Int. J. Open Problems Compt. Math., Vol. 5, No. 3, September 2012
ISSN 1998-6262; Copyright © ICSRS Publication, 2012
www.i-csrs.org

Solutions Of Differential Equations For Dual Curvatures Of Dual Biharmonic Curves With Spacelike Principal Normal in $\mathbb{D}_1^3$

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Abstract

In this paper, we study dual spacelike biharmonic curves with spacelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. We consider differential equations of dual Bishop curvatures of dual spacelike biharmonic curves with spacelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. This equations are seperated into dual and real parts such that the dual part of the equation is the higher order differential of each term in the real part.

Keywords: *Dual space curve, dual Bishop frame, biharmonic curve.*

1 Introduction

Differential equations of order n have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Consequently, considerable attention has been given to the solutions of ordinary differential equations, integral equations and fractional partial differential equations of physical interest.

A linear differential equation of order n, in the dependent variable and the independent variable , is an equation that can be expressed in the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1} y}{dx^{n-1}}+a_2(x)\frac{d^{n-2} y}{dx^{n-2}}+\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=f(x),$$

where $a_0(x)$ is not identically zero. An equation which is not linear is called a non-linear differential equation, [11].

On the other hand, harmonic maps $f:(M,g)\rightarrow(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler--Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\text{trace}\nabla df. \tag{1.2}$$

The bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [3], showing that the Euler--Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f)&=-\mathrm{J}^f(\tau(f))=-\Delta\tau(f)-\text{trace}R^N(df,\tau(f))df \\ &=0,\end{aligned} \tag{1.4}$$

where J^f is the Jacobi operator of f. The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study dual spacelike biharmonic curves with spacelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. We consider differential equations of dual Bishop curvatures of dual spacelike biharmonic curves with spacelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. This equations are seperated into dual and real parts such that the dual part of the equation is the higher order differential of each term in the real part.

2 Preliminaries

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

A dual point on $\mathbb{D}^3$ corresponds to a line in $\mathbb{E}^3$, two different points of $\mathbb{D}^3$ represents two skew lines in $\mathbb{E}^3$. A differentiable curve on $\mathbb{D}^3$ represents a ruled surface E^3. If φ and φ^* are real numbers and $\varepsilon^2 = 0$ the combination $\hat{\varphi} = \varphi + \varphi^*$ is called a dual number. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. Later, E.Study introduced the dual angle subtended by two nonparallel lines E^3, and defined it as $\hat{\varphi} = \varphi + \varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines.

By a dual number $\hat{x}$, we mean an ordered pair of the form (x, x^*) for all $x, x^* \in \mathsf{R}$. Let the set $\mathsf{R} \times \mathsf{R}$ be denoted as D. Two inner operations and an equality on $\mathsf{D} = \{(x, x^*) \mid x, x^* \in \mathsf{R}\}$ are defined as follows:

(i) $\oplus : \mathsf{D} \times \mathsf{D} \to \mathsf{D}$ for $\hat{x} = (x, x^*),\ \hat{y} = (y, y^*)$ defined as

$$\hat{x} \oplus \hat{y} = (x, x^*) \oplus (y, y^*) = \left(x + y, x^* + y^*\right)$$

is called the addition in D.

(ii) $\otimes : \mathsf{D} \times \mathsf{D} \to \mathsf{D}$ for $\hat{x} = (x, x^*),\ \hat{y} = (y, y^*)$ defined as

$$\hat{x} \otimes \hat{y} = (x, x^*) \otimes (y, y^*) = \left(xy, xy^* + x^* y\right)$$

is called the multiplication in D.

The set D of dual numbers is a commutative ring.

(iii) If $x = y$, $x^* = y^*$ for $\hat{x} = (x, x^*),\ \ \hat{y} = (y, y^*)\ \in\ \mathsf{D}$, $\hat{x}$ and $\hat{y}$ are equal, and it is indicated as $\hat{x} = \hat{y}$.

If the operations of addition, multiplication and equality on $\mathsf{D} = \mathsf{R} \times \mathsf{R}$ with set of real numbers R are defined as above, the set D is called the dual numbers system and the element (x, x^*) of D is called a dual number. In a dual number $\hat{x} = (x, x^*) \in\ \mathsf{D}$, the real number x is called the real part of $\hat{x}$ and the real number x^* is called the dual part of $\hat{x}$. The dual number $(1,0) = 1$ is called unit element of multiplication operation in D or real unit in D. The dual number $(0,1)$ is to be denoted with ε in short, and the $(0,1) = \varepsilon$ is to be called dual unit. In accordance with the definition of the operation of multiplication, it can easily be seen that $\varepsilon^2 = 0$. Also, the dual number $\hat{x} = (x, x^*) \in\ \mathsf{D}$ can be written as $\hat{x} = x + \varepsilon x^*$.

The set

$$\mathsf{D}^3 = \left\{\hat{x} : \hat{x} = x + \varepsilon x^*, x, x^* \in \mathsf{E}^3\right\}$$

is a module over the ring D.

The Lorentzian inner product of $\hat{x}$ and $\hat{y}$ is defined by

$$\langle \hat{x}, \hat{y} \rangle = \langle x, y \rangle + \varepsilon\left(\langle x, y^* \rangle + \langle x^*, y \rangle\right).$$

We call the dual space $\mathbb{D}^3$ together with the Lorentzian inner product the dual Lorentzian space and denote it by $\mathbb{D}_1^3$. The norm $\|\hat{x}\|$ of $\hat{x}$ is defined by

$$\|\hat{x}\| = \sqrt{|\langle \hat{x}, \hat{x}\rangle|} = \|x\| + \varepsilon \frac{\langle x, x^*\rangle}{\|x\|}.$$

A dual vector $\overleftarrow{\hat{x}}$ with norm 1 is called a dual unit vector.

Let $\overleftarrow{\hat{x}} = \overleftarrow{x} + \varepsilon \overleftarrow{x^*} \in \mathbb{D}_1^3$. The set

$$\mathbb{S}_1^2 = \left\{\hat{x} = x + \varepsilon x^* \mid \|\hat{x}\| = (1,0); x, x^* \in \mathbb{R}^3\right\}$$

is called the dual unit sphere with the center $\hat{O}$ in $\mathbb{D}_1^3$.

If every $x_i(s)$ and $x_i^*(s)$, $1 \le i \le 3$, real valued functions,are differentiable, the dual space curve

$$\hat{x} : I \subset \mathbb{R} \to \mathbb{D}_1^3.$$
$$s \to \hat{x}(s) = \left(x_1(s) + \varepsilon x_1^*(s), x_2(s) + \varepsilon x_2^*(s), x_3(s) + \varepsilon x_3^*(s)\right),$$

in $\mathbb{D}_1^3$ is differentiable.

3 Spacelike Dual Biharmonic Curves with Spacelike Principal Normal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\hat{\gamma}$ dual spacelike curve with spacelike principal normal by the dual arc length parameter $\hat{s}$. Then the unit tangent vector $\hat{\gamma}' = \hat{\mathbf{t}}$ is defined, and the principal normal is $\hat{\mathbf{n}} = \dfrac{1}{\hat{\kappa}} \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{t}}$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\|\nabla_{\hat{\mathbf{t}}} \hat{\mathbf{t}}\right\| = \kappa + \varepsilon \kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. Hence, the triple $\left\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\right\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{aligned}
&\nabla_{\hat{\mathbf{t}}(\hat{s})} \hat{\mathbf{t}}(\hat{s}) = \hat{\kappa}(\hat{s}) \hat{\mathbf{n}}(\hat{s}), \\
&\nabla_{\hat{\mathbf{t}}(\hat{s})} \hat{\mathbf{n}}(\hat{s}) = -\hat{\kappa}(\hat{s}) \hat{\mathbf{t}}(\hat{s}) + \hat{\tau}(\hat{s}) \hat{\mathbf{b}}(\hat{s}), \\
&\nabla_{\hat{\mathbf{t}}(\hat{s})} \hat{\mathbf{b}}(\hat{s}) = \hat{\tau}(\hat{s}) \hat{\mathbf{n}}(\hat{s}),
\end{aligned} \tag{3.1}$$

where $\hat{\tau}(\hat{s})$ is the dual torsion of the timelike dual curve $\hat{\gamma}(\hat{s})$. Here, we suppose that the dual torsion $\hat{\tau}(\hat{s})$ is never pure-dual. In addition,

$$g\left(\hat{\mathbf{t}}(\hat{s}),\hat{\mathbf{t}}(\hat{s})\right)=1,\, g\left(\hat{\mathbf{n}}(\hat{s}),\hat{\mathbf{n}}(\hat{s})\right)=1,\, g\left(\hat{\mathbf{b}}(\hat{s}),\hat{\mathbf{b}}(\hat{s})\right)=-1, \qquad (3.2)$$
$$g\left(\hat{\mathbf{t}}(\hat{s}),\hat{\mathbf{n}}(\hat{s})\right)=g\left(\hat{\mathbf{t}}(\hat{s}),\hat{\mathbf{b}}(\hat{s})\right)=g\left(\hat{\mathbf{n}}(\hat{s}),\hat{\mathbf{b}}(\hat{s})\right)=0.$$

In the rest of the paper, we suppose everywhere $\hat{\kappa}(\hat{s})\neq 0$ and $\hat{\tau}(\hat{s})\neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{t}}(\hat{s}) &= \hat{k}_1(\hat{s})\hat{\mathbf{m}}_1(\hat{s})-\hat{k}_2(\hat{s})\hat{\mathbf{m}}_2(\hat{s}),\\
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{m}}_1(\hat{s}) &= -\hat{k}_1(\hat{s})\hat{\mathbf{t}}(\hat{s}),\\
\nabla_{\hat{\mathbf{t}}(\hat{s})}\hat{\mathbf{m}}_2(\hat{s}) &= -\hat{k}_2(\hat{s})\hat{\mathbf{t}}(\hat{s}),
\end{aligned} \qquad (3.3)$$

where

$$g\left(\hat{\mathbf{t}}(\hat{s}),\hat{\mathbf{t}}(\hat{s})\right)=1,\, g\left(\hat{\mathbf{m}}_1(\hat{s}),\hat{\mathbf{m}}_1(\hat{s})\right)=1,\, g\left(\hat{\mathbf{m}}_2(\hat{s}),\hat{\mathbf{m}}_2(\hat{s})\right)=-1 \qquad (3.4)$$
$$g\left(\hat{\mathbf{t}}(\hat{s}),\hat{\mathbf{m}}_1(\hat{s})\right)=g\left(\hat{\mathbf{t}}(\hat{s}),\hat{\mathbf{m}}_2(\hat{s})\right)=g\left(\hat{\mathbf{m}}_1(\hat{s}),\hat{\mathbf{m}}_2(\hat{s})\right)=0.$$

Here, we shall call the set $\{\hat{\mathbf{t}},\hat{\mathbf{m}}_1,\hat{\mathbf{m}}_1\}$ as Bishop trihedra, $\hat{k}_1$ and $\hat{k}_2$ as Bishop curvatures and $\tau(s)=\hat{\theta}'(s)$, $\hat{\kappa}(s)=\sqrt{\left|\hat{k}_1^2-\hat{k}_2^2\right|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned}
\hat{k}_1(\hat{s}) &= \hat{\kappa}(\hat{s})\cosh\hat{\theta}(\hat{s}),\\
\hat{k}_2(\hat{s}) &= \hat{\kappa}(\hat{s})\sinh\hat{\theta}(\hat{s}).
\end{aligned} \qquad (3.5)$$

Theorem 3.1. *Let $\hat{\gamma}$ be spacelike dual curve with spacelike principal normal parametrized by dual arc length. $\hat{\gamma}$ is a spacelike dual biharmonic curve if and only if*

$$\begin{aligned}
\hat{k}_2^2(\hat{s})-\hat{k}_1^2(\hat{s}) &= \hat{\Omega},\\
\hat{k}_1''(\hat{s})+\hat{k}_1^3(\hat{s})-\hat{k}_2^2(\hat{s})\hat{k}_1 &= 0,\\
-\hat{k}_2''(\hat{s})+\hat{k}_2^3(\hat{s})-\hat{k}_1^2(\hat{s})\hat{k}_2 &= 0,
\end{aligned} \qquad (3.6)$$

where $\hat{\Omega}$ is dual constant of integration.

Proof. Using dual Bishop frame, we obtain above system.

Lemma 3.2. *Let $\hat{\gamma}$ be a spacelike dual curve with spacelike principal normal parametrized by dual arc length. $\hat{\gamma}$ is a spacelike dual biharmonic curve if and only if*

$$-\hat{k}_1^2(\hat{s})+\hat{k}_2^2(\hat{s})=\hat{\Omega},$$

$$\hat{k}_1''(\hat{s})+\hat{k}_1(\hat{s})\left(-\hat{k}_1^2(\hat{s})+\hat{k}_2^2(\hat{s})\right)=0, \tag{3.7}$$
$$\hat{k}_2''(\hat{s})+\hat{k}_2(\hat{s})\left(-\hat{k}_1^2(\hat{s})+\hat{k}_2^2(\hat{s})\right)=0,$$

where $\hat{\Omega}$ *is constant of integration.*

Definition 3.3. *If* x *and* y *are real variable and* $\varepsilon^2=0,$ *the combination*

$$\begin{aligned} F\left(X,Y,Y',...,Y^{(n)}\right) &= G(X) \\ &= F\left(x, f(x), f'(x),..., f^{(n)}(x)\right) \\ &+\varepsilon x^*[F\left(x, f(x), f'(x),..., f^{(n)}(x)\right)] \\ &= g(x)+\varepsilon x^* g'(x) \end{aligned} \tag{3.8}$$

is called a differential equation with dual variable. Here,

$$\begin{aligned} X &= x+\varepsilon x^*, \\ Y &= f(x)+\varepsilon x^* f'(x), \\ Y' &= f'(x)+\varepsilon x^* f''(x), \\ &... \\ Y^{(n)} &= f^{(n)}(x)+\varepsilon x^* f^{(n+1)}(s). \end{aligned}$$

If $y=f(x)$ *is solution for real part of differential equation (3.8),* $y=f(x)+c$ *(c = constant) is solution of dual part of differential equation (3.8).*

Lemma 3.4. *If* Y_{g_i} *is a general solution of a differential equation of order* i*, then*

$$\begin{aligned} &Y_{g_1} \\ &Y_{g_2} = Y_{g_1}+c_1, \\ &Y_{g_3} = Y_{g_2}+c_2 x, \\ &Y_{g_4} = Y_{g_3}+c_3 x^2, \\ &... \\ &Y_{g_n} = Y_{g_{n-1}}+c_{n-1}x^{n-2}, \end{aligned}$$

where dependent variable Y_{g_i} *and independent variable* s.

Theorem 3.5. *Let* $\hat{\gamma}$ *be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by dual arc length. Then, general solution of a differential equation* $\hat{k}_1''(\hat{s})+\hat{k}_1(\hat{s})\left(-\hat{k}_1^2(\hat{s})+\hat{k}_2^2(\hat{s})\right)=0$ *is*

$$\begin{aligned} \hat{k}_1(\hat{s})_g &= C_1\cos[\sqrt{-k_1^2+k_2^2}\,s]+C_2\sin[\sqrt{-k_1^2+k_2^2}\,s] \\ &+\varepsilon s^*[C_1\cos[\sqrt{-k_1^2+k_2^2}\,s]+C_2\sin[\sqrt{-k_1^2+k_2^2}\,s]+C_3 s], \end{aligned} \tag{3.9}$$

where $C_1,\ C_2,\ C_3$ *are constants of integration.*

Proof. Using second equation of (3.7), we have

$$\frac{d^2\hat{k}_1(\hat{s})}{d\hat{s}^2}+\hat{k}_1(\hat{s})\left(-\hat{k}_1^2(\hat{s})+\hat{k}_2^2(\hat{s})\right)=0. \qquad (3.10)$$

Using above method in (3.10) we obtain

$$\frac{d^2k_1}{ds^2}+\Omega k_1+\varepsilon s^*[\frac{d^3k_1}{ds^3}+\left(-k_1^2(\hat{s})+k_2^2(\hat{s})\right)\frac{dk_1}{ds}]=0.$$

If we calculate the real and dual parts of this equation, we get the following relations

$$\frac{d^2k_1}{ds^2}+\left(-k_1^2+k_2^2\right)k_1=0,$$
$$\frac{d^3k_1}{ds^3}+\left(-k_1^2+k_2^2\right)\frac{dk_1}{ds}=0.$$

Using Mathematica in above equations we obtain

$$k_1=C_1\cos[\sqrt{-k_1^2+k_2^2}\,s]+C_2\sin[\sqrt{-k_1^2+k_2^2}\,s], \qquad (3.11)$$

where $C_1,\ C_2$ are constants of integration.

On the other hand, solution of dual part of differential equation we have

$$k_1^*=k_1+C_3s, \qquad (3.12)$$

where C_3 is constant of integration.

From (3.12) we obtain

$$k_1^*=C_1\cos[\sqrt{-k_1^2+k_2^2}\,s]+C_2\sin[\sqrt{-k_1^2+k_2^2}\,s]+C_3s.$$

By substituting (3.11) and (3.12) in the last equation we get

$$\begin{aligned}\hat{k}_1(\hat{s})_g &= C_1\cos[\sqrt{-k_1^2+k_2^2}\,s]+C_2\sin[\sqrt{-k_1^2+k_2^2}\,s]\\ &+\varepsilon s^*[C_1\cos[\sqrt{-k_1^2+k_2^2}\,s]+C_2\sin[\sqrt{-k_1^2+k_2^2}\,s]+C_3s].\end{aligned}$$

Therefore, we have the equations (3.9), which it completes the proof.

Using Mathematica above Theorem, we obtain

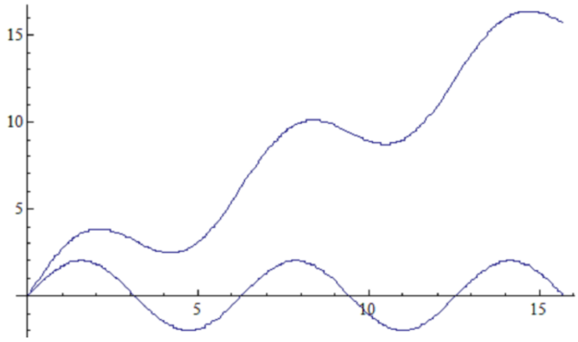

Fig1.

Corollary 3.6. *Let $\hat{\gamma}$ be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by dual arc length. Then, general solution of a differential equation*

$$\hat{k}_2(\hat{s})_g = C_4 \cos[\sqrt{-k_1^2 + k_2^2}s] + C_5 \sin[\sqrt{-k_1^2 + k_2^2}s]$$
$$+\varepsilon s^*[C_4 \cos[\sqrt{-k_1^2 + k_2^2}s] + C_5 \sin[\sqrt{-k_1^2 + k_2^2}s] + C_6 s],$$

where C_4, C_5, C_6 are constants of integration.

According to Lemma 3.4, the proof of the Corollary 3.6 is similar to the proof of Theorem 3.5.

Smilarly, using Mathematica above Corollary, we obtain

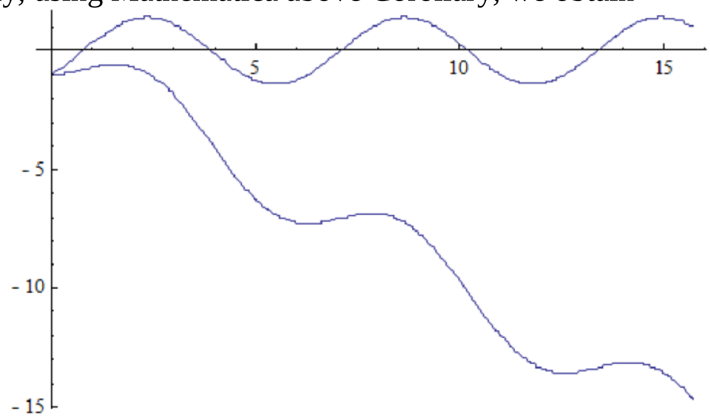

Fig2.

4 Open Problem

This Letter, we study dual spacelike biharmonic curves with spacelike principal normal for dual variable in dual Lorentzian space $\mathbb{D}_1^3$. We consider differential equations of dual Bishop curvatures of dual spacelike biharmonic curves with spacelike principal normal for dual variable in dual Lorentzian space D_1^3.

The authors can be presented this equations to Frenet curvatures in dual Lorentzian space D_1^3.

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