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The Total Number of Squares and Rectangles are there in Rectangle (Square) Boards

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Abstract

In the present paper we establish the general formulas for the total numbers of squares and rectangles are there in $n \times m$ ***and*** $n \times n$ ***boards. Some examples are also given to illustrate our new approaches.***

Keywords*:* A grid, Series, Combination, A polyomino.

1. Introduction

A grid or a board is a collection of such squares (rectangles) which are formed vertical and horizontal sides and describe details of pictures, letters and much more. The grids are very important in many applications such as in computer, engineering, mathematics, networks, communication systems, and many others various fields [1,2,3]. For example: In computer imaging, we have a grid of pixels and each pixel represents different colors together make the picture. Pixels are generally rectangles (squares). To save memory, we can combine rectangles (squares) together in to large rectangles (squares), this gives a picture. We need the number of rectangles (squares) of pixel to control of picture until to get a clear picture. Another example is that picture adding up squares as building a pyramid out of blocks, each block 1 unit on aside. The tip of the pyramid has 1 block, the next level down has 4 blocks (2×2 square), the next level has 9 blocks (3×3 square), and so on. Then the total number of blocks on a pyramid of height (n) is the same as the total numbers squares are there in $n\times n$ grid. Since its sometimes difficult to count the number of squares and rectangles are there in a square (a rectangle) board, its of great important to establish the general formulas for the number of squares and rectangles are there in $n\times n$ a square board and are there in $n\times m$ a rectangle board.

2. Number of Squares are there in $n\times n$ a Square Board

Consider the left hand vertical edge of a square of size $k\times k$. This edge can be in any on of (n) positions. Similarly, the top edge can occupy any one of (n) position for $k\times k$ square. So the total number of $k\times k$ squares are equal to $n\times n = n^2$.

For 2×2 a square, the left hand edge can be occupy $(n-1)$ positions and the top edge $(n-1)$ positions giving $(n-1)\times(n-1)=(n-1)^2$ squares of size 2×2. For 3×3 a square, the left hand edge can be occupy $(n-2)$ positions giving $(n-2)\times(n-2)=(n-2)^2$ squares of size 3×3. Continuing in this way, we get squares of size 4×4, 5×5, 6×6, ... , $n\times n$. We can summarize the result as follows [4]:

Size of Squares	Number of Squares
1×1	n^2
2×2	$(n-1)^2$
3×3	$(n-2)^2$
$\vdots$	$\vdots$
$(n-1)\times(n-1)$	2^2
$n\times n$	1^2

Note that the total number of squares are there in $n\times n$ a square board is derived from the sum of squares of the integers from 1 *1* to n.This notation conclude to the following result.

Theorem 1. Let $S(n)$ be the number of squares in a square board of size $n\times n$. Then the general formula for the number of squares are there in $n\times n$ a square board is given by:

$$S(n)=\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2.1}$$

Proof : The proof is by mathematical induction on n. If we have a square of size 1×1 (i.e., $n=1$) , then $S(1)=\frac{1(2)(3)}{6}=1$. Suppose that the number of squares are there in the first $(n-1)\times(n-1)$ a square board is: $S(n-1)=\sum_{r=1}^{n-1} r^2 = \frac{(n-1)(n)(2(n-1)+1)}{6}$, under this assumption, we must to show the result as in eq. (2.1). To prove this, let us add another square, so that the number of squares are there in the first $n\times n$ a square board is:

$$S(n)=\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\cdots+(n-1)^2+n^2$$

$$=\left(\sum_{r=1}^{n-1} r^2\right)+n^2 = \frac{(n-1)(n)(2(n-1)+1)}{6}+n^2$$

$$=\frac{(n-1)(n)(2(n-1)+1)+6n^2}{6}=\frac{n\left[(n-1)(2n-1)+6n\right]}{6}$$

$$= \frac{n[(2n^2 - 3n + 1 + 6n]}{6} = \frac{n[(2n^2 + 3n + 1]}{6} = \frac{n(n+1)(2n+1)}{6} .$$

So that the proof of Theorem 1 is completed.

Corollary 1. The number of squares are there in $n \times n$ a square board can be written as follows:

$$S(n) = 2\binom{n+1}{3} + \binom{n+1}{2} . \qquad (2.2)$$

Proof : By using the formula in eq. (2.1),

$$S(n) = \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(n+2+n-1)}{3\,!}$$

$$= \frac{n(n+1)(n+2)}{3\,!} + \frac{n(n+1)(n-1)}{3\,!} = \binom{n+2}{3} + \binom{n+1}{3}$$

$$= \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!} = 2\binom{n+1}{3} + \binom{n+1}{2}.$$

Problem 1. How many squares are there in the ordinary chess board [5]?

Solution: The ordinary chess board has an 8×8 a square board. Then the number of squares are there in the chess board is:

$$S(8) = \sum_{r=1}^{8} r^2 = \frac{8(8+1)(2(8)+1)}{6} = 2\binom{9}{3} + \binom{9}{2} = 204 .$$

3. Number of Rectangles are there in $n \times n$ a Square Board

Theorem 2. Let $Q(n)$ be the number of rectangles are there in a square board of size $n \times n$. Then the general formula for the number of rectangles (**including squares**) are there in $n \times n$ a square board is:

$$Q(n) = \binom{n+1}{2}^2 = \frac{n^2 (n+1)^2}{4} . \qquad (3.1)$$

Proof: There are $(n+1)$ vertical lines and $(n+1)$ horizontal lines .To form rectangle , we must choose 2 of the $(n+1)$ vertical lines and choose

2 of the $(n+1)$ horizontal lines. Each of these can be done in $\binom{n+1}{2}$ ways. So the number of rectangles are there in $n\times n$ a square board is:

$$Q(n)=\binom{n+1}{2}\binom{n+1}{2}=\binom{n+1}{2}^2=\left(\frac{(n+1)!}{2!(n-1)!}\right)^2=\left(\frac{(n+1)(n)}{2}\right)^2$$

$$=\frac{n^2(n+1)^2}{4}.$$

Corollary 2. The number of rectangles (**not including squares**) are there in $n\times n$ a square board is:

$$B(n)=\frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{6}=(n^2-1)\left(\frac{1}{4}n^2+\frac{1}{6}n\right). \qquad (3.2)$$

Problem 2. How many rectangles are there in the ordinary chess board?

Solution: The ordinary chess board is an 8×8 a square board. Then

(i) The number of rectangles (**including squares**) are there in the chessboard is:

$$Q(8)=\binom{9}{2}^2=\frac{9^2(10)^2}{4}=1296.$$

(ii) The number of rectangles (**not including squares**) are there in the chessboard is:

$$B(8)=1296-204=1092.$$

4. Number of Squares are there in $n\times m$ a Rectangle Board

Theorem 3. : Let $A(n,m)$ be $n\times m$ rectangle board with height n and width m and $S(n,m)$ be the number of squares in $A(n,m)$ where $m\geq n$. Then the general formulas for the number of squares are there in $n\times m$ a rectangle board is:

(i) $$S(n,m)=2\binom{n+1}{3}+(m-n+1)\binom{n+1}{2}, \qquad (4.1)$$

or

(ii) $S(n,m) = S(n,n+t) = \dfrac{n(n+1)(2n+3t+1)}{6}$, where $t = m - n$. (4.2)

Proof : (i) Straightforward by using Corollary 1.

(ii) Since $m \geq n$, then m can be written as $m = n + t$ and:

$$S(n,m) = \sum_{r=1}^{n} r(r+t) = \sum_{r=1}^{n}\left[r^2 + rt\right] = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} rt$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{tn(n+1)}{2} = \frac{n(n+1)(2n+3t+1)}{6}.$$

Remark 1. If $m = n$, then Theorem 3 is reduced to the Theorem 1 .

Problem 3. How many squares are in there in 8×9 a rectangle board ?

Solution: (i) By using the formula (4.1) in Theorem 3, then we have:

$$S(8,9) = 2\binom{9}{3} + 2\binom{9}{2} = 240$$

(ii) By using the formula (4.2) in Theorem 3, then we have:

$$S(8,9) = S(8,8+1) = \frac{8(8+1)(2\times 8 + 3\times 1 + 1)}{6} = 240.$$

5. Number of Rectangles are there in $n \times m$ a Rectangle Board

Theorem 4. Let $A(n,m)$ be $n \times m$ rectangle board with height n and width m and $Q(n,m)$ be the number of rectangles in $A(n,m)$ where $m \geq n$. Then the general formula for the number of rectangles (**including squares**) are there in $n \times m$ a rectangle board is :

$$Q(n,m) = \binom{n+1}{2}\binom{m+1}{2} = \frac{nm(n+1)(m+1)}{4}. \qquad (5.1)$$

Proof: There are $(n+1)$ vertical lines and $(m+1)$ horizontal lines . To form rectangle, we must choose 2 of the $(n+1)$ vertical lines and choose 2 of the $(m+1)$ horizontal lines. So the number of rectangles are in $n \times m$ rectangle board is as giving in eq (5.1).

Corollary 3. The number of rectangles (**not including squares**) are there in $n \times m$ a rectangle board is:

$$B(n,m) = Q(n,m) - S(n,m) = \binom{n+1}{2}\binom{m+1}{2} - 2\binom{n+1}{3} - (m-n+1)\binom{n+1}{2}.$$

Remark 2. If $m = n$, then Theorem 4 is reduced to the Theorem 2 .

Problem 4. How many rectangles are there in there in 8×9 a rectangle board ?

Solution: (i) The number of rectangles (**including squares**) are there in 8×9 a rectangle board is:

$$Q(8,9) = \binom{9}{2}\binom{10}{2} = 1620.$$

(ii)The number of rectangles (**not including squares**) are there in 8×9 a rectangle board is:

$$B(8,9) = Q(8,9) - S(8,9) = 1620 - 240 = 1380.$$

5. Open Problems

A polyomino is a connected interior-disjoint union of axis-aligned unit squares joined edge to edge , in other words, an edge-connected union of cells in the planar square lattice. The order of a polyomino is the number of a unit squares forming it. How many polyominoes on n squares are there? Do there exist rectangles that may be partitioned into a finite number n of rectangular pieces of equal area but all perimeters different?

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References

[1] S. Odell., "*Puzzies for Super Brains*", Sudha Publication, New Delhi, 1989.

[2] K. H. Rosen,"*Elementary Number Theory and Applications*", Addison Wesley, Reading, 1988.

[3] J. Daws, "*Squares in a Square Lattice*", Mathematical Gazette, 1972.

[4] F.M. Kamal Mohamad, Squares and Rectangles on A square, University Tecnology Mara, Malaysia, *Meneui Mat* . 23, 2001.

[5] Anthony & T. Henry, "*Squares, Rectangles on a Chessboard*", 1997.