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Range Kernel orthogonality of generalized derivations

A. Bachir

Department of Mathematics, Faculty of Science,King Khalid University, Abha,
P.O.Box 9004 Kingdom Saudi Arabia
e-mail:bachir_ahmed@hotmail.com

Abstract

We say that the operators A, B on Hilbert space satisfy the Fuglede-Putnam theorem if $AX = XB$ for some X implies $A^*X = XB^*$. We show that if A is $k-$quasihyponormal and B^* is an injective $p-$hyponormal operator, then A, B satisfy the Fuglede-Putnam theorem. As a consequence of this result, we obtain the range of the generalized derivation induced by the above classes of operators is orthogonal to its kernel.

Keywords: *Hyponormal Operators, Commutant, Derivation, Fuglede-Putnam Property.*

1 Introduction

Our aim is to extend the Fuglede-Putnam theorem [6]. Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces and $B(\mathcal{H}), B(\mathcal{K})$ the algebras of all bounded linear operators on $\mathcal{H}, \mathcal{K}$ respectively. The familiar Fuglede-Putnam Theorem is as follows:

Theorem 1.1 *if $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ be normal operators and $AX = XB$ for some $X \in B(\mathcal{H}, \mathcal{K})$, then $A^*X = XB^*$*

An operator T is called p-hyponormal [1, 4, 5, 8, 9, 11] if $(T^*T)^p \geq (TT^*)^p$. Throughout this paper, we consider the case $p \in (0, 1]$. This class is denoted $p - H$. A 1-hyponormal operator is called a hyponormal operator, which has been studied by many authors and it known that hyponormal operators have many interesting properties similar to those of normal operators (see [13]). By definition, the restriction of p-hyponormal operator to its invariant subspace is always p-hyponormal. Also T is called $k-$quasihyponormal operator if $T^{*k}(T^*T - TT^*)T^k \geq 0$ for a certain positive integer k, This class is denoted $Q(k)$. If $k = 1$, T is said quasinormal. The class $Q(k)$ contain strictly the class of hyponormal operators.

The organization of the paper is follows, in section 2, we recall some results will be used in the sequel. In section 3, we study the orthogonality of certain operators.

Let $A, B \in B(\mathcal{H})$, we define the generalized derivation $\delta_{A,B}$ induced by A and B by

$$\delta_{A,B}(X) = AX - XB, \text{ for all } X \in B(\mathcal{H}).$$

If $A = B$, we note $\delta_{A,B} = \delta_A$. Given subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Banach space $\mathcal{V}$ with norm $\| \, . \, \|$, $\mathcal{M}$ is said to be orthogonal to $\mathcal{N}$ if $\| \, m + n \, \| \geq \| \, n \, \|$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$(see [2]).

J.H.Anderson and Foias [3] proved that if A, B are normal operators, S is an operator such that $AS = SB$, then

$$\| \, \delta_{A,B}(X) - S \, \| \geq \| \, S \, \|, \text{for all } X \in B(\mathcal{H}).$$

Where $\| \, . \, \|$ is the usual operator norm. Hence the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. The orthogonality here is understood to be in the sense of the definition [2].

2 Preliminaries

The proof of the previous theorems proceeds through a number of steps, stated below as lemmas.

Definition 2.1 *Given $A, B \in B(\mathcal{H})$. We say that the pair (A, B) has $(FP)_{B(\mathcal{H})}$ the Fuglede-Putnam property if $AC = CB$ for some $C \in B(\mathcal{H})$, implies $A^*C = CB^*$.*

Lemma 2.2 *([1]) If $T \in p - H$ and $T = U \mid T \mid$ the polar decomposition of T, then $\mid T \mid^{1/2} U \mid T \mid^{1/2}$ is hyponormal for $1/2 \leq p \leq 1$.*

Lemma 2.3 *([12]) If $T \in p - H$ and M be an invariant subspace of T for which $T \mid_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces T.*

3 Main results

In this section, we prove that the Fuglede-Putnam's theorem holds when A is $k-$quasihyponormal and $B^* \in B(\mathcal{K})$ is an injective $p-$hyponormal operator $(p \in (0, 1])$. Before proving this result, we need the following lemmas which be used in the sequel.

Lemma 3.1 *Let $A \in B(\mathcal{H})$ be $k-$quasihyponormal and $\mathcal{M} \subset \mathcal{H}$ invariant subspace under A. If $A|_{\mathcal{M}}$ is an injective normal operator, then $\mathcal{M}$ reduces A.*

proof. Decompose A as

$$A = \begin{pmatrix} S & T \\ 0 & U \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$$

and $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Since $A|_{\mathcal{M}}$ is an injective normal operator, we have $\ker S = \ker S^* = \{0\}$. Hence $\mathcal{M} = [\operatorname{ran} S] \subset [\operatorname{ran} A]$. Then

$$0 \leq E(A^*A - AA^*)E = A|_{\mathcal{M}} = \begin{pmatrix} S^*S - SS^* - TT^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $T = 0$. $\square$

Lemma 3.2 *Let $A \in B(\mathcal{H})$ be $k-$quasihyponormal and B^* injective $p-$ hyponormal operator. If $AC = CB$ for some quasiaffinity $C \in B(\mathcal{H}, \mathcal{K})$, i.e., C is injective and has a dense range, then A, B are unitarily equivalent normal operators.*

Proof.(Case $\frac{1}{2} \leq p \leq 1$)

Let $B^* = U^* \mid B^* \mid$ be the polar decomposition of B^* and $\widetilde{B^*} =\mid B^* \mid^{\frac{1}{2}} U^* \mid B^* \mid^{\frac{1}{2}}$. Then B^* is an injective hyponormal operator by lemma 2.2 and $\mid B^* \mid^{\frac{1}{2}}$ is a quasiaffinity. Then $AC = CB$ implies

$$AC \mid B^* \mid^{\frac{1}{2}} = CB \mid B^* \mid^{\frac{1}{2}} = C \mid B^* \mid^{\frac{1}{2}} D,$$

where $D =\mid B^* \mid^{\frac{1}{2}} U \mid B^* \mid^{\frac{1}{2}}$ and $D^* = \widetilde{B^*}$ is hyponormal. Since $C \mid B^* \mid^{\frac{1}{2}}$ is a quasiaffinity, A, D are unitarily equivalent normal operators by [4]. Then $D^* =\mid B^* \mid^{\frac{1}{2}}$ is normal and $B^* = \widetilde{B^*}$ by [8]. Thus $A, D = B$ are unitarily equivalent normal operators.

(Case $0 < p \leq \frac{1}{2}$). Put $p' = p + \frac{1}{2}$, where $p' \in (\frac{1}{2}, 1]$. By using Aluthge transform, the proof follows from (Case $\frac{1}{2} \leq p \leq 1$). $\square$

Theorem 3.3 *Let $A \in B(\mathcal{H})$ be $k-$quasihyponormal and B^* injective $p-$ hyponormal operator. If $AC = CB$ for some $C \in B(\mathcal{H}, \mathcal{K})$, then $A^*C = CB^*$, $[\operatorname{ran} C]$ reduces A, $(\ker C)^{\perp}$ reduces B, and $A|_{[\operatorname{ran} C]}$, $B|_{(\ker C)^{\perp}}$ are unitarily equivalent normal operators.*

Proof. Decompose A, B, C as

$$\begin{aligned} A &= \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \text{ on } \mathcal{H} = [\operatorname{ran} C] \oplus \ker C^* \\ B &= \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix} \text{ on } \mathcal{K} = (\ker C)^\perp \oplus \ker C \\ C &= \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} : (\ker C)^\perp \oplus \ker C \to [\operatorname{ran} C] \oplus \ker C^*. \end{aligned}$$

Then $AC = CB$ implies $A_1C_1 = C_1B_1$ where A_1 is $k-$ quasihyponormal, B_1^* is an injective $p-$hyponormal operator and C_1 is a quasiaffinity. Then A_1, B_1 are unitarily equivalent normal operators by lemma 3.2 and $A_1^*C_1 = C_1B_1^*$. Since $B_1^* = B^*|_{(\ker C)^\perp}$ is injective, $A_1 = A|_{[\operatorname{ran} C]}$ is an injective normal operator. Hence $[\operatorname{ran} C]$ reduces A by lemma 2.2, and $(\ker C)^\perp$ reduces B^* by [12]. The rest follows from [9]. □

Remark 3.4 *The assumption B^* is injective in Theorem 3.3 cannot be relaxed to insure the Fuglede-Putnam property. For, if we take* $\dim \mathcal{H} = 3$ *and*

$$A = C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \text{ Then}$$

$$A^{*2}(A^*A - AA^*)A^2 = C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0 \cdot$$

Hence A is 2-hyponormal and B^* is not injective. On the otherhand, B^* is normal and

$$AC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = CB$$

but

$$A^*C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = CB^*$$

Theorem 3.5 *If $A \in B(\mathcal{H})$ is $k-$quasihyponormal and B^* is an injective $p-$ hyponormal operator in $B(\mathcal{H})$, then* $\operatorname{ran} \delta_{A,B}$ *is orthogonal to* $\ker \delta_{A,B}$.

Proof. The pair (A, B) has the $(FP)_{B(\mathcal{H})}$ property by Theorem 3.3. Let $C \in B(\mathcal{H})$ be such $AC = CB$. According to the following decompositions of $\mathcal{H}$.

$$\mathcal{H} = \mathcal{K} = \overline{\operatorname{ran} C} \oplus (\overline{\operatorname{ran} C})^{\perp},\ \ \mathcal{H} = \mathcal{L} = (\ker C)^{\perp} \oplus \ker C.$$

We can write A, B, C and X

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

where A_1 and B_1 are normal operators and X is an operator on $\mathcal{K}$ to $\mathcal{L}$. Since $AC = CA$, we obtain $A_1C_1 = C_1A_1$. Hence

$$AX - XA - C = \begin{bmatrix} A_1X_1 - X_1B_1 - C_1 & A_2X_2 - X_2B_2 \\ A_1X_3 - X_3B_1 & A_2X_4 - X_4B_2 \end{bmatrix}.$$

Since $C_1 \in \ker \delta_{A_1,B_1}$, A_1 and B_1 are normal, it follows by [3]

$$\|AX - XB - C\| \geq \|A_1X_1 - X_1B_1 - C_1\| \geq \|C_1\| = \|C\| \quad ,\forall X \in B(\mathcal{H})$$

This implies that $\operatorname{ran} \delta_{A,B}$ is orthogonal to $\ker \delta_{A,B}$. □

4 Open Problem

The open problem here is to find classes of nonnormal of operators satisfying the Fuglede-Putnam Property and consequently we obtain the range kernel orthogonality results.

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