Int. J. Open Problems Comput. Math. , Vol. 5, No. 3, September, 2012
ISSN 2074-2827; Copyright ©ICSRS Publication, 2012
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Note on some nonlinear integral inequalities in two independent variables on time scales and applications

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Abstract

Our aim in this note is to investigate some nonlinear integral inequalities in two independent variables on time scales.The inequalities given here can be used as handy tools to study the properties of certain partial dynamic equations on time scales.

Keywords: *Dynamic equations, time scale, integral inequality.*
Mathematics subject classification (2011): 26D15, 26D20, 39A12.

1 introduction

Motivated by the paper [1], many authors have extended some fundamental integral inequalities used in the theory of differential and integral equations on time scales.

In this paper,we investigate some nonlinear integral inequalities in two independent variables on time scales.The obtained inequalities can be used as important tools in the study of certain properties of partial dynamic equations on time scales.

1.1 Preliminaries on time scales

In this section, we begin by giving some necessary materials for our study.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers.The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T}$ for all $t \in \mathbb{T}$.In this definition we put $\inf(\emptyset) = \sup \mathbb{T}$,where $\emptyset$ is the empty set.If $\sigma(t) > t$,then we say that t is right-scattered.If

$\sigma(t) = t$ and $t < \sup \mathbb{T}$, then we say that t is right-dense.The backward jump operator, left–scattered and left–dense points are defined in a similar way.The graininess $\mu : \mathbb{T} \to [0, \infty)$ is de fined by $\mu(t) = \sigma(t) - t$.

C_{rd} denotes the set of rd-continuous functions.$\mathfrak{R}$ denotes the set of all regressive and $rd-$continuous functions.

We define the set of all positively regressive functions by

$$\mathfrak{R}^+ = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Throughout this paper,we always assume that $\mathbb{T}_1$and $\mathbb{T}_2$ are time scales,$\Omega = \mathbb{T}_1\times\ \mathbb{T}_2$ and we write $x^{\Delta_t}(t, s)$ for the partial delta derivatives of $x(t, s)$ with respect to t.

Theorem 1.1. *If $p \in \mathfrak{R}$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_p(., t_0)$is for the unique solution of the initial value problem*

$$x^{\Delta} = p(t)x, x(t_0) = 1 \text{ on } \mathbb{T}. \tag{1.1}$$

Theorem 1.2. *If $p \in \mathfrak{R}$ then*

1-$e_p(t, t) \equiv 1$ and $e_0(t, s) \equiv 1$;
2-If $p \in \mathfrak{R}^+$ then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.

Remark 1.3. *: Clearly, the exponential function is given by*

$$e_p(t, s) = e^{\int_s^t \ p(\eta)d\eta} \tag{1.2}$$

for $s, t \in \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function if $\mathbb{T} = \mathbb{R}$.

2 Main result

Before giving our main results,we introduce The following lemmas which are useful in our main results.

Lemma 2.1. *For $x \in \mathbb{R}_+,\ y \in \mathbb{R}_+,\ 1/p + 1/q = 1,$ with $p > 1$,we have,*

$$x^{1/p}y^{1/q} \leq x/p + y/q. \tag{2.1}$$

Lemma 2.2. *:Assume that $p \geq 1, a \geq 0.$Then*

$$a^{\frac{1}{p}} \leq \frac{1}{p}K^{\frac{1-p}{p}}a + \frac{p-1}{p}K^{\frac{1}{p}}. \tag{2.2}$$

for any $K > 0$

Proof. : see [6] . □

Lemma 2.3. *Assume that* $a \geq 0, p \geq q \geq 0$ *and* $p \neq 0$,*then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{2.3}$$

for any $K > 0$.

Proof. : if $q = 0$,it is easy to see that inequality (2.3) holds.So we only prove inequality (2.3) holds in the case of $q > 0$.

Let $b = \dfrac{p}{q}$,then $b \geq 1$ by Lemma 2.2, we obtain the result.

Theorem 2.4. (*Comparison Theorem* [1]) *.Suppose* $u, b \in C_{rd}$, $a \in \Re^+$ *. if* □

$$u^{\Delta}(t) \leq a(t)u(t) + b(t), t \in \mathbb{T} .$$

Then,

$$u(t) \leq u(t_0)e_a(t,t_0) + \int_{t_0}^{t} e_a(t,\sigma(\eta))b(\eta)\Delta\eta \ , \ t \in \mathbb{T} .$$

Now we state the main results of this work.

Theorem 2.5. *Let* $u(t,s), a(t,s), b(t,s)$ *and* $h_i(t,s) (i = 1,...n)$ *are non-negative functions defined for* $t, s \in \Omega$ *that are right-dense continuous for* $t, s \in \Omega$.*If there exists a series of positive real numbers* $p_1, p_2, ..., p_n$ *such that* $p \geq p_i > 0, i = 1, 2, ..., n$,*then*

$$u^p(t,s) \leq a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} \sum_{i=1}^{i=n} h_i(\tau,\eta) u^{p_i}(\tau,\eta)\Delta\eta\Delta\tau \ \textit{for} \ (t,s) \in \Omega \ , \tag{2.4}$$

implies

$$u(t,s) \leq (a(t,s) + b(t,s)m(t,s)e_{y_{(.,s)}}(t,t_0))^{\frac{1}{p}}, \tag{2.5}$$

where

$$m(t,s) = \int_{t_0}^{t} \int_{s_0}^{s} \sum_{i=1}^{i=n} h_i(\tau,\eta)(\frac{p_i}{p} a(\tau,\eta) + \frac{p-p_i}{p})\Delta\eta\Delta\tau, \tag{2.6}$$

$$y(t,s) = \int_{s_0}^{s} b(t,\eta) \sum_{i=1}^{i=n} \frac{p_i}{p} h_i(t,\eta)\Delta\eta, (t,s) \in \Omega. \tag{2.7}$$

Proof. Define a function $z(t,s)$ by

$$z(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}h_i(\tau,\eta)u^{p_i}(\tau,\eta)\Delta\eta\Delta\tau, \tag{2.8}$$

then (2.4) can be written as

$$u^p(t,s)\le a(t,s)+b(t,s)z(t,s). \tag{2.9}$$

From (2.9) , by Lemma 2.1, we get

$$u^{p_i}(t,s)=(u^p(t,s))^{\frac{p_i}{p}}\le(a(t,s)+b(t,s)z(t,s))^{\frac{p_i}{p}}\le\frac{p_i}{p}\,(a(t,s)+b(t,s)z(t,s))+\frac{p-p_i}{p}, \tag{2.10}$$

It follows from (2.8) and (2.10) that

$$\begin{aligned} z(t,s)\le\ & \int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}\ h_i(\tau,\eta)\left[\frac{p_i}{p}(a(\tau,\eta)+b(\tau,\eta)z(\tau,\eta))+\frac{p-p_i}{p}\right]\Delta\eta\Delta\tau,\\ =\ & \int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}(\frac{p_i}{p}\ h_i(\tau,\eta)b(\tau,\eta)z(\tau,\eta))\Delta\eta\Delta\tau+m(t,s), \end{aligned} \tag{2.11}$$

where $m(t,s)$ is defined by (2.6). □

Let $\varepsilon>0$ be given, and from (2.11), we obtain

$$\frac{z(t,s)}{m(t,s)+\varepsilon}\le 1+\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}(\frac{p_i}{p}h_i(\tau,\eta)b(\tau,\eta)\frac{z(\tau,\eta)}{m(\tau,\eta)+\varepsilon})\Delta\eta\Delta\tau, \tag{2.12}$$

Define a function $v(t,s)$ by

$$v(t,s)=1+\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}(\frac{p_i}{p}h_i(\tau,\eta)b(\tau,\eta)\frac{z(\tau,\eta)}{m(\tau,\eta)+\varepsilon})\Delta\eta\Delta\tau \text{ for } (t,s)\in\Omega, \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$z(t,s)\le(m(t,s)+\varepsilon)v(t,s). \tag{2.14}$$

From (2.13), a delta derivative with respect to t yields

$$\begin{aligned} v^{\Delta_t}(t,s) &= \int\limits_{s_0}^{s}\sum_{i=1}^{i=n}(\frac{p_i}{p}h_i(t,\eta)b(t,\eta)\frac{z(t,\eta)}{m(t,\eta)+\varepsilon})\Delta\eta, \\ &\leq \int\limits_{s_0}^{s}\sum_{i=1}^{i=n}(\frac{p_i}{p}h_i(t,\eta)b(t,\eta)v(t,\eta)\Delta\eta \leq (\int\limits_{s_0}^{s}\sum_{i=1}^{i=n}(\frac{p_i}{p}h_i(t,\eta)b(t,\eta)\Delta\eta)v(t,s), \\ &= y(t,s)v(t,s) \text{ for } (t,s)\in\Omega, \end{aligned} \tag{2.15}$$

where $y(t,s)$ is is defined by (2.7) with $v(t_0,s)=1$ and $y(t,s)>0$.
Using Theorem 2.4 from (2.15), we obtain

$$v(t,s) \leq e_{y_{(.,s)}}(t,t_0),(t,s)\in\Omega. \tag{2.16}$$

It follows from (2.9), (2.14) and (2.16) that

$$\begin{aligned} u(t,s) &\leq (a(t,s)+b(t,s)z(t,s))^{\frac{1}{p}}, \\ &\leq (a(t,s)+b(t,s)(m(t,s)+\varepsilon)v(t,s))^{\frac{1}{p}}, \\ &\leq (a(t,s)+b(t,s)(m(t,s)+\varepsilon)e_{y_{(.,s)}}(t,t_0))^{\frac{1}{p}} \text{for } (t,s)\in\Omega. \end{aligned} \tag{2.17}$$

Letting $\varepsilon\to 0$ in (2.17) the Theorem is proved.

Remark 2.6. *:If we take $n=2$, $p\geq 1$ and $p_1=p, p_2=1$, $h_1=g$ and $h_2=h$, then the inequality established in Theorem 2.5 becomes the inequality given by in [7, Theorem 2.3].*

Theorem 2.7. *Assume that all assumptions of Theorem 2.5 hold. If $a(t,s)>0$ and is nondecreasing for $(t,s)\in\Omega$, then*

$$u^p(t,s) \leq a^p(t,s)+b(t,s)\int_{t_0}^{t}\int\limits_{s_0}^{s}\sum_{i=1}^{i=n}h_i(\tau,\eta)u^{p_i}(\tau,\eta)\Delta\eta\Delta\tau \text{ for } (t,s)\in\Omega\ , \tag{2.18}$$

implies

$$u(t,s)\leq a(t,s)(1+b(t,s)n(t,s)e_{w_{(.,s)}}(t,t_0))^{\frac{1}{p}}, \tag{2.19}$$

where

$$n(t,s)=\int_{t_0}^{t}\int\limits_{s_0}^{s}\sum_{i=1}^{i=n\prime\prime}H_i(\tau,\eta)\Delta\eta\Delta\tau, \tag{2.20}$$

$$w(t,s)=\int_{s_0}^{s} b(t,\eta)\sum_{i=1}^{i=n\prime\prime}\frac{p_i}{p}H_i(t,\eta)\Delta\eta, \tag{2.21}$$

and

$$H_i(t,s)=h_i(t,s)a^{p_i-p}(t,s). \tag{2.22}$$

Proof. : Nothing that $a(t,s)>0$ is nondecreasing for $(t,s)\in\Omega$, from (2.18) we have

$$\left(\frac{u(t,s)}{a(t,s)}\right)^p \le 1+b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}h_i(\tau,\eta)a(\tau,\eta)^{p_i-p}\left(\frac{u(\tau,\eta)}{a(\tau,\eta)}\right)^{p_i}\Delta\eta\Delta\tau. \tag{2.23}$$

By Theorem 2.5, and from (2.23), we easily obtain the result. This completes the proof of Theorem 2.7. □

Remark 2.8. *Letting $n=2$, $p\ge 1$ and $p_1=p, p_2=1$, $h_1=g$ and $h_2=h$, then the inequality established in Theorem 2.7 becomes the inequality given in [7, Theorem 2.5].*

Now by using Lemmas 2.2 and 2.3, other estimates are established.

Theorem 2.9. *: Assume that all assumptions of Theorem 2.5 hold,then*

$$u^p(t,s)\le a(t,s)+b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}h_i(\tau,\eta)u^{p_i}(\tau,\eta)\Delta\eta\Delta\tau \ \textit{for}\ (t,s)\in\Omega\ , \tag{2.24}$$

implies

$$u(t,s)\le \left(a(t,s)+b(t,s)m^*(t,s)e_{y^*_{(.,s)}}(t,t_0)\right)^{\frac{1}{p}}, \tag{2.25}$$

where

$$m^*(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}h_i(\tau,\eta)\left(\frac{p_i}{p}K^{\frac{p_i-p}{p}}a(\tau,\eta)+\frac{p-p_i}{p}K^{\frac{p_i}{p}}\right)\Delta\eta\Delta\tau, \tag{2.26}$$

and

$$y^*(t,s)=\int_{s_0}^{s}\sum_{i=1}^{i=n}\frac{p_i}{p}K^{\frac{p_i-p}{p}}h_i(t,\eta)b(t,\tau)\Delta\eta, (t,s)\in\Omega, \tag{2.27}$$

Proof. Define a function $z(t,s)$ by

$$z(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}h_i(\tau,\eta)u^{p_i}(\tau,\eta)\Delta\eta\Delta\tau,$$

□

By Lemma 2.3, we get

$$\begin{aligned} u^{p_i}(t,s)= & \quad (u^p(t,s))^{\frac{p_i}{p}}\le(a(t,s)+b(t,s)z(t,s))^{\frac{p_i}{p}} \qquad (2.28)\\ \le & \quad \frac{p_i}{p}\,K^{\frac{p_i-p}{p}}(a(t,s)+b(t,s)z(t,s))+\frac{p-p_i}{p}K^{\frac{p_i}{p}},\end{aligned}$$

from the proof of Theorem 2.5,we obtain the required inequality in (2.25) where $m^*(t,s)$ is defined by (2.26) and $y^*(t,s)$ is defined by (2.27)

Theorem 2.10. *Assume that all assumptions of Theorem 2.5 hold. If $a(t,s)>0$ is nondecreasing for $(t,s)\in\Omega$, then*

$$u^p(t,s)\le a^p(t,s)+b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n}h_i(\tau,\eta)u^{p_i}(\tau,\eta)\Delta\eta\Delta\tau \text{ for } (t,s)\in\Omega,$$

implies

$$u(t,s)\le a(t,s)\left[1+b(t,s)n^*(t,s)e_{w^*_{(.,s)}}(t,t_0))^{\frac{1}{p}}\right], \qquad (2.29)$$

where

$$n^*(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}\sum_{i=1}^{i=n\prime\prime}H_i(\tau,\eta)(\frac{p_i}{p}K^{\frac{p_i-p}{p}}+\frac{p-p_i}{p}K^{\frac{p_i}{p}})\Delta\eta\Delta\tau \text{ for any } K>0, \qquad (2.30)$$

$$w^*(t,s)=\int_{s_0}^{s}\sum_{i=1}^{i=n\prime\prime}\frac{p_i}{p}K^{\frac{p_i-p}{p}}H_i(t,\eta)b(t,\eta)\Delta\eta,$$

and

$$H_i(t,s)=h_i(t,s)a^{p_i-p}(t,s).$$

Proof. : The proof is similar to the proof given in theorem 2.7. □

Theorem 2.11. *: Assume that $u(t,s)$, $a(t,s)$ and $b(t,s)$ are nonnegative functions defined for $(t,s)\in\Omega$ that are right-dense continuous for $(t,s)\in\Omega$, and $p>1$ is a real constant. If $f:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is right-dense continuous on Ω and continuous on $\mathbb{R}_+$ such that*

$$0\le f(t,s,x)-f(t,s,y)\le h(t,s,y)(x-y), \tag{2.31}$$

for $(t,s)\in\Omega$, $x\ge y\ge 0$ where $h:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is right-dense continuous on Ω and continuous on $\mathbb{R}_+$, if

$$u^p(t,s)\le a(t,s)+b(t,s)\int_{t_0}^{t}\int_{s_0}^{s} f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau \text{ for } (t,s)\in\Omega, \tag{2.32}$$

then

$$u(t,s)\le (a(t,s)+b(t,s)l^*(t,s)e_{w^*_{(.,s)}}(t,t_0))^{\frac{1}{p}}, \tag{2.33}$$

where

$$l^*(t,s)=\quad \int_{t_0}^{t}\int_{s_0}^{s} f(\tau,\eta,\frac{K^{\frac{1-p}{p}}a(\tau,\eta)+(p-1)K^{\frac{1}{p}}}{p})\Delta\eta\Delta\tau, \tag{2.34}$$

$$w^*(t,s)=\quad \int_{s_0}^{s} h(t,\eta,\frac{K^{\frac{1-p}{p}}a(t,\eta)+(p-1)K^{\frac{1}{p}}}{p})\frac{K^{\frac{1-p}{p}}b(t,\eta)}{p}\Delta\eta, \tag{2.35}$$

Proof. : Define a function $z(t,s)$ by □

$$z(t,s)=\int_{t_0}^{t}\int_{s_0}^{s} f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau \text{ for } (t,s)\in\Omega,$$

Fom (2.32) ,we have

$$u(t,s)\le (a(t,s)+b(t,s)z(t,s))^{\frac{1}{p}},$$

using Lemma 2.2, we obtain

$$u(t,s)\le \frac{1}{p}K^{\frac{1-p}{p}}(a(t,s)+b(t,s)z(t,s))+\frac{p-1}{p}K^{\frac{1}{p}}$$

noting the assumptions on f, we have :

$$z(t,s)\le\int_{t_0}^{t}\int\limits_{s_0}^{s}\left[\begin{array}{c} f(\tau,\eta,\frac{K^{\frac{1-p}{p}}a(\tau,\eta)+(p-1)K^{\frac{1}{p}}}{p}+\frac{K^{\frac{1-p}{p}}b(\tau,\eta)z(\tau,\eta)}{p})- \\ -f(\tau,\eta,\frac{K^{\frac{1-p}{p}}a(\tau,\eta)+(p-1)K^{\frac{1}{p}}}{p})+ \\ +f(\tau,\eta,\frac{K^{\frac{1-p}{p}}a(\tau,\eta)+(p-1)K^{\frac{1}{p}}}{p})\Delta\eta\Delta\tau, \end{array}\right]$$

$$\le l^*(t,s)+\int_{t_0}^{t}\int\limits_{s_0}^{s}h(\tau,\eta,\frac{K^{\frac{1-p}{p}}a(\tau,\eta)+(p-1)K^{\frac{1}{p}}}{p})\frac{K^{\frac{1-p}{p}}b(\tau,\eta)}{p}z(\tau,\eta)\Delta\eta\Delta\tau,$$

where $l^*(t,s)$ is defined by (2.34) and is nonnegative, right-dense continuous, and nondecreasing for $(t,s)\in\Omega$. The remainder of the proof is similar to that of Theorem 2.5.

Theorem 2.12. *: Assume that $u(t,s)$,$a(t,s)$ and $b(t,s)$ are nonnegative functions defined for $(t,s)\in\Omega$ that are right-dense continuous for $(t,s)\in\Omega$, and $p>1$ is a real constant. If $f:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is right-dense continuous on Ω and continuous on $\mathbb{R}_+$ such that*

$$0\le f(t,s,x)-f(t,s,y)\le h(t,s,y)\Phi^{-1}(x-y),$$

for $(t,s)\in\Omega$, $x\ge y\ge 0$ where $h:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is right-dense continuous on Ω and continuous on $\mathbb{R}_+$, $\Phi^{-1}$is the inverse function of Φ, and

$$\Phi^{-1}(x.y)\le\Phi^{-1}(x)\Phi^{-1}(y)\ ,$$

if

$$u^p(t,s)\le a(t,s)+b(t,s)\Phi(\int_{t_0}^{t}\int\limits_{s_0}^{s}f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau)\text{ for }(t,s)\in\Omega,\quad(2.36)$$

then

$$u(t,s)\le\left\{a(t,s)+b(t,s)\Phi(l^*(t,s)e_{w^*_{(.,s)}}(t,t_0))\right\}^{\frac{1}{p}},\qquad(2.37)$$

where $l^*(t,s)$ is define by (2.34) and

$$w^*(t,s)=\int\limits_{s_0}^{s}h(t,\eta,\frac{K^{\frac{1-p}{p}}a(t,\eta)+(p-1)K^{\frac{1}{p}}}{p})\Phi^{-1}(\frac{K^{\frac{1-p}{p}}b(t,\eta)}{p})\Delta\eta.$$

Proof. :Define a function $z(t,s)$ by

$$z(t,s)=\int_{t_0}^{t}\int_{s_0}^{s} f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau \text{ for } (t,s)\in\Omega,$$

$$u(t,s)\le (a(t,s)+b(t,s)\Phi(z(t,s)))^{\frac{1}{p}},$$

using Lemma 2.2,we obtain :

$$u(t,s)\le \frac{1}{p}K^{\frac{1-p}{p}}(a(t,s)+b(t,s)\Phi(z(t,s)))+\frac{p-1}{p}K^{\frac{1}{p}},$$

by the assumptions on f and Φ,we have
$z(t,s)\le l^{*}(t,s)+$

$$+\int_{t_0}^{t}\int_{s_0}^{s} h(\tau,\eta,\frac{K^{\frac{1-p}{p}}a(\tau,\eta)+(p-1)K^{\frac{1}{p}}}{p})\Phi^{-1}(\frac{\frac{1}{p}K^{\frac{1-p}{p}}b(\tau,\eta)}{p})z(\tau,\eta)\Delta\eta\Delta\tau,$$

where $l^{*}(t,s)$ is defined by (2.34).The remainder of the proof is similar to that of Theorem 2.5. □

3 application :

In this section we give an application of Theorem 2.9. We Consider the following partial dynamic equation on time scales

$$(u^{p}(t,s))^{\Delta_t\Delta_s}=H(t,s,u(t,s))+r(t,s) \text{ for } (t,s)\in\Omega \tag{3.1}$$

with the initial boundary conditions

$$u(t,s_0)=\alpha(t),u(t_0,s)=\beta(s),u(t_0,s_0)=\omega \tag{3.2}$$

where $H:\mathbb{T}_1\times\mathbb{T}_2\times\mathbb{R}\to\mathbb{R}$ is right-dense continuous on Ω and continuous on $\mathbb{R}$, $r:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ is right-dense continuous on Ω, $\alpha:\mathbb{T}_1\to\mathbb{R}$ and $\beta:\mathbb{T}_2\to\mathbb{R}$ are right-dense continuous, and $\omega\in\mathbb{R}$ is a constant.

Assume that

$$|H(t,s,u)|\le\sum_{i=1}^{i=n}h_i(t,s)\,|u|^{p_i}, \tag{3.3}$$

$h_i(t,s)(i=1,...n)$ are nonnegative right-dense continuous functions defined for $(t,s)\in\Omega$.

If $u(t,s)$ is a solution of (3.1) , (3.2) then $u(t,s)$ satisfies

$$|u(t,s)| \leq \left\{a(t,s) + m(t,s)e_{y(.s)}(t,t_0)\right\}^{\frac{1}{p}}, \tag{3.4}$$

where

$$a(t,s) = |\alpha^p(t) + \beta^p(s) - \omega^p| + \int_{t_0}^{t} \int_{s_0}^{s} |r(\tau,\eta)| \,\Delta\eta\Delta\tau, \tag{3.5}$$

$$y(t,s) = \int_{s_0}^{s} \sum_{i=1}^{i=n\prime\prime} \frac{p_i}{p} K^{\frac{p_i-p}{p}} h_i(t,\eta)\Delta\eta, \text{for } (t,s) \in \Omega, \tag{3.6}$$

and

$$m(t,s) = \int_{t_0}^{t} \int_{s_0}^{s} \sum_{i=1}^{i=n} h_i(\tau,\eta)(\frac{p_i}{p} K^{\frac{p_i-p}{p}} a(\tau,\eta) + \frac{p-p_i}{p} K^{\frac{p_i}{p}})\Delta\eta\Delta\tau,$$

Proof. : the solution of (3.1),(3.2) satisfies : □

$$u^p(t,s) = \alpha^p(t) + \beta^p(s) - \omega^p + \int_{t_0}^{t} \int_{s_0}^{s} H(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau + \int_{t_0}^{t} \int_{s_0}^{s} r(\tau,\eta)\Delta\eta\Delta\tau$$

Therefore,

$$|u^p(t,s)| \leq a(t,s) + \int_{t_0}^{t} \int_{s_0}^{s} h_i(\tau,\eta) \,|u^{p_i}(\tau,\eta)| \,\Delta\eta\Delta\tau.$$

Applying theorem 2.9, we easily obtain (3.4).

4 Open problem

In this work, we have established some nonlinear integral inequalities in two independent variables on time scales.The inequalities given here can be used as handy tools to study the properties of certain partial dynamic equations on time scales, for example the partial dynamic equation on time scales $(3.1) - (3.2)$.

It will be interesting to estimate the solution of $(3.1) - (3.2)$ in the cases $p_\star = \min\{p_i, i = 1, ...n\} \leq p < p^\star = \max\{p_i, i = 1, ...n\}$ and $p < p_\star$.

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