpha^* = K^*(C)/I^*(C)$, and the Ultra Generalized Linear continuous Functionals in the Space $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ are defined. Their important properties are also proved.

**Key words:** New Generalized Function Space, Rome- Helfand- Shilov Spaces

## 1 Introduction

In [6] the Ultra Generalized Function Space $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ were defined in the following way: if $\alpha$ and $\beta$ are nonnegative real numbers and $k, q \in \mathbb{N}$, define the following sets [2]:

$S_\alpha^\beta = \{f \in C^\infty(R) : \exists\ A > 0, \exists\ B > 0,\ \forall k\ \forall q\ \exists\ C > 0$
such that $|x^k f^{(q)}(x)| \leq CA^k B^q K^{k\alpha} q^{q\beta}\}$

If $\alpha > 0$ and $\beta = \alpha$, then the Space $S_\alpha^\alpha(\mathbb{R})$ is said to be Rome-Helfand-Shilov Space.

The Topology in $S_\alpha^\alpha(\mathbb{R})$ is defined by the system of semi norms in the following way :

$$p_{n,l} = \sup_{\substack{k \leq n \\ m \leq l}} q_{k,m}$$

where

$$q_{k,m} = \sup_{x\in\mathbb{R}} \frac{x^k f^{(m)}(x)}{A^k B^m k^{\alpha k} m^{\alpha m}}$$

The following theorem is true see [3,6]

**Theorem 1.1** If $f, g \in S^{\alpha}_{\alpha}(\mathbb{R})$, then for each $n, l$ there is a constant $C_{n,l} > 0$ such that $p_{n,l}(fg) \leq C_{n,l}\ p_{n,l}(f)\ p_{n,l}(g), \forall f, g \in S^{\alpha}_{\alpha}(\mathbb{R})$

Now , let $X$ be separated complete locally convex algebra [1] with topology defined by the family of semi norms $P_{i\epsilon I}$ such that for each $i \in I$ there is $j \in I$ and a constant $C_i > 0$ for which

$$p_i(xy) \leq C_i\ p_j(x)\ p_j(y) \ \ \forall\ x, y \in X \quad (*)$$

if we denote by $G(X)$ the set of all possible sequences $(x_k)$ in $X$ , then $G(X)$ is an algebra with operations of coordinate wise multiplication.

Let $\alpha > 1$ be positive real number , define the following sets [6]:

$$G_\alpha(X) = \{x = (x_k) \in G(X) : \exists\ m,\ \forall\ i \in I,\ \exists C_i > 0,\ p_i(x_k) \leq C_i\ exp(mk^{\frac{1}{\alpha}}),\ \forall k\}$$

$$N_\alpha(X) = \{x = (x_k) \in G(X) : \exists\ m,\ \forall\ i \in I,\ \exists C_i > 0,\ p_i(x_k) \leq C_i\ exp(-mk^{\frac{1}{\alpha}}),\ \forall k\}$$

**Theorem 1.2** The space $G_\alpha(X)$ is a subalgebra of the Algebra $G(X)$ , and $N_\alpha(X)$ is an ideal of $G_\alpha(X)$.

Define the Ultra space $L_\alpha(X)$ as a factor space $L_\alpha(X) = G_\alpha(X)/N_\alpha(X)$ . Since, $S^{\alpha}_{\alpha}(\mathbb{R})$ is complete separated locally convex algebra and $p_{n,l}$ satisfy $(*)$ , then the Ultra Generalized Functions Space is defined in the following way:

$$L_\alpha(S^{\alpha}_{\alpha}(\mathbb{R})) = G_\alpha(S^{\alpha}_{\alpha}(\mathbb{R}))/N_\alpha(S^{\alpha}_{\alpha}(\mathbb{R}))$$

The embedding of the spaces $S^{\alpha}_{\alpha}(\mathbb{R})$ and $[S^{\alpha}_{\alpha}(\mathbb{R})]'$ in to the Algebra $L_\alpha(S^{\alpha}_{\alpha}(\mathbb{R}))$ have been defined [6].Therefore , we can write $S^{\alpha}_{\alpha}(\mathbb{R}) \subset L_\alpha(S^{\alpha}_{\alpha}(\mathbb{R})$ , $[S^{\alpha}_{\alpha}(\mathbb{R})]' \subset L_\alpha(S^{\alpha}_{\alpha}(\mathbb{R})$.

## 2 Ultra Generalized Complex Numbers

Now our aim is to construct tools in the space $L_\alpha(S_\alpha^\alpha(\mathbb{R})$ . For example we need Ultra Generalized Numbers $C^*$ to study mathematical models as Cauchy's

$$\begin{cases} Df = fg & f(0) = z^* \\ f(0) = z^* & u, v \in L_\alpha(S_\alpha^\alpha(\mathbb{R})) \\ u, v \in L_\alpha(S_\alpha^\alpha(\mathbb{R})) \end{cases}$$

or

$$\begin{cases} Df = \delta^n f & f(a) = b \\ f(a) = b & a, b \in C^* \\ f \in L_\alpha(S_\alpha^\alpha(\mathbb{R})) \end{cases}$$

We define the Ultra Generalized Complex Numbers corresponding to the Space of the Ultra Generalized Functions $L_\alpha(S_\alpha^\alpha(\mathbb{R})$ in the following way:

let $K(C)$ be the set of all sequences of complex numbers . Define $K^*(C)$ as the set of all sequences $(z_k) \in K(C)$ such that there is a natural numbers $m \in N$ and a constant $C > 0$, such that $| z_k |\leq C.exp(mk^{\frac{1}{\alpha}})$, for each $k$. Define $I^*(C)$ as the set of all sequences $(\eta_k) \in K(C)$ such that for each $m \in N$ and for each $k$ there is a constant $d > 0$ such that $| \eta_k |\leq d.exp(-mk^{\frac{1}{\alpha}})$, for each $k$ in the domain of the sequence $(\eta_k)$.

**Theorem 2.1**

(a) The set $K^*(C)$ is an algebra

(b) The set $I^*(C)$ be an ideal in $K^*(C)$.

**Proof.** a) Suppose $z_1 = (z_k), z_2 = (z_k)'$ are elements in $K^*(C)$ , then there are natural numbers $m_1, m_2$ and the constants $C_1 > 0, C_2 > 0$ such that $| z_k |\leq C_1.exp(m_1 k^{\frac{1}{\alpha}})$ and $| z'_k |\leq C_2.exp(m_2 k^{\frac{1}{\alpha}})$. Then, $| z_k z'_k |\leq C_1 C_2.exp((m_1 + m_2)k^{\frac{1}{\alpha}})$ and hence $z_1.z_2 \in K^*(C)$.

b) Now suppose that $z = (z_k) \in K^*(C)$, then there is a natural number $m_1$ and a constant $C > 0$ such that $| z_k |\leq C.exp(m_1 k^{\frac{1}{\alpha}})$

, for each $k$. Now if $\eta = (\eta_k) \in I^*(C)$ then for each $m \in N$ there is a constant $d > 0$ such that $\mid \eta \mid = \mid \eta_k \mid \leq d.exp(-mk^{\frac{1}{\alpha}})$ for each $k$. Now consider $\mid \eta z \mid = \mid \eta_k z_k \mid \leq Cd.exp((m_1 - m)k^{\frac{1}{\alpha}})$, that is $z\eta \in I^*(C)$.

**Theorem 2.2**

(a) If $h = (h_k) \in G_\alpha(S_\alpha^\alpha(R))$ and $\mu_0 \in R$, then $h(\mu_0) = (h_k(\mu_0)) \in K^*(C)$

(b) If $h = (h_k) \in N_\alpha(S_\alpha^\alpha(R))$ and $\mu_0 \in R$, then $\eta(\mu_0) = (\eta_k(\mu_0)) \in I^*(C)$

**Proof.** The theorem is proved by using definitions of $G_\alpha(S_\alpha^\alpha(R)$, $N_\alpha(S_\alpha^\alpha(R))$ , $K^*(C)$, and $I^*(C)$

**Definition** The Space of Ultra Generalized Complex Numbers is defined as a factor Algebras

$$C_\alpha^* = K^*(C)/I^*(C).$$

The definition of $C_\alpha^*$ and theorem 2.2 play important role when we study the models like Cauchy's in the Space of Ultra Generalized Functions $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$.

Moreover , we define embedding of the set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{R}$ into the Space of Ultra Generalized Complex Numbers $C_\alpha^*$ by the following

$$\begin{aligned} j_1 &: x \in R \to (x_k + 0i) \in C_\alpha^*, \text{where } x_k = x \ \forall k \\ j_2 &: z \in C \to (z_k) \in C_\alpha^*, \text{where } z_k = z \ \forall k \end{aligned}$$

## 3 Ultra Linear Continuous Functionals in $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$

Let $A : S_\alpha^\alpha(\mathbb{R}) \to S_\alpha^\alpha(\mathbb{R})$ be a linear continuous operator , then [1] for each $i \in I$ there exists $j$ and a constant $C_i > 0$ such that $p_i(A(\varphi(x))) \leq C_i p_j(\varphi(x))$, $\forall \varphi \in S_\alpha^\alpha(R)$. The operator $A$ is lifted coordinate wise to a map which we denote by $A^* : G(S_\alpha^\alpha(\mathbb{R})) \to G(S_\alpha^\alpha(\mathbb{R}))$.

**Theorem 3.1**

(a) $A^*[G_\alpha(S_\alpha^\alpha(\mathbb{R}))] \subset G_\alpha(S_\alpha^\alpha(\mathbb{R}))$

(b) $A^*[N_\alpha(S_\alpha^\alpha(\mathbb{R}))] \subset N_\alpha(S_\alpha^\alpha(\mathbb{R}))$.

**Proof.** The proof of this theorem follows from the definition of sets $G_\alpha(S_\alpha^\alpha(\mathbb{R}))$, $N_\alpha(S_\alpha^\alpha(\mathbb{R}))$, and by using the continuity of the operator $A$.

Now the operator $A$ can be lifted to a map which we will denote by

$$A_\alpha^* : L_\alpha(S_\alpha^\alpha(\mathbb{R})) \to L_\alpha(S_\alpha^\alpha(\mathbb{R}))$$

**Theorem 3.2**

(a) The Operator $A_\alpha^*$ is independent on a representative

(b) if $A : S_\alpha^\alpha(\mathbb{R}) \to S_\alpha^\alpha(\mathbb{R})$ is an isomorphism of $S_\alpha^\alpha(\mathbb{R})$ , then $A_\alpha^* : L_\alpha(S_\alpha^\alpha(\mathbb{R})) \to L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ is an isomorphism of $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$

**Proof.** a) Let $f \in L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ and let $(f_k)$ and $(g_k)$ are two representatives of $f$, then

$$(f_k - g_k) \in N_\alpha(S_\alpha^\alpha(\mathbb{R})).$$

Since $A^*$ is continuous, then

$$p_i[A_\alpha^*(f_k) - A_\alpha^*(g_k)] = p_i(A_\alpha^*(f_k - g_k)) \leq C_j p_j(f_k - g_k) \leq C_i C_j exp(-mk^{\frac{1}{\alpha}}), \forall k,$$

that is $[A_\alpha^*(f_k) - A_\alpha^*(g_k)] \in N_\alpha(S_\alpha^\alpha(\mathbb{R}))$.

b) The proof follows immediately from the definition of $A_\alpha^*$ and by the fact that $A : S_\alpha^\alpha(\mathbb{R}) \to S_\alpha^\alpha(\mathbb{R})$ is an isomorphism of $S_\alpha^\alpha(\mathbb{R})$

Now , if $h : S_\alpha^\alpha(\mathbb{R}) \to C$ be linear continuous functional , then it is lifted coordinate wise to $h : G(S_\alpha^\alpha(\mathbb{R})) \to C$ and $h_\alpha^* : L_\alpha(S_\alpha^\alpha(\mathbb{R})) \to C$ and the functional $h_\alpha^*$ is well defined by virtue of the following results.

**Corollary 3.3**

(a) $h^*_\alpha[G^*_\alpha(S^\alpha_\alpha(\mathbb{R}))] \subset K^*(C)$

(b) $h^*_\alpha[N^*_\alpha(S^\alpha_\alpha(\mathbb{R}))] \subset I^*(C)$ ,

(c) The functional $h^*_\alpha$ is independent on a representative.

Now we can define the Lebege's Integral on $L_\alpha(S^\alpha_\alpha(\mathbb{R}))$ as a linear continuous functional by the following way: Let $K \subset \mathbb{R}$ be any compact set and $\varphi \in L_\alpha(S^\alpha_\alpha(\mathbb{R}))$ , then define $\int_K \varphi(x)d\mu = \int_K \varphi_k(x)d\mu$, where, $\varphi_k(x)$ be any representative of $\varphi$.

**Definition** The ultra generalized complex number $z^* = \int_K \varphi(x)d\mu = \int_K \varphi_k(x)d\mu$ is called the generalized integral of ultra generalized function $\varphi \in L_\alpha(S^\alpha_\alpha(\mathbb{R}))$ over the compact $K$ .

The generalized integral defined above preserve many properties of usual Lebege's integral defined in $S^\alpha_\alpha(\mathbb{R})$ , for example, the following properties are preserved :

(1) $\int_K[\lambda(x) \pm \eta(x)]d\mu = \int_K \lambda(x)d\mu \pm \int_K \eta(x)d\mu$

(2) $\int_K a\lambda(x)d\mu = a\int_K \lambda(x)d\mu, \forall a \in C^*_\alpha$

## 4 Open Problem

How to define the Extended Laplace Transform in the Ultra Generalized Function Space $S^\alpha_\alpha(R)$

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