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A new characterization of Smarandache $\mathbf{ts\alpha}$ curves according to Sabban frame in Heisenberg group Heis^3

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Abstract

In this paper, we study Smarandache $\mathbf{ts\alpha}$ curves according to Sabban frame in the Heisenberg group Heis^3. Finally, we find explicit parametric equations of Smarandache $\mathbf{ts\alpha}$ curves according to Sabban Frame.

***Keywords**: Bienergy, Biharmonic curve, Heisenberg group, Smarandache $\mathbf{ts\alpha}$ curves.*

1 Introduction

In differential geometry, the theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by E. Almansi, T. Levi-Civita and M. Nicolescu. In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations.

This study is organised as follows: Firstly, we study Smarandache $\mathbf{ts\alpha}$ curves according to Sabban frame in the Heisenberg group Heis^3. Finally, we find explicit parametric equations of Smarandache $\mathbf{ts\alpha}$ curves according to Sabban Frame.

2 Preliminaries

Heisenberg group $Heis^3$ can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\bar{x},\bar{y},\bar{z})(x,y,z)=(\bar{x}+x,\bar{y}+y,\bar{z}+z-\frac{1}{2}\bar{x}y+\frac{1}{2}x\bar{y}) \tag{2.1}$$

$Heis^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of $Heis^3$ has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial x},\ \mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\ \mathbf{e}_3=\frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,\ [\mathbf{e}_2,\mathbf{e}_3]=[\mathbf{e}_3,\mathbf{e}_1]=0$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1.$$

We obtain

$$\begin{aligned}
&\nabla_{\mathbf{e}_1}\mathbf{e}_1=\nabla_{\mathbf{e}_2}\mathbf{e}_2=\nabla_{\mathbf{e}_3}\mathbf{e}_3=0,\\
&\nabla_{\mathbf{e}_1}\mathbf{e}_2=-\nabla_{\mathbf{e}_2}\mathbf{e}_1=\frac{1}{2}\mathbf{e}_3,\\
&\nabla_{\mathbf{e}_1}\mathbf{e}_3=\nabla_{\mathbf{e}_3}\mathbf{e}_1=-\frac{1}{2}\mathbf{e}_2,\\
&\nabla_{\mathbf{e}_2}\mathbf{e}_3=\nabla_{\mathbf{e}_3}\mathbf{e}_2=\frac{1}{2}\mathbf{e}_1.
\end{aligned} \tag{2.3}$$

3 Biharmonic $S-$Helices According To Sabban Frame In The Heisenberg Group $Heis^3$

Let $\gamma:I\to Heis^3$ be a non geodesic curve on the Heisenberg group $Heis^3$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group $Heis^3$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion,

$$g(\mathbf{T},\mathbf{T}) = 1, g(\mathbf{N},\mathbf{N}) = 1, g(\mathbf{B},\mathbf{B}) = 1,$$
$$g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0.$$

In the rest of the paper, we suppose everywhere

$$\kappa \neq 0 \text{ and } \tau \neq 0.$$

Now we give a new frame different from Frenet frame. Let $\alpha : I \to \mathbb{S}^2_{Heis^3}$ be unit speed spherical curve. We denote σ as the arc-length parameter of α. Let us denote $\mathbf{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of α. We now set a vector $\mathbf{s}(\sigma) = \alpha(\sigma) \times \mathbf{t}(\sigma)$ along α. This frame is called the Sabban frame of α on the Heisenberg group Heis^3. Then we have the following spherical Frenet-Serret formulae of α :

$$\begin{aligned} \nabla_{\mathbf{t}}\alpha &= \mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{t} &= -\alpha + \kappa_g\mathbf{s}, \\ \nabla_{\mathbf{t}}\mathbf{s} &= -\kappa_g\mathbf{t}, \end{aligned} \tag{3.2}$$

where κ_g is the geodesic curvature of the curve α on the $\mathbb{S}^2_{Heis^3}$ and

$$g(\mathbf{t},\mathbf{t}) = 1, g(\alpha,\alpha) = 1, g(\mathbf{s},\mathbf{s}) = 1,$$
$$g(\mathbf{t},\alpha) = g(\mathbf{t},\mathbf{s}) = g(\alpha,\mathbf{s}) = 0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned} \alpha &= \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3, \\ \mathbf{t} &= t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3, \\ \mathbf{s} &= s_1\mathbf{e}_1 + s_2\mathbf{e}_2 + s_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

To separate a biharmonic curve according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $\mathbb{S}$-curve.

Lemma 3.1. *α is a biharmonic $\mathbb{S}$-curve if and only if*

$$\kappa_g = \text{constant} \neq 0,$$
$$1+\kappa_g^2 = -[\frac{1}{4}-s_3^2]+\kappa_g[-\alpha_3 s_3], \qquad (3.4)$$
$$\kappa_g^3 = -\alpha_3 s_3 - \kappa_g[\frac{1}{4}-\alpha_3^2].$$

Then the following result holds.

Theorem 3.2. ([9]) *All of biharmonic* S*-curves in* $\mathrm{S}^2_{Heis^3}$ *are helices.*

Theorem 3.3. ([9]) *Let* $\alpha : I \to \mathrm{S}^2_{Heis^3}$ *be a unit speed non-geodesic biharmonic* S*-curve. Then, the position vector of* α *is*

$$\begin{aligned}\alpha(\sigma) = {} & [-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]\mathbf{e}_1 + [\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]\mathbf{e}_2 \\ & +[\cos\mathrm{E}\sigma - \frac{\mathrm{V}\sigma+\mathrm{M}_1}{2\mathrm{V}^2}\sin^4\mathrm{E} - \frac{\sin 2[\mathrm{M}\sigma+\mathrm{M}_1]}{4\mathrm{V}^2}\sin^4\mathrm{E} \\ & -[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3][-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2] \\ & +\frac{\mathrm{M}_2}{\mathrm{V}}\sin^3\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_4]\mathbf{e}_3,\end{aligned} \qquad (3.5)$$

where $\mathrm{M}_1,\mathrm{M}_2,\mathrm{M}_3,\mathrm{M}_4$ *are constants of integration and*

$$\mathrm{M} = (\frac{\sqrt{1+\kappa_g^2}}{\sin\mathrm{E}} - \cos\mathrm{E}) \text{and} \mathrm{V} = \sqrt{1+\kappa_g^2} - \frac{1}{2}\sin 2\mathrm{E}.$$

4 Smarandache $\mathbf{ts\alpha}$ Curves Of Biharmonic S-Curves According To Sabban Frame In The Heisenberg Group Heis ^{3}

Definition 4.1. *Let* $\alpha : I \to \mathrm{S}^2_{Heis^3}$ *be a unit speed regular curve in the Heisenberg group* $Heis^3$ *and* $\{\alpha, \mathbf{t}, \mathbf{s}\}$ *be its moving Bishop frame. Smarandache* $\mathbf{ts\alpha}$ *curves are defined by*

$$\gamma_{\mathbf{ts\alpha}} = \frac{1}{\sqrt{\left(1-\kappa_g\right)+1+\kappa_g^2}}\left(\mathbf{t}+\mathbf{s}+\boldsymbol{\alpha}\right). \tag{4.1}$$

Theorem 4.2. *Let $\alpha : I \to \mathbb{S}^2_{Heis^3}$ be a unit speed non-geodesic biharmonic $\mathbb{S}$-curve $\gamma_{\mathbf{ts\alpha}}$ its Smarandache $\mathbf{ts\alpha}$ curve. Then, the position vector of Smarandache $\mathbf{ts\alpha}$ curve is*

$$\begin{aligned}
\gamma_{\mathbf{ts\alpha}}(\sigma) &= \Pi \sin \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1] - \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2 \\
&+ \frac{1}{\kappa_g}[\sin \mathsf{E} \cos[\mathsf{M}\sigma + \mathsf{M}_1](\mathsf{M} + \cos \mathsf{E}) - \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2]]\mathbf{e}_1 \\
&+ \Pi \sin \mathsf{E} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_3 \\
&+ \frac{1}{\kappa_g}[-\sin \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1](\mathsf{M} + \cos \mathsf{E}) + \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_3]]\mathbf{e}_2 \qquad (4.2) \\
&+ \Pi \cos \mathsf{E} + \cos \mathsf{E}\sigma - \frac{\mathsf{V}\sigma + \mathsf{M}_1}{2\mathsf{V}^2} \sin^4 \mathsf{E} - \frac{\sin 2[\mathsf{M}\sigma + \mathsf{M}_1]}{4\mathsf{V}^2} \sin^4 \mathsf{E} \\
&- [\frac{\sin^2 \mathsf{E}}{\mathsf{V}} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_3][-\frac{\sin^2 \mathsf{E}}{\mathsf{V}} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2] \\
&+ \frac{\mathsf{M}_2}{\mathsf{V}} \sin^3 \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_4 \\
&+ \frac{1}{\kappa_g}[\cos \mathsf{E}\sigma - \frac{\mathsf{V}\sigma + \mathsf{M}_1}{2\mathsf{V}^2} \sin^4 \mathsf{E} - \frac{\sin 2[\mathsf{M}\sigma + \mathsf{M}_1]}{4\mathsf{V}^2} \sin^4 \mathsf{E} \\
&- [\frac{\sin^2 \mathsf{E}}{\mathsf{V}} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_3][-\frac{\sin^2 \mathsf{E}}{\mathsf{V}} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2] \\
&+ \frac{\mathsf{M}_2}{\mathsf{V}} \sin^3 \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_4]\mathbf{e}_3,
\end{aligned}$$

where $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4$ *are constants of integration and*

$$\begin{aligned}
\mathsf{M} &= (\frac{\sqrt{1+\kappa_g^2}}{\sin \mathsf{E}} - \cos \mathsf{E}), \\
\mathsf{V} &= \sqrt{1+\kappa_g^2} - \frac{1}{2}\sin 2\mathsf{E}, \qquad (4.3) \\
\Pi &= \frac{1}{\sqrt{\left(1-\kappa_g\right)+1+\kappa_g^2}}.
\end{aligned}$$

Proof. From definition of S-helix, we obviously obtain

$$\mathbf{t} = \sin \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1]\mathbf{e}_1 + \sin \mathsf{E} \cos[\mathsf{M}\sigma + \mathsf{M}_1]\mathbf{e}_2 + \cos \mathsf{E}\mathbf{e}_3. \quad (4.4)$$

We can easily verify that

$$\nabla_{\mathbf{t}}\mathbf{t} = (t_1' + t_2 t_3)\mathbf{e}_1 + (t_2' - t_1 t_3)\mathbf{e}_2 + t_3'\mathbf{e}_3. \quad (4.5)$$

Since, we immediately arrive at

$$\nabla_{\mathbf{t}}\mathbf{t} = \sin \mathsf{E} \cos[\mathsf{M}\sigma + \mathsf{M}_1](\mathsf{M} + \cos \mathsf{E})\mathbf{e}_1$$
$$- \sin \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1](\mathsf{M} + \cos \mathsf{E})\mathbf{e}_2.$$

Obviously, we also obtain

$$\mathbf{s}(\sigma) = \frac{1}{\kappa_g}[\sin \mathsf{E} \cos[\mathsf{M}\sigma + \mathsf{M}_1](\mathsf{M} + \cos \mathsf{E}) - \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2]\mathbf{e}_1$$

$$+ \frac{1}{\kappa_g}[- \sin \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1](\mathsf{M} + \cos \mathsf{E}) + \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_3]\mathbf{e}_2$$
$$+ \frac{1}{\kappa_g}[\cos \mathsf{E}\sigma - \frac{\mathsf{V}\sigma + \mathsf{M}_1}{2\mathsf{V}^2} \sin^4 \mathsf{E} - \frac{\sin 2[\mathsf{M}\sigma + \mathsf{M}_1]}{4\mathsf{V}^2} \sin^4 \mathsf{E} \quad (4.6)$$
$$- [\frac{\sin^2 \mathsf{E}}{\mathsf{V}} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_3][- \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2]$$
$$+ \frac{\mathsf{M}_2}{\mathsf{V}} \sin^3 \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_4]\mathbf{e}_3,$$

where

$$\mathsf{M} = (\frac{\sqrt{1+\kappa_g^2}}{\sin \mathsf{E}} - \cos \mathsf{E}) \text{and} \mathsf{V} = \sqrt{1+\kappa_g^2} - \frac{1}{2}\sin 2\mathsf{E}.$$

Substituting (4.4) and (4.6) in (4.1) we have (4.3), which completes the proof.

Corollary 4.3. *Let* $\alpha : I \to \mathsf{S}^2_{Heis^3}$ *be a unit speed non-geodesic biharmonic* S *-curve* $\gamma_{\mathbf{ts\alpha}}$ *its Smarandache* **tsα** *curve. Then, the parametric equations of of Smarandache* **tsα** *curve are*

$$x_{\mathbf{ts\alpha}}(\sigma) = \Pi \sin \mathsf{E} \sin[\mathsf{M}\sigma + \mathsf{M}_1] - \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2$$
$$+ \frac{1}{\kappa_g}[\sin \mathsf{E} \cos[\mathsf{M}\sigma + \mathsf{M}_1](\mathsf{M} + \cos \mathsf{E}) - \frac{\sin^2 \mathsf{E}}{\mathsf{V}} \cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2]],$$

$$y_{\mathbf{ts\alpha}}(\sigma)=\Pi\sin\mathsf{E}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3 \tag{4.7}$$
$$+\frac{1}{\kappa_g}[-\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})+\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]],$$

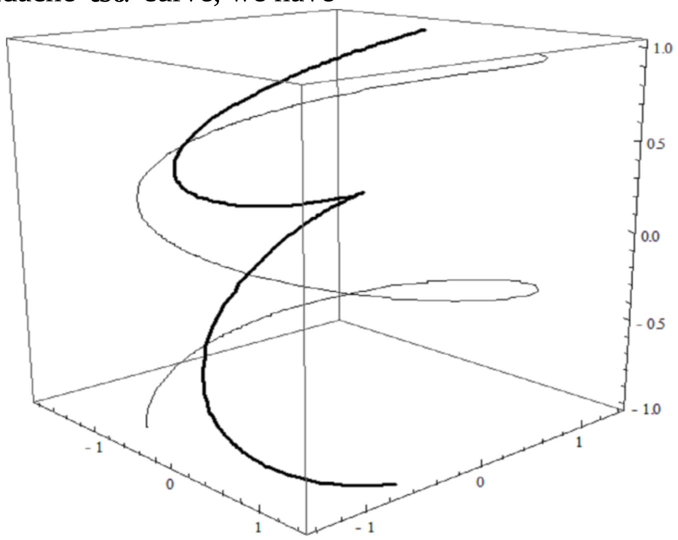
$$z_{\mathbf{ts\alpha}}(\sigma)=\Pi\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2$$
$$+\frac{1}{\kappa_g}[\sin\mathsf{E}\cos[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]$$
$$+\mathsf{M}_2]]\Pi\sin\mathsf{E}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]$$
$$+\mathsf{M}_3+\frac{1}{\kappa_g}[-\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})+\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]$$
$$+\mathsf{M}_3]]+\Pi\cos\mathsf{E}+\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}$$
$$\sin^4\mathsf{E}-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$
$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4$$
$$+\frac{1}{\kappa_g}[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}$$
$$-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$
$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4],$$

where $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3,\mathsf{M}_4$ *are constants of integration and*

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E}),$$
$$\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E},$$
$$\Pi=\frac{1}{\sqrt{(1-\kappa_g)+1+\kappa_g^2}}.$$

Proof. Substituting (2.1) to (4.2), we have (4.7) as desired.

If we use Mathematica both unit speed non-geodesic biharmonic $\mathbb{S}$-curve and its Smarandache $\mathbf{ts\alpha}$ curve, we have



5 Open Problem

In this work, we study Smarandache $\mathbf{ts\alpha}$ curves according to Sabban frame in the Heisenberg group $Heis^3$. We have given some explicit characterizations of this curves. Additionally, problems such as; investigation timelike biharmonic curves or extending such kind curves to higher dimensional Heisenberg group can be presented as further researches.

References

[1] V. Asil, T. Körpinar and S. Bas : *Construction of dual focal curves in* $\mathbb{D}^3$, Advanced Modeling and Optimization, 14 (2) (2012), 293-296.

[2] R. Caddeo and S. Montaldo: *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. 12(8) (2001), 867--876.

[3] B. Y. Chen: *Some open problems and conjectures on submanifolds of finite type,* Soochow J. Math. 17 (1991), 169--188.

[4] I. Dimitric: *Submanifolds of* $\mathbb{E}^m$ *with harmonic mean curvature vector,* Bull. Inst. Math. Acad. Sinica 20 (1992), 53--65.

[5] J. Eells and L. Lemaire: *A report on harmonic maps,* Bull. London Math. Soc. 10 (1978), 1--68.

[6] J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109--160.

[7] S. Izumiya and N. Takeuchi: *Special Curves and Ruled Surfaces,* Contributions to Algebra and Geometry 44 (2003), 203-212.

[8] G. Y. Jiang: *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A 7(4) (1986), 389--402.

[9] T. Körpınar and E. Turhan: *Biharmonic S-Curves According to Sabban Frame in Heisenberg Group* $Heis^3$, Bol. Soc. Paran. Mat. 31 (1) (2013), 205--211.

[10] E. Loubeau and S. Montaldo: *Biminimal immersions in space forms,* preprint, 2004, math.DG/0405320 v1.

[11] B. O'Neill: *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[12] I. Sato: *On a structure similar to the almost contact structure,* Tensor, (N.S.), 30 (1976), 219-224.

[13] T. Takahashi: Sasakian ϕ-symmetric spaces, Tohoku Math. J., 29 (1977), 91-113.

[14] M. Turgut, S. Yilmaz: *Smarandache Curves in Minkowski Space-time,* International Journal of Mathematical Combinatorics 3 (2008), 51-55.

[15] E. Turhan and T. Körpınar: *Timelike Biharmonic Curves Of AW(k)-Type In The Lorentzian Heisenberg Group* $Heis^3$, Int. J. Open Problems Compt. Math. 4 (1) (2011), 183-190.

[16] E. Turhan and T. Körpınar: *Position vectors of spacelike biharmonic curves with spacelike bihormal in Lorentzian Heisenberg group* $Heis^3$, Int. J. Open Problems Compt. Math. 3 (3) (2010), 413-422.

[17] J.D. Watson, F.H. Crick: *Molecular structures of nucleic acids,* Nature 171 (1953) 737 738.