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Construction of Focal Curves of Spacelike Biharmonic Curves with Timelike Binormal in the Lorentzian Heisenberg Group Heis^3

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Abstract

In this paper, we study focal curve of spacelike biharmonic curve with a timelike binormal in the Heis^3***. We characterize focal curve of spacelike biharmonic curve with a timelike binormal in terms of curvature and torsion of biharmonic curve in the*** Heis^3***. Finally, we construct parametric equations of focal curve of spacelike biharmonic curve with a timelike binormal curve.***

Keywords: *Heisenberg group, biharmonic curve, focal curve.*

1 Introduction

Lorentzian geometry helps to bridge the gap between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. The fact that relativity theory is expressed in terms of Lorentzian geometry is attractive for geometers, who can penetrate surprising quickly into cosmology (redshift, expanding universe and big bang) and a topic no less interesting geometrically, the gravitation of a single star (perihelion procession, bending of light and black holes); see [15].

For any unit speed curve γ, the focal curve C_γ is defined as the centers of the osculating spheres of γ. Since the center of any sphere tangent to at a point lies on the normal plane to γ at that point, the focal curve of γ may be parameterized using the Frenet frame $(\mathbf{t}(s),\mathbf{n}(s),\mathbf{b}(s))$ of γ as follows:

$$C_\gamma(s) = (\gamma + c_1\mathbf{n} + c_2\mathbf{b})(s),$$

where the coefficients c_1, c_2 are smooth functions that are called focal curvatures of γ.

The aim of this paper is to study focal curve of spacelike biharmonic curve with a timelike binormal in the Lorentzian Heisenberg group Heis^3.

Harmonic maps $f:(M,g)\to(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler--Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\mathrm{trace}\nabla df. \tag{1.2}$$

The bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

The first and the second variation formula for the bienergy, showing that the Euler--Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f)&=-\mathrm{J}^f(\tau(f))=-\Delta\tau(f)-\mathrm{trace}R^N(df,\tau(f))df \\ &=0,\end{aligned} \tag{1.4}$$

where J^f is the Jacobi operator of f. The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study focal curve of spacelike biharmonic curve with a timelike binormal in the Heisenberg group Heis^3. We characterize focal curve of spacelike biharmonic curve with a timelike binormal in terms of curvature and torsion of biharmonic curve in the Heisenberg group Heis^3. Finally, we construct parametric equations of focal curve of spacelike biharmonic curve with a timelike binormal curve.

2 Preliminaries

The Lorentzian Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\overline{x}y+x\overline{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x}, \tag{2.1}$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = 1,\ g(\mathbf{e}_3,\mathbf{e}_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric* g *, defined above, the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i,j) *-element in the table above equals* $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = -g(R(X,Y)Z,W).$$

Moreover we put

$$R_{abc} = R(\mathbf{e}_a,\mathbf{e}_b)\mathbf{e}_c, R_{abcd} = R(\mathbf{e}_a,\mathbf{e}_b,\mathbf{e}_c,\mathbf{e}_d),$$

where the indices a,b,c and d take the values $1,2$ and 3.

$$R_{232} = -3R_{131} = -3\mathbf{e}_3,$$
$$R_{133} = -R_{122} = \mathbf{e}_1,$$
$$R_{233} = -3R_{121} = -3\mathbf{e}_2,$$

and

$$R_{1212} = -1,\ R_{1313} = 1,\ R_{2323} = -3. \tag{2.3}$$

3 Spacelike Biharmonic Curves with Timelike Binormal In The Lorentzian Heisenberg Group Heis^3

Let $\gamma : I \to Heis^3$ be a spacelike biharmonic curve with a timelike binormal on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ), and $\mathbf{b}$ is chosen so that $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} &\nabla_{\mathbf{t}}\mathbf{t} = \kappa\mathbf{n}, \\ &\nabla_{\mathbf{t}}\mathbf{n} = -\kappa\mathbf{t} + \tau\mathbf{b}, \\ &\nabla_{\mathbf{t}}\mathbf{b} = \tau\mathbf{n}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion

$$\begin{aligned} &g(\mathbf{t},\mathbf{t}) = g(\mathbf{n},\mathbf{n}) = 1, g(\mathbf{b},\mathbf{b}) = -1, \\ &g(\mathbf{t},\mathbf{n}) = g(\mathbf{n},\mathbf{b}) = (\mathbf{t},\mathbf{b}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} &\mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3, \\ &\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3, \\ &\mathbf{b} = \mathbf{t}\times\mathbf{n} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3. \end{aligned}$$

Theorem 3.1. *(see [18]) Let $\gamma : I \to Heis^3$ be a non-geodesic spacelike curve with a timelike binormal on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is a timelike non-geodesic biharmonic curve if and only if*

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^2 - \tau^2 = 1 + 4(b_1)^2, \\ &\tau' = -2n_1b_1. \end{aligned} \tag{3.2}$$

Corollary 3.2. *(see [18]) Let $\gamma : I \to Heis^3$ be a non-geodesic spacelike curve with a timelike binormal on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is biharmonic if and only if*

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\tau = \text{constant}, \\ &n_1b_1 = 0, \\ &\kappa^2 - \tau^2 = 1 + 4(b_1)^2. \end{aligned} \tag{3.3}$$

Theorem 3.3. *(see [18]) Let $\gamma : I \to Heis^3$ be a non-geodesic spacelike biharmonic curve with a timelike binormal on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Then*

$$\mathbf{t}(s) = \cosh \Pi \mathbf{e}_1 + \sinh \Pi \sinh \psi(s) \mathbf{e}_2 + \sinh \Pi \cosh \psi(s) \mathbf{e}_3, \tag{3.4}$$

where $\Pi \in \mathbb{R}$.

4 Focal Curve of Spacelike Biharmonic Curve in the Lorentzian Heisenberg Group Heis^3

For a unit speed curve γ, the curve consisting of the centers of the osculating spheres of γ is called the parametrized focal curve of γ. The hyperplanes normal to γ at a point consist of the set of centers of all spheres tangent to γ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_γ, we can write

$$C_\gamma(s) = (\gamma + c_1 \mathbf{n} + c_2 \mathbf{b})(s), \tag{4.1}$$

where the coefficients c_1, c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa}, c_2 = -\frac{c_1'}{\tau}, \kappa \neq 0, \tau \neq 0. \tag{4.2}$$

Lemma 4.1. *Let $\gamma : I \to Heis^3$ be a unit speed spacelike biharmonic curve and C_γ its focal curve on $Heis^3$. Then,*

$$c_1 = \frac{1}{\kappa} = \text{constant and } c_2 = 0. \tag{4.3}$$

Proof. Using (3.3) and (4.2), we get (4.3).

Lemma 4.2. *Let $\gamma : I \to Heis^3$ be a unit speed spacelike biharmonic curve and C_γ its focal curve on $Heis^3$. Then,*

$$C_\gamma(s) = (\gamma + c_1 \mathbf{n})(s). \tag{4.4}$$

Theorem 4.3. *Let $\gamma : I \to Heis^3$ be a unit speed spacelike biharmonic curve and C_γ its focal curve on $Heis^3$. Then, the parametric equations of C_γ are*

$$x_{C_\gamma}(s) = \frac{c_1}{\kappa}\sinh\Pi(\Omega + 2\cosh\Pi)\sinh(\Omega s + \zeta) + \frac{1}{\Omega}\sinh\Pi\sinh(\Omega s + \zeta) + a_1,$$

$$y_{C_\gamma}(s) = \frac{c_1}{\kappa}\sinh\Pi(\Omega + 2\cosh\Pi)\cosh(\Omega s + \zeta) + \frac{1}{\Omega}\sinh\Pi\cosh(\Omega s + \zeta) + a_2,$$

$$\begin{aligned} z_{C_\gamma}(s) = {} & \frac{c_1}{\kappa}\sinh\Pi(\Omega + 2\cosh\Pi)(-\frac{1}{\Omega^2}\sinh(\Omega s + \zeta)\cosh(\Omega s + \zeta) \\ & - d_1 s\cosh(\Omega s + \zeta) - d_2\cosh(\Omega s + \zeta)) + (\cosh\Pi + \frac{1}{2\Omega}\sinh^2\Pi)s \\ & - \frac{1}{4\Omega^2}\sinh^2\Pi\sinh 2(\Omega s + \zeta) - \frac{a_1}{2\Omega}\sinh\Pi\cosh(\Omega s + \zeta) + a_3, \end{aligned} \tag{4.5}$$

where $\zeta,\ a_1,\ a_2,\ a_3,\ d_1,\ d_2$ are constants of integration and $\Omega = \dfrac{\kappa - \sinh 2\Pi}{\sinh\Pi}$.

Proof. Since $|\nabla_{\mathbf{t}}\mathbf{t}| = \kappa,$ we obtain

$$\psi(s) = (\frac{\kappa - \sinh 2\Pi}{\sinh\Pi})s + \zeta, \tag{4.6}$$

where $\zeta \in \mathbb{R}$.

Thus (3.4) and (4.6), imply

$$\mathbf{t} = \cosh\Pi\mathbf{e}_1 + \sinh\Pi\sinh(\Omega s + \zeta)\mathbf{e}_2 + \sinh\Pi\cosh(\Omega s + \zeta)\mathbf{e}_3, \tag{4.7}$$

where $\Omega = \dfrac{\kappa - \sinh 2\Pi}{\sinh\Pi}$.

Using (2.1) in (4.7), we obtain

$$\begin{aligned} \mathbf{t} = {} & (\sinh\Pi\cosh(\Omega s + \zeta), \sinh\Pi\sinh(\Omega s + \zeta), \cosh\Pi \\ & - \frac{1}{\Omega}\sinh^2\Pi\sinh^2(\Omega s + \zeta) - a_1\sinh\Pi\sinh(\Omega s + \zeta)), \end{aligned} \tag{4.8}$$

where a_1 is constant of integration.

From (4.8), we have

$$\begin{aligned} \gamma(s) = {} & (\frac{1}{\Omega}\sinh\Pi\sinh(\Omega s + \zeta) + a_1, \frac{1}{\Omega}\sinh\Pi\cosh(\Omega s + \zeta) + a_2, \\ & \left[\cosh\Pi + \frac{1}{2\Omega}\sinh^2\Pi\right]s - \frac{1}{4\Omega^2}\sinh^2\Pi\sinh 2(\Omega s + \zeta) \\ & - \frac{a_1}{2\Omega}\sinh\Pi\cosh(\Omega s + \zeta) + a_3), \end{aligned} \tag{4.9}$$

where a_1, a_2, a_3 are constants of integration.

On the other hand, using Frenet formulas (3.1) and (4.7), we have

$$\mathbf{n} = \frac{1}{\kappa}[\sinh\Pi(\Omega + 2\cosh\Pi)\cosh(\Omega s + \zeta)\mathbf{e}_2 \tag{4.10}$$
$$+\sinh\Pi(\Omega + 2\cosh\Pi)\sinh(\Omega s + \zeta)\mathbf{e}_3].$$

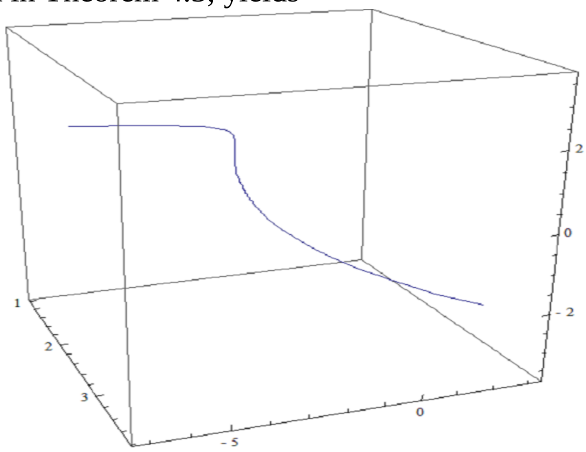
Smilarly, using (2.1) in (4.10), we obtain

$$\mathbf{n} = \frac{1}{\kappa}\sinh\Pi(\Omega + 2\cosh\Pi)(\sinh(\Omega s + \zeta), \cosh(\Omega s + \zeta),$$
$$-\frac{1}{\Omega^2}\sinh(\Omega s + \zeta)\cosh(\Omega s + \zeta) \tag{4.11}$$
$$-d_1 s\cosh(\Omega s + \zeta) - d_2\cosh(\Omega s + \zeta)),$$

where d_1, d_2 are constants of integration.

Next, we substitute (4.9) and (4.11) into (4.4), we get (4.5). The proof is completed.

Using Mathematica in Theorem 4.3, yields



5 Open Problem

This Letter, we study focal curve of spacelike biharmonic curve with a timelike binormal in the Lorentzian Heisenberg group Heis^3. We characterize focal curve of spacelike biharmonic curve with a timelike binormal in terms of curvature and torsion of biharmonic curve in the Lorentzian Heisenberg group Heis^3.

The authors can be presented this curves in the Heisenberg group Heis^3.

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