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Properties of Three Functions Relating to the Exponential Function and the Existence of Partitions of Unity

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Abstract

In the paper, the authors study properties of three functions relating to the exponential function and the existence of partitions of unity, including accurate and explicit computation of their derivatives, analyticity, complete monotonicity, logarithmically complete monotonicity, absolute monotonicity, and the like. Finally, the authors pose an open problem.

Keywords: *Exponential function; derivative; analyticity; complete monotonicity; absolute monotonicity; logarithmically complete monotonicity.*

1 Introduction

Throughout this paper, we denote the set of all positive integers by $\mathbb{N}$, and the set of all real numbers by $\mathbb{R}$.

In the theory of differentiable manifolds, the function

$$f(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \le 0 \end{cases} \tag{1}$$

plays an indispensable role in the proof of the existence of partitions of unity. See, for example, [6, p. 10, Lemma 1.10].

Similarly, we define

$$g(t) = e^{-1/t}, \quad t \in \mathbb{R} \setminus \{0\} \tag{2}$$

and

$$h(t) = e^{1/t}, \quad t \in \mathbb{R} \setminus \{0\}. \tag{3}$$

It is clear that $g(-t) = h(t)$ for $t \neq 0$ and $f(t) = g(t)$ for $t > 0$.

The task of this paper is to study properties of these three functions, including accurate and explicit computation of their derivatives, analyticity, complete monotonicity, absolute monotonicity, logarithmically complete monotonicity, and the like.

2 Properties

In this section, we will study properties of the functions $f(t)$, $g(t)$, and $h(t)$.

2.1 Derivatives

We first accurately compute derivatives of the i-th order for $i \in \mathbb{N}$.

Theorem 2.1 *For $i \in \mathbb{N}$ and $t \neq 0$, we have*

$$h^{(i)}(t) = (-1)^i e^{1/t} \frac{1}{t^{2i}} \sum_{k=0}^{i-1} a_{i,k} t^k, \tag{4}$$

where

$$a_{i,k} = \binom{i}{k} \binom{i-1}{k} k! \tag{5}$$

for all $0 \leq k \leq i-1$.

Proof. We prove this theorem by induction on $i \in \mathbb{N}$.

From $h'(t) = -\frac{1}{t^2} e^{1/t}$, it follows that $a_{1,0} = 1$. This means that the equality (4) is valid for $i = 1$.

Assume the equality (4) is valid for some $i > 2$.

By virtue of the inductive hypothesis, a direct differentiation gives

$$\begin{aligned}
h^{(i+1)}(t) &= \left[h^{(i)}(t)\right]' \\
&= \left[(-1)^i e^{1/t}\frac{1}{t^{2i}}\sum_{k=0}^{i-1} a_{i,k}t^k\right]' \\
&= (-1)^{i+1}e^{1/t}\frac{1}{t^{2(i+1)}}\left(\sum_{k=0}^{i-1} a_{i,k}t^k + 2i\sum_{k=0}^{i-1} a_{i,k}t^{k+1} - \sum_{k=1}^{i-1} k a_{i,k}t^{k+1}\right)
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{i+1}e^{1/t}\frac{1}{t^{2(i+1)}}\left[\sum_{k=0}^{i-1} a_{i,k}t^k + 2i\sum_{k=1}^{i} a_{i,k-1}t^k - \sum_{k=2}^{i}(k-1)a_{i,k-1}t^k\right] \\
&= (-1)^{i+1}e^{1/t}\frac{1}{t^{2(i+1)}}\left\{a_{i,0} + \sum_{k=1}^{i-1}[a_{i,k} + (2i-k+1)a_{i,k-1}]t^k + (i+1)a_{i,i-1}t^i\right\} \\
&= (-1)^{i+1}e^{1/t}\frac{1}{t^{2(i+1)}}\left(a_{i+1,0} + \sum_{k=1}^{i-1} a_{i+1,k}t^k + a_{i+1,i}t^i\right) \\
&= (-1)^{i+1}e^{1/t}\frac{1}{t^{2(i+1)}}\sum_{k=0}^{i} a_{i+1,k}t^k.
\end{aligned}$$

The proof of Theorem 2.1 is completed. ■

Theorem 2.2 *For $i \in \mathbb{N}$ and $t \neq 0$,*

$$g^{(i)}(t) = \frac{1}{e^{1/t}t^{2i}}\sum_{k=0}^{i-1}(-1)^k a_{i,k}t^k, \tag{6}$$

where $a_{i,k}$ is determined by (5).

Proof. This follows readily from combination of

$$g^{(i)}(t) = [h(-t)]^{(i)} = (-1)^i h^{(i)}(-t)$$

with Theorem 2.1. ■

Theorem 2.3 *For $i \in \mathbb{N}$,*

$$f^{(i)}(t) = \begin{cases} \dfrac{1}{e^{1/t}t^{2i}}\displaystyle\sum_{k=0}^{i-1}(-1)^k a_{i,k}t^k, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{7}$$

where $a_{i,k}$ is determined by (5).

Proof. This follows from the definition of $f(t)$ by (1) and Theorem 2.2. ■

2.2 Analyticity

A real function $q(t)$ is said to be analytic at a point t_0 if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of t_0.

Theorem 2.4 *The function $f(t)$ defined by (1) is infinitely differentiable on $\mathbb{R}$ and*

$$f^{(i)}(0) = 0 \tag{8}$$

for all $i \in \{0\} \cup \mathbb{N}$, but it is not analytic at $t = 0$.

Proof. By Theorem 2.3, it is obvious that, to prove the infinite differentiability of $f(t)$, it suffices to show the infinite differentiability of $f(t)$ at $t = 0$. For this, we just need to compute the limit $f^{(i)}(0) = \lim_{t \to 0^+} f^{(i)}(t) = 0$ for all $i \geq 0$, which can be calculated directly from (7). Consequently, the function $f(t)$ is infinitely differentiable on $(0, \infty)$.

Suppose that the function $f(t)$ is analytic at $t = 0$, then it should have a Taylor series

$$f(t) = \sum_{\ell=0}^{\infty} \frac{f^{(\ell)}(0)}{\ell!} t^{\ell} \tag{9}$$

for $t \in (-\delta, \delta)$, where $\delta > 0$. Using (8), the right hand side of (9) becomes 0 for all $t \in (-\delta, \delta)$. This contradicts with the positivity of $f(t)$ on $(0, \delta) \subset (0, \infty)$. As a result, the function $f(t)$ is not analytic at $t = 0$. The proof is thus complete. ■

Theorem 2.5 *The point $t = 0$ is a jump-infinite discontinuous point of the functions $g(t)$ and $h(t)$ on $\mathbb{R}$.*

Proof. This follows from

$$\lim_{t \to 0^-} h(t) = \lim_{t \to 0^-} e^{1/t} = 0 = \lim_{t \to 0^+} e^{-1/t} = \lim_{t \to 0^+} g(t)$$

and

$$\lim_{t \to 0^+} h(t) = \lim_{t \to 0^+} e^{1/t} = \infty = \lim_{t \to 0^-} e^{-1/t} = \lim_{t \to 0^-} g(t).$$

The proof is thus complete. ■

2.3 Complete monotonicity

A function $q(x)$ is said to be completely monotonic on an interval I if $q(x)$ has derivatives of all orders on I and $(-1)^n q^{(n)}(x) \geq 0$ for $x \in I$ and $n \geq 0$. See [7, Chapter IV].

Theorem 2.6 *The function $h(t)$ is completely monotonic on $(0, \infty)$.*

Proof. By Theorem 2.1, it follows that

$$(-1)^i h^{(i)}(t) = e^{1/t} \frac{1}{t^{2i}} \sum_{k=0}^{i-1} a_{i,k} t^k > 0$$

on $(0, \infty)$. So the function $h(t)$ is completely monotonic on $(0, \infty)$. ■

2.4 Logarithmically complete monotonicity

A positive function $f(x)$ is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\ln f(x)$ satisfies $(-1)^k [\ln f(x)]^{(k)} \geq 0$ for $k \in \mathbb{N}$ on I. A logarithmically completely monotonic function on I must be completely monotonic on I, but not conversely. See [1, 2, 4], the expository article [3] and closely-related references therein.

Theorem 2.7 *The function* $h(t)$ *and* $\frac{1}{g(t)}$ *are logarithmically completely monotonic on* $(0, \infty)$*, and so are the functions* $g(t)$ *and* $\frac{1}{h(t)}$ *on* $(-\infty, 0)$.

Proof. This may be deduced from $\ln h(t) = \frac{1}{t}$ and $\ln g(t) = -\frac{1}{t}$ and standard arguments. ■

2.5 Absolute monotonicity

An infinitely differentiable function $q(t)$ defined on an interval $I \subseteq \mathbb{R}$ is said to be absolutely monotonic if $q^{(i)}(t) \geq 0$ holds on I for all $i \in \{0\} \cup \mathbb{N}$. See [2, 7].

Theorem 2.8 *The function* $g(t)$ *is absolutely monotonic on* $(-\infty, 0)$.

Proof. This comes from (6) in Theorem 2.2. ■

3 Remarks

Remark 3.1 *To the best of my knowledge, the derivatives* (4) *were described but without explicit and accurate expressions in many books.*

Remark 3.2 *By definitions, it is easy to see that if* $h(t)$ *is completely monotonic on an interval* I *then* $h(-t)$ *is absolutely monotonic on* $-I$*, the symmetrical interval of* I *with respect to* 0*. This implies that Theorem 2.6 and Theorem 2.8 are equivalent to each other.*

Remark 3.3 *This paper is a slightly revised version of the preprint [5].*

4 An open problem

The famous Bernstein-Widder's Theorem [7, p. 161, Theorem 12b] reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\alpha(t), \tag{10}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. When replacing $f(x)$ in (10) by $h(x)$ defined by (3) on $(0, \infty)$, can one find the measure $\alpha(t)$ in (10)? In other words, can one find the integral representation like (10) of the function $e^{1/x}$ on $(0, \infty)$?

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