Note on a nonlocal boundary value problem
with solutions positive on an interval

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Abstract
We reconsider the boundary value problem studied in [1] and prove the existence of sign changing solutions under more general conditions on the nonlinear term.

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1 Introduction
The following problem is studied in [1]

\[ u'' + g(t)f(u(t)) = 0, \quad 0 < t < 1 \] (1)
\[ u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta) \] (2)

where \( \eta \in (0,1) \), \( g \in C([0,1],[0,\infty)) \), \( f \in C(\mathbb{R}, [0,\infty)) \). The parameters \( \alpha \) and \( \beta \) are such that \( \alpha > 0, \beta > 0 \) and \( 1 + \alpha \neq \beta \). In a personal communication, Prof. J.R.L. Webb remarked that proper account was not taken of the fact that \( G \) is both discontinuous and changes sign, and, in fact, positive solutions may not exist under the given conditions, the main result in [1] is therefore not valid. We give a correction here and provide new results on existence of solutions that are positive on a sub-interval of \([0,1]\).

This type of problem has been studied by Infante-Webb in [2,3]. The case \( \alpha = 0, 0 \leq \beta < 1 - \eta \) was studied in detail in [2]. They established, using fixed point index theory, the existence of multiple nonzero solutions that are positive on a subinterval of \([0,1]\) but can change sign. Similar problems were
studied in [3] and in the interesting paper [4] which studied the model of a thermostat. In [4] the authors proved that there is loss of positivity as the parameter decreases, and proved a uniqueness result. For more results on this subject we refer to [3,5,6].

2 Preliminaries

Let $E = C[0,1]$, with the supremum norm $||y|| = \sup \{|y(t)|, t \in [0,1]\}$. To study the problem (1)-(2), we write it as an equivalent fixed point problem for the Hammerstein integral operator

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s)))ds$$

where the Green’s function is defined by

$$G(t,s) = \frac{(t + \alpha)(1-s)}{1 + \alpha - \beta} \left\{ \begin{array}{ll} \beta \frac{t+\alpha}{1+\alpha-\beta}, & s \leq \eta \\ 0, & s > \eta \end{array} \right.$$  \(\tag{3}\)

We give a correction to [1] here and provide new results for the case for $\alpha > 0$ and $0 < \beta < 1 - \eta$. We replace the previous assumptions that $f$ is either sublinear or superlinear by more general conditions and prove existence of sign changing solutions. The method used is to apply the theory of [2,3] which is based on the fixed point index for the compact map $T$ defined on a cone in the Banach space $E$.

3 Main Results

Following the theory of [2,3], an important step is to show that

$$|G(t,s)| \leq \Phi(s), \forall (t,s) \in [0,1] \times [0,1],$$  \(\tag{3}\)

$$G(t,s) \geq c\Phi(s), \forall t \in [0,b], \forall s \in [0,1].$$  \(\tag{4}\)

**Theorem 3.1** If $0 < \beta < 1 - \eta$ and $\alpha > 0$, then there exists a continuous function $\Phi(s) = \frac{(1+\alpha)}{1+\alpha-\beta} (1-s)$ on $[0,1]$, a real number $b \in (0,1 - \beta)$ and a constant $c = \frac{\alpha \min((1-b-\beta),(1-\eta-\beta))}{1+\alpha} \in (0,1)$ such that inequalities (3) and (4) hold.

Proof. We find upper and lower bounds of $G$.

**Upper bounds. Case 1.** $s > \eta$. If $t < s$ then $G(t,s) \geq 0$ and

$$G(t,s) = \frac{(t + \alpha)(1-s)}{1 + \alpha - \beta} \leq \frac{(1 + \alpha)}{1 + \alpha - \beta} (1-s) = \Phi(s)$$
If \( t \geq s \) then \( G(t, s) \geq 0 \) since

\[
G(t, s) = \frac{(t + \alpha)(1 - s)}{1 + \alpha - \beta} - (t - s) = \frac{(s + \alpha)(1 - t) + \beta(t - s)}{1 + \alpha - \beta} \geq 0
\]

and we have

\[
G(t, s) = \frac{(t + \alpha)(1 - s)}{1 + \alpha - \beta} - (t - s) \leq \frac{(t + \alpha)(1 - s)}{1 + \alpha - \beta}
\]

\[
\leq \frac{1 + \alpha}{1 + \alpha - \beta} (1 - s) = \Phi(s)
\]

**Case 2.** \( s < \eta \). If \( t < s \), then \( G(t, s) = \frac{(t + \alpha)(1 - s - \beta)}{1 + \alpha - \beta} \) and the function \( G \) is positive because we are taking \( \beta < 1 - \eta \). Consequently we have

\[
G(t, s) = \frac{(t + \alpha)(1 - s - \beta)}{1 + \alpha - \beta} \leq \frac{(1 + \alpha)}{1 + \alpha - \beta} (1 - s) = \Phi(s),
\]

(the case \( \eta > s > 1 - \beta \) is impossible since by hypothesis we have \( \beta < 1 - \eta \).)

If \( t \geq s \) then

\[
G(t, s) = \frac{(t + \alpha)(1 - s - \beta)}{1 + \alpha - \beta} - (t - s) = \frac{(s + \alpha)(1 - t - \beta)}{1 + \alpha - \beta}
\]

the function \( G \) is positive if \( t \leq 1 - \beta \), in this case we get

\[
G(t, s) = \frac{(s + \alpha)(1 - t - \beta)}{1 + \alpha - \beta} \leq \frac{(1 + \alpha)}{1 + \alpha - \beta} (1 - s) = \Phi(s)
\]

if \( t > 1 - \beta \) then \( G(t, s) \leq 0 \) and

\[
-G(t, s) = \frac{(s + \alpha)(-1 + t + \beta)}{1 + \alpha - \beta} \leq \frac{(1 + \alpha)}{1 + \alpha - \beta} (1 - \eta)
\]

\[
\leq \frac{(1 + \alpha)}{1 + \alpha - \beta} (1 - s) = \Phi(s)
\]

therefore \(|G(t, s)| \leq \frac{1 + \alpha}{1 + \alpha - \beta} (1 - s) = \Phi(s)\).

**Lower bounds.** Let \( b \in (0, 1 - \beta) \) and \( \alpha > 0 \), then for \( t \in [0, b] \) and \( s \in [0, 1] \) it yields: **Case 1.** \( s > \eta \), if \( t < s \) then

\[
G(t, s) = \frac{(t + \alpha)(1 - s)}{1 + \alpha - \beta} \geq \frac{\alpha(1 - s)}{1 + \alpha - \beta} = \frac{\alpha}{1 + \alpha} \Phi(s)
\]

If \( t \geq s \) then

\[
G(t, s) = \frac{(s + \alpha)(1 - t) + \beta(t - s)}{1 + \alpha - \beta} \geq \frac{\alpha(1 - t)}{1 + \alpha - \beta}
\]

\[
\geq \frac{\alpha(1 - b)}{1 + \alpha - \beta} (1 - s) = \frac{\alpha(1 - b)}{1 + \alpha} \Phi(s)
\]
Case 2. $s < \eta$. If $t < s$ then
\[
G(t, s) = \frac{(t+\alpha)(1-s-\beta)}{1+\alpha-\beta} \geq \frac{\alpha(1-\eta-\beta)}{1+\alpha-\beta} (1-s) = \frac{\alpha(1-\eta-\beta)}{1+\alpha} \Phi(s).
\]

If $t \geq s$ then
\[
G(t, s) = \frac{(s+\alpha)(1-t-\beta)}{1+\alpha-\beta} \geq \frac{\alpha(1-b-\beta)}{1+\alpha} \Phi(s). \square
\]

Define the operator $T : E \to E$ by $Tu(t) = \int_0^1 G(t, s)g(s)f(u(s))ds$.

**Notations.** Let $K$ be the cone $K = \left\{ u \in C[0, 1] : \min_{t \in [0, b]} u(t) \geq c \|u\| \right\}$ and define the following subsets of $K$,

$K_r = \{ u \in K : \|u\| < r \}$, $K_r = \{ u \in K : \|u\| \leq r \}$,

$K_{\rho, r} = \{ u \in K : \rho \leq \|u\| \leq r \}$, where $0 < \rho < r < \infty$.

We also let
\[
f^0 = \limsup_{u \to 0} \frac{f(u)}{u}, \quad f^\infty = \limsup_{|u| \to \infty} \frac{f(u)}{|u|}, \quad f_0 = \liminf_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \liminf_{u \to \infty} \frac{f(u)}{u},
\]

and
\[
f^{-\rho, \rho} = \sup_{u \in [-\rho, \rho]} \frac{f(u)}{\rho}, \quad f_{cp, \rho} = \inf_{u \in [cp, \rho]} \frac{f(u)}{\rho}.
\]

Let $m = \left(\max_{0 \leq t \leq 1} \int_0^1 |G(t, s)| g(s)ds\right)^{-1}$ and $M = \left(\min_{t \in [0, b]} \int_0^1 G(t, s)g(s)ds\right)^{-1}$.

As in [5] define the continuous function $q : E \to \mathbb{R}$, $q(u) = \min_{t \in [0, b]} u(t)$ and the set $\Omega_\rho = \{ u \in K : q(u) < c\rho \}$. It is clear that if $u \in \partial \Omega_\rho$, (the boundary relative to $K$) then $c\rho \leq u(t) \leq \rho$ for all $t \in [0, b]$. The set $\Omega_\rho$ was introduced in [5] for cones of positive functions, the case here was used in [2], for further properties of $\Omega_\rho$ see [5].

Now we state the existence results from [2] specialized to our case.

**Theorem 3.2** Assume that $0 < \beta < 1 - \eta$, and let $b \in (0, 1 - \beta)$ and suppose that $\int_0^b \Phi(s)g(s)ds > 0$. Then for $\alpha > 0$, the problem (1)-(2) has at least one nonzero solution, positive on $[0, b]$, if either

- **H1** $0 \leq f^0 < m$ and $M < f_\infty \leq \infty$, or
- **H2** $0 \leq f^\infty < m$ and $M < f_0 \leq \infty$,

and has two nonzero solutions, positive on $[0, b]$, if there is $\rho > 0$ such that either

- **S1** $0 \leq f^0 < m$, $f_{cp, \rho} \geq cM$, $u \neq Tu$ for $u \in \partial \Omega_\rho$ and $0 \leq f^\infty < m$ or
- **S2** $M < f_0 \leq \infty$, $f^{-\rho, \rho} \leq m$, $u \neq Tu$ for $u \in \partial K_\rho$ and $M < f_\infty \leq \infty$. 


Example 3.3 Consider the following BVP

\[ u'' + 1 = 0, \quad t \in (0, 1), \quad u(0) = u'(0), \quad u(1) = \frac{1}{3} u'(\frac{1}{2}). \]

Here \( \alpha = 1, \eta = 1/2, \beta = 1/3, \ g(t) = 1 \) on \( [0, 1] \) and \( f = 1 \) on \( \mathbb{R} \). Then \( f_0 = \infty, \ f^\infty = 0 \) so this is a sublinear case. Choosing \( b = \frac{3}{5} < 2/3 \), then \( \int_0^1 \Phi(s)g(s)ds > 0 \), \( (H_2) \) holds and the Theorem gives that the BVP has at least one solution which is positive on \( [0, 3/5] \). In fact, the solution is easily found to be \( u(t) = \frac{1}{5} + \frac{1}{5} t - \frac{t^2}{2} \) which is positive on \( [0, (1 + \sqrt{11})/5] \approx [0, 0.863] \).

Example 3.4 Let \( f(u) = \begin{cases} 10^3 u^2, |u| < 1 \\ -999u + 1999, 1 \leq |u| < 2 \\ 1, |u| \geq 2 \end{cases} \)

Choosing \( g = 1, \alpha = 1, \eta = 1/2, \beta = 1/3 \) and \( b = 3/5 \) as in Example 2.4, then \( f^0 = 0 \). \( f^\infty = 0, \ c = \frac{1}{30}, \ M = \frac{1250}{79} \approx 15.823 \). Let \( \rho = \frac{1}{2} \) then \( f_{c\rho, \rho} = \frac{10-10}{18} \approx 0.5556 \geq cM = \frac{1250}{79 \times 30} \approx 0.52743 \). Assume that there exists \( u \in \partial \Omega_\rho \) such that \( u = Tu \), then for \( t \in [0, 3/5] \) it yields by property of \( \Omega_\rho \),

\[
\int_0^t G(t,s)f(u(s))ds \geq \frac{3}{5} \int_0^3 G(t,s)f(u(s))ds
\]

\[
= 10^3 \int_0^\frac{3}{5} G(t,s)u^2(s)ds \geq 10^3 c^2 \rho^2 \int_0^\frac{3}{5} G(t,s)ds.
\]

Taking the minimum over \( t \in [0, \frac{3}{5}] \), we get

\[ c\rho \geq 10^3 c^2 \rho^2 / M, \text{ that is } c\rho \leq M/10^3. \]

Since \( c\rho = 1/60 > M/10^3 \), this is a contradiction, consequently \( (S1) \) holds and the BVP has at least two nonzero solutions positive on \( [0, \frac{3}{5}] \).

4 Open problem

In the present note we have established the existence of nonzero solutions changing sign and positive on a subinterval of \( [0, 1] \), in the case \( \alpha > 0, 0 < \beta < 1 - \eta \) and under more general conditions on \( f \). The existence of nonzero solutions for problem (1)-(2) could be investigated for other cases such as \( \alpha > 0 \) and \( \beta < 0 \).

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References


