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Related Fixed Point Theorems In Fuzzy Metric Spaces

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Abstract

In this paper we establish the existence and uniqueness of related fixed points for mappings with different contractive conditions in two complete fuzzy metric spaces.

Keywords: *Fuzzy Metric Space , Fixed Point , Related Fixed Points .*

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1 Introduction

The concepts of fuzzy sets was initially investigated by Zadeh [7] in 1965 as a new way to represent imprecise facts or uncertainties or vagueness in everyday life. Subsequently , it was developed extensively by many authors and used in population dynamics , chaos control , computer programming , medicine , etc. In 1975 Kramosil and Michalek [9] introduced the concept of fuzzy metric spaces (briefly , FM-spaces) , which opened a new avenue for further development of analysis in such spaces. Later on it is modified and a few concepts of mathematical analysis have been developed by George and Veeramani [2]. Fisher [3], Aliouche and Fisher [1], Telci [8] proved some related fixed point theorems in compact metric spaces. Recently, Rao et.al [5] and [6] proved some related fixed point theorems in sequentially compact fuzzy metric spaces. However , the study of related fixed points for two pairs of mappings is also interesting. In this paper we extend this concept to fuzzy metric space and establish the existence of related fixed points theorems for two pairs of mappings. This research modifies and generalizes the results of Fisher [3], R.K.Namdeo,

S. Jain and B.Fisher [10] under a different contraction condition in two fuzzy metric spaces .

2 Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1 *[4] A binary operation* $*\,:\,[0\,,\,1]\times[0\,,\,1]\;\longrightarrow\;[0\,,\,1]$ *is continuous* t *- norm if* $*$ *satisfies the following conditions* :
(i) $*$ *is commutative and associative ,*
(ii) $*$ *is continuous ,*
(iii) $a * 1 = a \quad \forall\; a \,\varepsilon\, [0\,,\,1]$,
(iv) $a * b \,\leq\, c * d$ *whenever* $a \leq c\,,\, b \leq d$ *and* $a\,,\,b\,,\,c\,,\,d \,\varepsilon\, [0\,,\,1]$.

Definition 2.2 *[2] The* 3*-tuple* $(X\,,\,\mu\,,\,*)$ *is called a fuzzy metric space if* X *is an arbitrary non-empty set,* $*$ *is a continuous t-norm and* μ *is a fuzzy set in* $X^2 \times (0,\infty)$ *satisfying the following conditions :*
(i) $\mu(x\,,\,y\,,\,t) > 0$;
(ii) $\mu(x\,,\,y\,,\,t) = 1$ *if and only if* $x = y$
(iii) $\mu(x\,,\,y\,,\,t) = \mu(y\,,\,x\,,\,t)$;
(iv) $\mu(x\,,\,y\,,\,s) * \mu(y\,,\,z\,,\,t) \leq \mu(x\,,\,z\,,\,s + t)$;
(v) $\mu(x\,,\,y\,,\,\cdot) : (0\,,\,\infty) \to (0\,,\,1]$ *is continuous;*
for all $x\,,y\,,z \in X$ *and* $t,\,s > 0$.

Definition 2.3 *[11] Let* $(X\,,\,\mu\,,\,*)$ *be a fuzzy metric space. A sequence* $\{x_n\}_n$ *in* X *is said to converge to* $x \in X$ *if and only if*

$$\lim_{n\to\infty}\mu(x_n\,,\,x\,,\,t) = 1 \;\; for\; each\; t > 0$$

A sequence $\{x_n\}_n$ *in* X *is called Cauchy sequence if and only if*

$$\lim_{n\to\infty}\mu(x_n\,,\,x_{n+p}\,,\,t) = 1 \; for\; each\; t > 0 \; and\; p = 1\,,\,2\,,\,3\,,\,\cdots$$

A fuzzy metric space $(X\,,\,\mu\,,\,*)$ *is said to be complete if and only if every Cauchy sequence in* X *is convergent in* X .

Definition 2.4 *Let* $(X\,,\,\mu\,,\,*)$ *and* $(Y\,,\,\nu\,,\,*)$ *be two fuzzy metric spaces. Suppose* T *is a mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *such that* $z \in X$ *is a fixed point of* ST *and* $w \in Y$ *is a fixed point of* TS *for which* $Tz = w$ *and* $Sw = z$ *. Then* T *and* S *is called a pair of mapping for related fixed point .*

3 Related Fixed Point Theorems

Theorem 3.1 *Let* $(X, \mu, *)$ *and* $(Y, \nu, *)$ *be complete fuzzy metric spaces. If* T *is a continuous mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *satisfying the inequalities*

$$k\,\mu(STx, STx', t) \geq \min\{\mu(x, x', t), \mu(x, STx, t), \mu(x', STx', t), \nu(Tx, Tx', t)\}$$

$$k\,\nu(TSy, TSy', t) \geq \min\{\nu(y, y', t), \nu(y, TSy, t), \nu(y', TSy', t), \mu(Sy, Sy', t)\}$$

for all x, x' *in* X *and* y, y' *in* Y*, where* $k \in (0, 1)$*, then* ST *has a unique fixed point* z *in* X *and* TS *has a unique fixed point* w *in* Y*. Further,* $Tz = w$ *and* $Sw = z$

Proof. Let x be an arbitrary point in X. Let $x_1 = (ST)^1x$, $x_2 = (ST)^2x = (ST)x_1, \cdots, x_n = (ST)^nx = (ST)x_{n-1}$
and
$y_1 = Tx$, $y_2 = T(STx) = Tx_1, \cdots, y_n = T(ST)^{n-1}x = Tx_{n-1}$
for all $n \in N$. By induction

$$k\,\mu(x_n, x_{n+1}, t) = k\,\mu(STx_{n-1}, STx_n, t)$$

$$\geq \min\{\mu(x_{n-1}, x_n, t), \mu(x_{n-1}, STx_{n-1}, t), \mu(x_n, STx_n, t), \nu(Tx_{n-1}, Tx_n, t)\}$$

$$= \min\{\mu(x_{n-1}, x_n, t), \mu(x_{n-1}, x_n, t), \mu(x_n, x_{n+1}, t), \nu(y_n, y_{n+1}, t)\}$$

$$= \min\{\mu(x_{n-1}, x_n, t), \mu(x_n, x_{n+1}, t), \nu(y_n, y_{n+1}, t)\} \quad \cdots \quad (1)$$

Hence

$$\mu(x_n, x_{n+1}, t)$$

$$\geq \frac{1}{k}\min\{\mu(x_{n-1}, x_n, t), \mu(x_n, x_{n+1}, t), \nu(y_n, y_{n+1}, t)\} \quad \cdots \quad (2)$$

By putting (2) in (1) we obtain that

$$k\,\mu(x_n, x_{n+1}, t)$$

$$\geq \min\{\mu(x_{n-1}, x_n, t), \nu(y_n, y_{n+1}, t), \frac{1}{k}\mu(x_{n-1}, x_n, t), \frac{1}{k}\mu(x_n, x_{n+1}, t), \frac{1}{k}\nu(y_n, y_{n+1}, t)\}$$

$$= \min\{\mu(x_{n-1}, x_n, t), \nu(y_n, y_{n+1}, t), \frac{1}{k}\mu(x_n, x_{n+1}, t)\}$$

$$\vdots$$

$\geq \min\{\mu(x_{n-1}, x_n, t), \nu(y_n, y_{n+1}, t), \frac{1}{k^m}\mu(x_n, x_{n+1}, t)\}$

Taking lim as $m \longrightarrow \infty$ we have

$$k\,\mu(x_n, x_{n+1}, t) \geq \min\{\mu(x_{n-1}, x_n, t), \nu(y_n, y_{n+1}, t)\} \quad \cdots \qquad (3)$$

Again,

$k\,\nu(y_n, y_{n+1}, t) = k\,\nu(Tx_{n-1}, Tx_n, t)$

$= k\,\nu(T(STx_{n-2}), T(STx_{n-1}), t) = k\,\nu(TSy_{n-1}, TSy_n, t)$

$\geq \min\{\nu(y_{n-1}, y_n, t), \nu(y_{n-1}, TSy_{n-1}, t), \nu(y_n, TSy_n, t), \mu(Sy_{n-1}, Sy_n, t)\}$

$= \min\{\nu(y_{n-1}, y_n, t), \nu(y_{n-1}, y_n, t), \nu(y_n, y_{n+1}, t), \mu(x_{n-1}, x_n, t)\}$

$= \min\{\nu(y_{n-1}, y_n, t), \nu(y_n, y_{n+1}, t), \mu(x_{n-1}, x_n, t)\}$

$$\Longrightarrow \; k\,\nu(y_n, y_{n+1}, t) \geq \min\{\nu(y_{n-1}, y_n, t), \mu(x_{n-1}, x_n, t)\} \cdots \quad (4)$$

From (3) and (4), by induction we get

$$\mu(x_n, x_{n+1}, t) = \frac{1}{k}\min\{\mu(x_{n-1}, x_n, t), \nu(y_n, y_{n+1}, t)\}$$
$$\vdots$$
$$\geq \frac{1}{k^n}\min\{\mu(x, x_1, t), \nu(y_1, y_2, t)\}$$

We now verify that x_n is a cauchy sequence. Let $t_1 = \frac{t}{p}$.

$\mu(x_n, x_{n+p}, t)$

$\geq \mu(x_n, x_{n+1}, t_1) * \mu(x_{n+1}, x_{n+2}, t_1) * \cdots * \mu(x_{n+p-1}, x_{n+p}, t_1)$

$\geq \frac{1}{k^n}\min\{\mu(x, x_1, t), \nu(y_1, y_2, t)\} * \cdots * \frac{1}{k^{n+p-1}}\min\{\mu(x, x_1, t), \nu(y_1, y_2, t)\}$

$$\Longrightarrow 1 \geq \lim_{n\to\infty}\mu(x_n, x_{n+p}, t) \geq \lim_{n\to\infty}\frac{1}{k^n}\min\{\mu(x, x_1, t), \nu(y_1, y_2, t)\} > 1$$

$$\Longrightarrow \lim_{n\to\infty}\mu(x_n, x_{n+p}, t) = 1$$

Hence $\{x_n\}$ is a Cauchy sequence with a limit z in X and similarly , $\{y_n\}$ is a cauchy sequence with a limit w in Y.
We have on using the continuity of T

$$w = \lim_{n \to \infty} y_n = \lim_{n \to \infty} Tx_n = Tz$$

Further,

$k\,\mu(STz, x_n, t) = k\,\mu(STz, STx_{n-1}, t)$

$\geq \min\{\mu(z, x_{n-1}, t), \mu(z, STz, t), \mu(x_{n-1}, x_n, t), \nu(Tz, y_n, t)\}$
and on letting $n \longrightarrow \infty$ we have

$$k\,\mu(STz, z, t) \geq \min\{\mu(z, STz, t), \nu(Tz, w, t)\} = \mu(z, STz, t)$$

it follows that $STz = z$. Hence we have

$$STz = Sw = z$$

Now suppose that ST has a second fixed point z'. Then

$k\,\mu(z, z', t) = k\,\mu(STz, STz', t)$

$\geq \min\{\mu(z, z', t), \mu(z, STz, t), \mu(z', STz, t), \nu(Tz, Tz', t)\}$
$\Longrightarrow \; k\,\mu(z, z', t) \geq \min\{\mu(z, z', t), 1, \nu(Tz, Tz', t)\}$
$\Longrightarrow \; k\,\mu(z, z', t) \geq \nu(Tz, Tz', t)$
But

$k\,\nu(Tz, Tz', t) = \nu(TSTz, TSTz', t)$

$\geq \min\{\nu(Tz, Tz', t), \nu(Tz, TSTz, t), \nu(Tz', TSTz', t), \mu(STz, STz', t)\}$
$= \min\{\nu(Tz, Tz', t), 1, 1, \mu(z, z', t)\}$
$\Longrightarrow \; \nu(Tz, Tz', t) \geq \dfrac{1}{k}\mu(z, z', t)$
Hence ,

$\mu(z, z', t) \geq \dfrac{1}{k^2}\mu(z, z', t) \geq \cdots \geq \dfrac{1}{k^n}\mu(z, z', t) \longrightarrow \infty \; as \; n \longrightarrow \infty$

$\Longrightarrow 1 \geq \mu(z, z', t) \geq \lim_{n \to \infty} \dfrac{1}{k^n}\mu(z, z', t) > 1$

$\Longrightarrow \mu(z, z', t) = 1$
which implies $z = z'$.
Similarly , w is the unique fixed point of TS. This completes the proof of the theorem.

Theorem 3.2 *Let* $(X, \mu, *)$ *and* $(Y, \nu, *)$ *be two complete fuzzy metric spaces. Let* A, B *be mappings of* X *into* Y *and let* S, T *be mappings of* Y *into* X *satisfying the inequalities*

$$k\,\mu(SAx, TBx', t) \geq \frac{f(x, x', y, y', t)}{h(x, x', y, y', t)} \qquad \cdots \qquad (5)$$

$$k\,\nu(BSy, ATy', t) \geq \frac{g(x, x', y, y', t)}{h(x, x', y, y', t)} \qquad \cdots \qquad (6)$$

for all x, x' *in* X *and* y, y' *in* Y *for which* $f(x, x', y, y', t)$ *and* $g(x, x', y, y', t) < h(x, x', y, y', t) < 1$ *where*

$$f(x, x', y, y', t) = \min\{\mu(x, x', t)\nu(Ax, Bx', t), \mu(x, x', t)\mu(Sy, Ty', t), \mu(x, Ty', t)\nu(Ax, ATy', t), \mu(x', Sy, t)\nu(Bx', BSy, t)\}$$

$$g(x, x', y, y', t) = \min\{\nu(y, y', t)\mu(Sy, Ty', t), \nu(y, y', t)\nu(Ax, Bx', t), \nu(y, Bx', t)\mu(Sy, TBx', t), \nu(y', Ax, t)\mu(Ty', SAx, t)\}$$

$$h(x, x', y, y', t) = \min\{\nu(Ax, Bx', t), \mu(SAx, TBx', t), \mu(Sy, Ty', t), \nu(BSy, ATy', t)\}$$

and $0 < k < 1$. *If one of the mappings* A, B, S *and* T *is continuous , then* SA *and* TB *have a unique common fixed point* z *in* X *and* BS *and* AT *have a unique common fixed point* w *in* Y. *Further ,* $Az = Bz = w$ *and* $Sw = Tw = z$.

Proof. Let $x = x_0$ be an arbitrary point in X. let

$$Ax_0 = y_1, Sy_1 = x_1, Bx_1 =, y_2, Ty_2 = x_2 \text{ and } Ax_2 = y_3$$

and in general let

$$y_{2n-1} = Ax_{2n-2}, x_{2n-1} = Sy_{2n-1}, y_{2n} = Bx_{2n-1} \text{ and } x_{2n} = Ty_{2n}$$

for $n = 1, 2, \cdots$
We will first of all suppose that for some n ,

$$h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)$$

$$= \min\{\nu(Ax_{2n}, Bx_{2n-1}, t), \mu(SAx_{2n}, TBx_{2n-1}, t), \mu(Sy_{2n-1}, Ty_{2n}, t), \nu(BSy_{2n-1}, ATy_{2n}, t)\}$$

$$= \min\{\nu(y_{2n+1}, y_{2n}, t), \mu(x_{2n+1}, x_{2n}, t), \mu(x_{2n-1}, x_{2n}, t), \nu(y_{2n}, y_{2n+1}, t)\}$$

$= 1$

Then putting $x_{2n-1} = x_{2n} = x_{2n+1} = z$ and $y_{2n} = y_{2n+1} = w$, we see that

$$SAz = TBz = z\,,\ ATw = w\,,\ Az = Bz = w\,,\ Tw = z \quad \cdots \qquad (7)$$

from which it follows that $Sw = z\,,\ BSw = w$.
Similarly , $h(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}, t) = 1$ for some n implies that there exist points z in X and w in Y such that

$$SAz = TBz = z\,,\ BSw = ATw = w\,,\ Az = Bz = w\,,\ Sw = Tw = z.$$

We will now suppose that

$$h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t) < 1$$

and

$$h(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}, t) < 1$$

Applying inequality (5) , we get

$$k\,\mu(x_{2n+1}, x_{2n}, t) = k\,\mu(SAx_{2n}, TBx_{2n-1}, t) \geq \frac{f(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)}{h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)}$$

where

$$f(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)$$

$$= \min\{\mu(x_{2n}, x_{2n-1}, t)\,\nu(Ax_{2n}, Bx_{2n-1}, t)\,,\ \mu(x_{2n}, x_{2n-1}, t)\,\mu(Sy_{2n-1}, Ty_{2n}, t)\,,$$
$$\mu(x_{2n}, Ty_{2n}, t)\nu(Ax_{2n}, ATy_{2n}, t)\,,\ \mu(x_{2n-1}, Sy_{2n-1}, t)\nu(Bx_{2n-1}, BSy_{2n-1}, t)\}$$
$$= \min\{\mu(x_{2n}, x_{2n-1}, t)\,\nu(y_{2n+1}, y_{2n}, t)\,,\ \mu(x_{2n}, x_{2n-1}, t)\,\mu(x_{2n-1}, x_{2n}, t)\,,$$
$$\mu(x_{2n}, x_{2n}, t)\,\nu(y_{2n+1}, y_{2n+1}, t)\,,\ \mu(x_{2n-1}, x_{2n-1}, t)\,\nu(y_{2n}, y_{2n}, t)\}$$
$$= \min\{\mu(x_{2n}, x_{2n-1}, t)\nu(y_{2n+1}, y_{2n}, t)\,,\ \mu^2(x_{2n}, x_{2n-1}, t)\,,\ 1\,,\ 1\}$$
$$= \min\{\mu(x_{2n}, x_{2n-1}, t)\nu(y_{2n+1}, y_{2n}, t)\,,\ \mu^2(x_{2n}, x_{2n-1}, t)\}$$

and

$$h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)$$

$$= \min\{\nu(y_{2n+1}, y_{2n}, t)\,,\ \mu(x_{2n+1}, x_{2n}, t)\,,\mu(x_{2n-1}, x_{2n}, t)\,,\ \nu(y_{2n}, y_{2n+1}, t)\}$$

then

$$k\,\mu(x_{2n+1}, x_{2n}, t)$$

$$\geq \frac{\min\{\mu(x_{2n}, x_{2n-1}, t)\nu(y_{2n+1}, y_{2n}, t), \mu^2(x_{2n}, x_{2n-1}, t)\}}{\min\{\nu(y_{2n+1}, y_{2n}, t), \mu(x_{2n+1}, x_{2n}, t), \mu(x_{2n-1}, x_{2n}, t)\}}$$

from which it follows that

$$k\,\mu(x_{2n}, x_{2n+1}, t) \geq \min\{\mu(x_{2n-1}, x_{2n}, t), \nu(y_{2n}, y_{2n+1}, t)\} \quad \cdots \; (8)$$

Using inequality (5) again, we get

$$k\,\mu(x_{2n-1}, x_{2n}, t)$$

$$= k\,\mu(SAx_{2n-2}, TBx_{2n-1}, t)$$

$$\geq \frac{f(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}, t)}{h(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}, t)}$$

$$= \frac{\min\{\mu(x_{2n-2}, x_{2n-1}, t)\nu(y_{2n-1}, y_{2n}, t), \mu^2(x_{2n-2}, x_{2n-1}, t)\}}{\min\{\nu(y_{2n-1}, y_{2n}, t), \mu(x_{2n-1}, x_{2n}, t), \mu(x_{2n-1}, x_{2n-2}, t)\}}$$

$$k\,\mu(x_{2n-1}, x_{2n}, t) \geq \min\{\mu(x_{2n-2}, x_{2n-1}, t), \nu(y_{2n-1}, y_{2n}, t)\} \quad (9)$$

Again , on using inequality (6)

$$k\,\nu(y_{2n}, y_{2n+1}, t) = k\,\nu(BSy_{2n-1}, ATy_{2n}, t)$$

$$\geq \frac{g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)}{h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)}$$

where

$$g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)$$

$$= \min\{\nu(y_{2n-1}, y_{2n}, t)\,\mu(Sy_{2n-1}, Ty_{2n}, t), \nu(y_{2n-1}, y_{2n}, t)\,\nu(Ax_{2n}, Bx_{2n-1}, t),$$
$$\nu(y_{2n-1}, Bx_{2n-1}, t)\,\mu(Sy_{2n-1}, TBx_{2n-1}, t), \nu(y_{2n}, Ax_{2n}, t)\,\mu(Ty_{2n}, SAx_{2n}, t)\}$$

$$= \min\{\nu(y_{2n-1}, y_{2n}, t)\,\mu(x_{2n-1}, x_{2n}, t), \nu(y_{2n-1}, y_{2n}, t)\,\nu(y_{2n+1}, y_{2n}, t),$$
$$\nu(y_{2n-1}, y_{2n}, t)\,\mu(x_{2n-1}, x_{2n}, t), \nu(y_{2n}, y_{2n+1}, t)\,\mu(x_{2n}, x_{2n+1}, t)\}$$

$$= \min\{\nu(y_{2n-1}, y_{2n}, t)\,\mu(x_{2n-1}, x_{2n}, t), \nu(y_{2n-1}, y_{2n}, t)$$
$$\nu(y_{2n+1}, y_{2n}, t), \nu(y_{2n}, y_{2n+1}, t)\,\mu(x_{2n}, x_{2n+1}, t)\}$$

We then have either

$$g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)$$
$$= \nu(y_{2n-1}, y_{2n}, t)\,\min\{\mu(x_{2n-1}, x_{2n}, t), \nu(y_{2n+1}, y_{2n}, t)\}$$

or

$$g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)$$

$$= \nu(y_{2n+1}, y_{2n}, t) \min\{\mu(x_{2n}, x_{2n+1}, t), \nu(y_{2n-1}, y_{2n}, t)\}$$

Further,

$$h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}, t)$$

$$= \min\{\nu(y_{2n+1}, y_{2n}, t), \mu(x_{2n+1}, x_{2n}, t), \mu(x_{2n-1}, x_{2n}, t)\}$$
$$= \min\{\nu(y_{2n+1}, y_{2n}, t), \mu(x_{2n-1}, x_{2n}, t)\}$$

on using inequality (8) . It follows that either

$$k\,\nu(y_{2n}, y_{2n+1}, t) \geq \nu(y_{2n}, y_{2n-1}, t)$$

or

$$k\,\nu(y_{2n}, y_{2n+1}, t) \geq \min\{\mu(x_{2n+1}, x_{2n}, t), \nu(y_{2n-1}, y_{2n}, t)\}$$

Thus, we have

$$k\,\nu(y_{2n}, y_{2n+1}, t) \geq \min\{\mu(x_{2n+1}, x_{2n}, t), \nu(y_{2n-1}, y_{2n}, t)\} \cdots \quad (10)$$

Using inequality (6) again, we get

$$\begin{aligned} k\,\nu(y_{2n}, y_{2n-1}, t) &= k\,\nu(BSy_{2n-1}, ATy_{2n-2}, t) \\ &\geq \frac{g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}, t)}{h(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}, t)} \end{aligned}$$

from which it follows that

$$k\,\nu(y_{2n}, y_{2n-1}, t) \geq \min\{\mu(x_{2n}, x_{2n-1}, t), \nu(y_{2n-2}, y_{2n-1}, t)\} \cdots \quad (11)$$

It now follows from inequalities (8) and (10) that

$$\begin{aligned} \mu(x_n, x_{n+1}, t) &\geq \frac{1}{k} \min\{\mu(x_{n-1}, x_n, t), \frac{1}{k}\mu(x_{n+1}, x_n, t), \frac{1}{k}\nu(y_{n-1}, y_n, t)\} \\ &> \frac{1}{k} \min\{\mu(x_{n-1}, x_n, t), \nu(y_{n-1}, y_n, t)\} \\ &\vdots \\ &\geq \frac{1}{k^{n-1}} \min\{\mu(x_1, x_2, t), \nu(y_1, y_2, t)\} \end{aligned}$$

Let $t_1 = \frac{t}{p}$.

$$\mu(x_n, x_{n+p}, t) \geq \mu(x_n, x_{n+1}, t_1) * \cdots * \mu(x_{n+p-1}, x_{n+p}, t_1)$$

$$\geq \frac{1}{k^{n-1}} \min\{\mu(x_1, x_2, t_1), \nu(y_1, y_2, t_1)\} * \cdots *$$
$$\frac{1}{k^{n+p-2}} \min\{\mu(x_1, x_2, t_1), \nu(y_1, y_2, t_1)\}$$

which implies that

$$\lim_{n \to \infty} \mu(x_n, x_{n+p}, t) > 1 * \cdots * 1 = 1$$

Similarly ,

$$\lim_{n \to \infty} \nu(y_n, y_{n+p}, t) = 1$$

$\Longrightarrow$ $\{x_n\}$ is a cauchy sequence in X with a limit z and $\{y_n\}$ is a cauchy sequence in Y with a limit w .
Now suppose that A is continuous . Then

$$w = \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} Ax_{2n} = Az \qquad \cdots \qquad (12)$$

and

$$\lim_{n \to \infty} f(z, x_{2n-1}, w, y_{2n}, t)$$
$$= \lim_{n \to \infty} \min\{\mu(z, x_{2n-1}, t)\nu(Az, Bx_{2n-1}, t), \mu(z, x_{2n-1}, t)\mu(Sw, Ty_{2n}, t),$$
$$\mu(z, Ty_{2n}, t)\nu(Az, ATy_{2n}, t), \mu(x_{2n-1}, Sw, t)\nu(Bx_{2n-1}, BSw, t)\}$$
$$= \min\{\mu(z, Sw, t)\nu(w, BSw, t), \mu^2(z, Sw, t)\} \qquad \cdots \qquad (13)$$

$$\lim_{n \to \infty} g(z, x_{2n-1}, w, y_{2n}, t)$$
$$= \min\{\nu(w, y_{2n}, t)\mu(Sw, Ty_{2n}, t), \nu(w, y_{2n}, t)\nu(Az, Bx_{2n-1}, t),$$
$$\nu(w, Bx_{2n-1}, t)\mu(Sw, TBx_{2n-1}, t), \nu(y_{2n}, Az, t)\mu(Ty_{2n}, SAz, t)\}$$
$$= \min\{\mu(z, Sw, t), \mu(z, SAz, t)\} \qquad \cdots \qquad (14)$$

$$\lim_{n \to \infty} h(z, x_{2n-1}, w, y_{2n}, t) = \min\{\mu(Sw, z, t), \nu(BSw, w, t)\} \quad \cdots \qquad (15)$$

If

$$\min\{\mu(Sw, z, t), \nu(BSw, w, t)\} = 1 \qquad \cdots \qquad (16)$$

then

$$Sw = z, \; BSw = w, \; Bz = w. \qquad \cdots \qquad (17)$$

If it were possible that

$$\min\{\mu(Sw, z, t), \nu(BSw, w, t)\} < 1 \qquad \cdots \qquad (18)$$

then we have on using inequality (5) and equations (13) and (15)

$$\mu(Sw\,,\,z\,,\,t) \;=\; \lim_{n\to\infty}\mu(SAz\,,\,TBx_{2n-1}\,,\,t) \;\geq\; \frac{1}{k}\lim_{n\to\infty}\frac{f(z\,,\,x_{2n-1}\,,\,w\,,\,y_{2n}\,,\,t)}{h(z\,,\,x_{2n-1}\,,\,w\,,\,y_{2n}\,,\,t)}$$

$$= \frac{\min\{\mu(z\,,\,Sw,\,t)\,\nu(w\,,\,BSw,\,t)\,,\,\mu(z\,,\,Sw,\,t)\}}{k\,\min\{\mu(Sw\,,\,z,\,t)\,,\,\nu(BSw\,,\,w,\,t)\}} \;\geq\; \frac{1}{k^2}\;\mu(Sw\,,\,z,\,t)$$

$$\Longrightarrow \mu(Sw\,,\,z,\,t) \;=\; 1$$

and so $Sw = z$. Further, using inequality (6) and equations (14) and (15), we have equation (17).
To complete the proof, We now prove that $Tw = z$. Then

$$\lim_{n\to\infty} f(x_{2n}\,,\,z\,,\,w\,,\,w\,,\,t)$$

$$= \lim_{n\to\infty}\min\{\mu(x_{2n}\,,\,z,\,t)\,\nu(Ax_{2n}\,,\,Bz,\,t)\,,\,\mu(x_{2n}\,,\,z,\,t)\,\mu(Sw\,,\,Tw,\,t)\,,$$
$$\mu(x_{2n}\,,\,Tw,\,t)\,\nu(Ax_{2n}\,,\,ATw,\,t)\,,\,\mu(z\,,\,Sw\,,\,t)\,\nu(Bz\,,\,BSw\,,\,t)\}$$
$$= \min\{\mu(z\,,\,Tw,\,t)\,\nu(w\,,\,ATw,\,t)\,,\,\nu(ATw\,,\,w,\,t)\} \qquad \cdots \qquad (19)$$

If

$$\lim_{n\to\infty} h(x_{2n}\,,\,z\,,\,w\,,\,w\,,\,t) \;=\; \min\{\mu(z\,,\,Tw,\,t)\,,\,\nu(w\,,\,ATw,\,t)\} \;=\; 1\,,$$

then obviously $Tw = z$. So we suppose that

$$\lim_{n\to\infty} h(x_{2n}\,,\,z\,,\,w\,,\,w\,,\,t) \;=\; \min\{\mu(z\,,\,Tw,\,t)\,,\,\nu(w\,,\,ATw,\,t)\} \;<\; 1 \quad\cdots \qquad (20)$$

Then we have on using inequality (5) and equations (19) and (20)

$$\mu(z\,,\,Tw\,,\,t) \;=\; \lim_{n\to\infty}\mu(SAx_{2n}\,,\,TBz\,,\,t) \;\geq\; \frac{1}{k}\lim_{n\to\infty}\frac{f(x_{2n}\,,\,z\,,\,w\,,\,w\,,\,t)}{h(x_{2n}\,,\,z\,,\,w\,,\,w\,,\,t)}$$

$$= \frac{\min\{\mu(z\,,\,Tw,\,t)\,\nu(w\,,\,ATw,\,t)\,,\,\nu(ATw\,,\,w,\,t)\}}{k\,\min\{\mu(z\,,\,Tw,\,t)\,,\,\nu(w\,,\,ATw,\,t)\}} \;\geq\; \frac{1}{k}\,\mu(z\,,\,Tw,\,t)$$

$$\Longrightarrow \mu(z\,,\,Tw,\,t) \;=\; 1$$

We must therefore have $Tw = z$ and equations (7) again follow.
By the symmetry, the same results again hold if one of the mappings B, S, T is continuous, instead of A.
To prove the uniqueness, suppose that TB and SA have a second common fixed point z'. Then, using inequality (5), we have

$$\mu(z\,,\,z'\,,\,t) \;=\; \mu(SAz\,,\,TBz'\,,\,t) \;\geq\; \frac{1}{k}\,\frac{f(z\,,\,z'\,,\,Az\,,\,Bz'\,,\,t)}{h(z\,,\,z'\,,\,Az\,,\,Bz'\,,\,t)}$$

$$= \frac{\min\{\mu(z, z', t)\nu(w, Bz', t), \mu^2(z, z', t), \mu(z, z', t)\nu(w, Az', t)\}}{k \min\{\nu(w, Bz', t), \mu(z, z', t), \nu(w, Az', t)\}}$$

$$\geq \frac{1}{k}\mu(z, z', t)$$

$$\Longrightarrow \mu(z, z', t) = 1$$

we can prove similarly that w is the unique common fixed point of BS and AT. This completes the proof of the theorem.

Corollary 3.3 *Let A, B, S and T be self mappings on the complete fuzzy metric space $(X, \mu, *)$ satisfying the inequalities*

$$k\,\mu(SAx, TBy, t) \geq \frac{f(x, y, t)}{h(x, y, t)} \quad \cdots \tag{21}$$

$$k\,\mu(BSx, ATy, t) \geq \frac{g(x, y, t)}{h(x, y, t)} \quad \cdots \tag{22}$$

for all x, y in X for which $f(x, y, t), g(x, y, t) < h(x, y, t) < 1$ where

$$f(x, y, t) = \min\{\mu(Sx, Ty, t)\mu(Ax, BSx, t), \mu(Sx, TBy, t)\mu(x, Sx, t),$$
$$\mu(x, y, t)\mu(SAx, Ty, t), \mu(x, Ty, t)\mu(x, ATy, t)\}$$

$$g(x, y, t) = \min\{\mu(x, Sx, t)\mu(x, y, t), \mu(y, TBy, t)\mu(y, Ax, t),$$
$$\mu(SAx, Ty, t)\mu(Ax, By, t), \mu(Ax, ATy, t)\mu(SAx, Sx, t)\}$$

$$h(x, y, t) = \min\{\mu(Ax, BSx, t), \mu(x, SAx, t), \mu(Sx, TBy, t),$$
$$\mu(By, ATy, t)\}$$

and $0 < k < 1$. If one of the mappings A, B, S or T is continuous , then SA and TB have a unique common fixed point u and BS and AT have a unique common fixed point v. Further , $Au = Bu = v$ and $Sv = Tv = u$.

4 Open Problem

Under what sufficient conditions one can prove related fixed points theorem for n pair of mappings. One can try to prove the existence of related fixed points theorem for two pair of mappings using an implicit relation which is a good idea since it covers several contractive conditions rather than one contractive condition.

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