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Nabla $1-$Forms on $n-$Dimensional Time Scales

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Abstract

In this paper, nabla $1-$***forms for multivariable functions on*** $n-$***dimensional time scale is presented.***

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1 Introduction

The unification and extension of continuous calculus, discrete calculus, $q-$calculus, and indeed arbitrary real-number calculus to time scale calculus was first accomplished by Hilger in his PhD thesis [11]. This theory is very important and useful in the mathematical modelling of several important dynamic processes. As a result the theory of dynamic systems on time scales is developed in ([1]-[10]).

There are a number of differences between the calculus one and of two variables. The calculus of functions of three or more variables differs only slightly from that of two variables. Bohner and Guseinov have published a paper about the partial differentiation on time scale. Here, authors introduced partial delta and nabla derivative and the chain rule for two variables functions on time scale and also the concept of the directional derivative [8].

In [2], we have investigated the calculus of multivariable functions on $n-$dimensional time scale. In that paper, we introduced partial delta derivative and the chain rule for $n-$variables functions on $n-$dimensional time scale and also the concept of the directional derivative and to application a Differential geometry. In [9], the authors study some geometrical structures such that tangent vector, vector fields, curves and mappings on $n-$dimensional time scales. Moreover, they investigated some properties of these structures.

The present paper deals with the nabla $1-$forms which is another geometrical structure on $n-$dimensional time scale $\wedge^n$. The paper is organized as follows. In Section 2, we give a brief account of time scale calculus, partial nabla derivatives for multivariable functions on $n-$dimensional time scales $\wedge^n$ and offer several concepts related to $\nabla-$differentiability which will be use later. Section 3 is devoted to nabla1$-$forms and its properties on $\wedge^n$. In Section, 4 an open problem is given.

2 Preliminaries

The following definitions and theorems will serve as a short primer on time scale calculus; they can be found ([6], [7]). A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. Within that set, define the jump operators $\rho, \sigma : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} \quad \text{and} \quad \sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\rho(t) = t$ and $\rho(t) < t$, then the point $t \in \mathbb{T}$ is left-dense, left-scattered. If $\sigma(t) = t$ and $\sigma(t) > t$, then the point $t \in \mathbb{T}$ is right-dense, right-scattered. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T}-\{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T}-\{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The so-called graininess functions are $\mu(t) := \sigma(t) - t$ and $v(t) := t - \rho(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$, the delta derivative of f at t, denoted $f^{\nabla}(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\nabla}(t)[\sigma(t) - s] \right| \leq \varepsilon \left| \rho(t) - s \right|,$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, $f^{\nabla} = f'$, the usual derivative; for $\mathbb{T} = \mathbb{Z}$ the delta derivative is the backward difference operator, $f^{\Delta}(t) = f(t+1) - f(t)$; in the case of $q-$difference equations with $q>1$,

$$f^{\nabla}(t)=\frac{f(qt)-f(t)}{(q-1)t}, \qquad f^{\nabla}(0)=\lim_{s\to 0}\frac{f(s)-f(0)}{s}.$$

If $f,g:\mathbb{T}\to\mathbb{R}$ are $\nabla-$differentiable at $t\in\mathbb{T}_k$, then

(i) $f+g$ is $\nabla-$ differentiable at t and
$$(f+g)^{\nabla}(t)=f^{\nabla}(t)+g^{\nabla}(t).$$

(ii) For any constant c, $c.f$ is $\nabla-$ differentiable at t and
$$(cf)^{\nabla}(t)=cf^{\nabla}(t).$$

(iii) $f.g$ is $\nabla-$ differentiable at t and
$$\begin{aligned}(fg)^{\nabla}(t)&=f^{\nabla}(t)g(t)+f(\rho(t))g^{\nabla}(t)\\&=g^{\nabla}(t)f(t)+g(\rho(t))f^{\nabla}(t).\end{aligned}$$

(iv) If $g(t).g(\rho(t))\neq 0$ then $\dfrac{f}{g}$ is $\nabla-$ differentiable at t and
$$\left(\frac{f}{g}\right)^{\nabla}(t)=\frac{f^{\nabla}(t)g(t)-f(t)g^{\nabla}(t).}{g(t).g(\rho(t))}.$$

Let T be a time scale and $\nu:\mathbb{T}\to\mathbb{R}$ be a strictly increasing function such that $\overline{\mathbb{T}}=\nu(\mathbb{T})$ is also a time scale. By $\overline{\rho}$ we denote the jump function on $\overline{\mathbb{T}}$, and by $\overline{\Delta}$ we denote the derivative on $\overline{\mathbb{T}}$. Then

$$\nu\circ\rho=\overline{\rho}\circ\nu.$$

(Chain Rule) Assume $\nu:\mathbb{T}\to\mathbb{R}$ is strictly increasing and $\overline{\mathbb{T}}=\nu(\mathbb{T})$ is a time scale. Let $\omega:\overline{\mathbb{T}}\to\mathbb{R}$. If $\nu^{\nabla}(t)$ and $\omega^{\overline{\nabla}}(\nu(t))$ exist for $t\in\mathbb{T}_k$, then $(\omega\circ\nu)^{\nabla}$ exist at t and satisfy the chain rule

$$(\omega\circ\nu)^{\nabla}=(\omega^{\overline{\nabla}}\circ\nu)\nu^{\nabla} \text{ at } t.$$

Many other information concerning time scales and dynamic equations on time scales can be found in the books ([6], [7]).

After already, in this section, for the convenience of readers, we repeat the relevant material from [2] and [9].

2.1 Partial Differentiation on $n-$Dimensional Time Scales

Let $n\in N$ be fixed and for each $i\in\{1,2,...,n\}$, $\mathbb{T}_i$ denote a time scale. Let us set

$$\wedge^n=\mathbb{T}_1\times\mathbb{T}_2\times...\times\mathbb{T}_n=\{(l_1,l_2,...,l_n):l_i\in\mathbb{T}_i \text{ for all } i\in\{1,2,...,n\}\}.$$

We call $\wedge^n$ an $n-$ dimensional time scale. the set $\wedge^n$ is a complete metric space with the metric d defined by

$$d(t,s)=\left(\sum_{i=1}^{n}|t_i-s_i|^2\right)^{\frac{1}{2}},\ \forall t,s\in\wedge^n.$$

Let σ_i and ρ_i denote the backward and backward jump operators in $\mathbb{T}$, respectively. Remember that for $u\in\mathbb{T}_i$ the backward jump operator $\sigma_i:\mathbb{T}_i\to\mathbb{T}_i$ is defined by

$$\sigma_i(u)=\inf\{v\in\mathbb{T}_i:v>u\},$$

and the backward jump operator $\rho_i:T_i\to T_i$ is defined by

$$\rho_i(u)=\sup\{v\in\mathbb{T}_i:v<u\}.$$

In this definition, we put $\sigma_i(\max\mathbb{T}_i)=\max\mathbb{T}_i$ if $\mathbb{T}_i$ has a finite maximum, and $\rho_i(\min\mathbb{T}_i)=\min\mathbb{T}_i$ if $\mathbb{T}_i$ has a finite minimum. If $\sigma_i(u)>u$, then we say that u is right-scattered in $\mathbb{T}_i$, while any u with $\rho_i(u)<u$ is called left-scattered in $\mathbb{T}_i$. Also, if $u<\max\mathbb{T}_i$ and $\sigma_i(u)=u$, then u is called right-dense in $\mathbb{T}_i$, and if $u>\min\mathbb{T}_i$ and $\rho_i(u)=u$ then u is called left-dense in $\mathbb{T}_i$. If $\mathbb{T}_i$ has a left-scattered maximum M, then we define $\mathbb{T}_i^k=\mathbb{T}_i-\{M\}$, otherwise $\mathbb{T}_i^k=\mathbb{T}_i$. If $\mathbb{T}_i$ has a right-scattered minimum m, then we define $(\mathbb{T}_i)_k=\mathbb{T}_i-\{m\}$, otherwise $(\mathbb{T}_i)_k=\mathbb{T}_i$.

Let $f:\wedge^n\to\mathbb{R}$ be a function. The partial nabla derivative of f with respect to $t_i\in(\mathbb{T}_i)_k$ is defined as the limit

$$\lim_{\substack{s_i\to t_i\\ s_i\neq\rho_i(t_i)}}\frac{f(t_1,t_2,...,t_{i-1},\rho_i(t_i),t_{i+1},...,t_n)-f(t_1,t_2,...,t_{i-1},s_i,t_{i+1},...,t_n)}{\rho_i(t_i)-s_i}$$
$$=\frac{\partial f(t)}{\nabla_i t_i}.$$

Higher order partial nabla derivatives are defined similarly.

We say that a function $f:\wedge^n\to\mathbb{R}$ is completely $\nabla-$differentiable at the point $t^0\in(\mathbb{T}_1)_k\times(\mathbb{T}_2)_k\times...\times(\mathbb{T}_n)_k$ if there exist numbers $A_1,...,A_n$ independent of $t=(t_1,...,t_n)\in\wedge^n$(but, generally, dependent on $(t_1^0,...,t_n^0)$) such that all $t\in U_\delta(t^0)$,

$$f(t_1^0,t_2^0,...,t_n^0)-f(t_1,t_2,...,t_n)=\sum_{i=1}^{n}A_i(t_i^0-t_i)+\sum_{i=1}^{n}\alpha_i(t_i^0-t_i),\qquad(1)$$

and, for $j\in\{1,...,n\}$ and all $t\in U_\delta(t^0)$,

$$f(t_1^0,...,t_{j-1}^0,\rho_j(t_j^0),t_{j+1}^0...,t_n^0)-f(t_1,...,t_{i-1},t_i,t_{i+1}...,t_n)=$$
$$A_j\left[\rho_j(t_j^0)-t_j\right]+\sum_{\substack{i=1\\ i\neq j}}^{n}A_i\left[t_i^0-t_i\right]+\beta_j\left[\rho_j(t_j^0)-t_j\right]+\sum_{\substack{i=1\\ i\neq j}}^{n}\beta_i\left[t_i^0-t_i\right],\qquad(2)$$

where δ is a sufficiently small positive number, $U_\delta(t^0)$ is the the $\delta-$neighborhood of t^0 in $\wedge^n$, $\alpha_i = \alpha_i(t^0,t)$ and $\beta_i = \beta_i(t^0,t)$ are defined on $U_\delta(t^0)$ such that they are equal to zero at $t = t^0$ and such that

$$\lim_{t\to t^0} \alpha_i(t^0,t) = 0 \quad \text{and} \quad \lim_{t\to t^0} \beta_i(t^0,t) = 0 \quad \text{for all } i \in \{1,...,n\}.$$

We say that a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n \to \mathbb{R}$ is ρ_j-completely $\nabla-$differentiable at a point $t^0 = (t_1^0,...,t_n^0) \in (\mathbb{T}_1)_k \times (\mathbb{T}_2)_k \times ... \times (\mathbb{T}_n)_k$ if it is completely $\nabla-$differentiable at that point in the sense of conditions (1), (2) and moreover, along with the numbers $A_1,...,A_n$ presented in (1) and (2) there exists also numbers $B_1,...,B_n$ independent of $t = (t_1,...,t_n) \in \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n$(but, generally, dependent on $(t_1^0,...,t_n^0)$) such that for $j \in \{1,...,n\}$

$$\begin{aligned} & f(\rho_1(t_1^0),\rho_2(t_2^0)...,\rho_n(t_n^0)) - f(t_1,t_2,...,t_n) = A_j\left[\rho_j(t_j^0) - t_j\right] \\ & + \sum_{\substack{i=1\\ i\neq j}}^{n} B_i\left[\rho_i(t_i^0) - t_i\right] + \gamma_j\left[\rho_j(t_j^0) - t_j\right] + \sum_{\substack{i=1\\ i\neq j}}^{n} \gamma_i\left[\rho_i(t_i^0) - t_i\right], \end{aligned} \tag{3}$$

for all $t = (t_1,...,t_n) \in V^{\rho_j}(t_1^0,...,t_n^0)$, where $V^{\rho_j}(t_1^0,...,t_n^0)$ is a union of some neighborhoods of the points $(t_1^0,...,t_n^0)$ and $(\rho_1(t_1^0),...,t_i^0...,\rho_n(t_n^0))$, and the functions $\gamma_j = \gamma_j(t^0;t)$ and $\gamma_i = \gamma_i(t^0;t_i)$ are equal to zero for $(t_1,...,t_n) = (t_1^0,...,t_n^0)$ and

$$\lim_{t\to t^0} \gamma_j(t^0;t) = 0 \quad \text{and} \quad \lim_{t_i\to t^0} \gamma_i(t^0;t_i) = 0.$$

2.2 The Chain Rule

The chain rule for one-variable and two-variable functions on time scales has been investigated in ([1], [6], [8]). In order to get an extension to $n-$variable functions on time scales, we start with a time scale $\mathbb{T}$. Denote its backward jump operator by ρ_i and its nabla differentiation operator by ∇_i for $i = 1,..,n$. Moreover, let $n-$functions

$$\varphi_i : \mathbb{T} \to \mathbb{R} \quad \text{for} \quad i = 1,...,n,$$

be given. Let us set

$$\varphi_i(\mathbb{T}) = \mathbb{T}_i \quad \text{for} \quad i = 1,...,n.$$

We will assume that $\mathbb{T}_1,...,\mathbb{T}_n$ are time scales. $\rho_1,\nabla_1,...,\rho_n,\nabla_n$ are denoted by the backward jump operators and nabla operators for $\mathbb{T}_1,...,\mathbb{T}_n$, respectively. Take a point $\xi^0 \in \mathbb{T}^k$ and put

$$t_i^0 = \varphi_i(\xi^0) \quad \text{for } i = 1,...,n.$$

We will also assume that

$$\varphi_i(\rho(\xi^0)) = \rho_i(\varphi_i(\xi^0)) \quad \text{for} \quad i = 1, ..., n, \tag{4}$$

Under the assumptions above, let a function $f : \mathbb{T}_1 \times ... \times \mathbb{T}_n \to \mathbb{R}$ be given.

Let the function f be ρ_j-completely $\nabla-$differentiable at the point $(t_1^0, ..., t_n^0)$. If the function φ_i $(i = 1, ..., n)$ has nabla derivatives at the point ξ^0, then the composite function

$$F(\xi) = f(\varphi_1(\xi), ..., \varphi_n(\xi)) \quad \text{for} \quad \xi \in \mathbb{T}, \tag{5}$$

has a nabla derivative at that point which is expressed by the formula

$$F^{\nabla}(\xi^0) = \frac{\partial f(t_1^0, ..., t_n^0)}{\nabla_j t_j}\varphi_j^{\nabla}(\xi^0) + \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i}\varphi_i^{\nabla}(\xi^0), \tag{6}$$

for each $j \in \{1, ..., n\}$.

Let the function f be ρ_j-completely $\nabla-$differentiable at the point $(t_1^0, ..., t_n^0)$. If the function φ_i $(i = 1, ..., n)$ has first order partial nabla derivatives at the point $\xi^0 = (\xi_1^0, ..., \xi_n^0)$, then the composite function

$$F(\xi^0) = f(\varphi_1(\xi^0), ..., \varphi_n(\xi^0)) \quad \text{for} \quad \xi^0 = (\xi_1^0, ..., \xi_n^0) \in \mathbb{T}_{(1)} \times ... \times \mathbb{T}_{(n)}, \tag{7}$$

has a nabla derivative at that point which is expresses by the formula

$$\begin{aligned} \frac{\partial F(\xi_1^0, ..., \xi_n^0)}{\nabla_{(k)}\xi_k} = & \frac{\partial f(t_1^0, ..., t_m^0)}{\nabla_j t_j}\frac{\partial \varphi_j(\xi_1^0, ..., \xi_n^0)}{\nabla_{(k)}\xi_k}) \\ & + \sum_{\substack{i=1 \\ i\neq j}}^{m} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_m(t_m^0))}{\nabla_i t_i}\frac{\partial \varphi_i(\xi_1^0, ..., \xi_n^0)}{\nabla_{(k)}\xi_k}, \end{aligned} \tag{8}$$

for each $k \in \{1, ..., n\}$.

2.3 The Directional ∇−Derivative

Let $\mathbb{T}$ be a time scale with the backward jump operator ρ and the nabla operator ∇. We will assume that $0 \in \mathbb{T}$. Furthermore, let $\omega = (\omega_1, ..., \omega_n) \in \mathbb{R}^n$ be a unit vector and let $(t_1^0, ..., t_n^0)$ be a fixed point in $\mathbb{R}^n$. Let us set

$$\mathbb{T}_i = \{t_i = t_i^0 + \xi\omega_i : \ \xi \in \mathbb{T}\}, \quad i = 1, ..., n.$$

Then $\mathbb{T}_1, ..., \mathbb{T}_n$ are time scales and $t_i^0 \in \mathbb{T}_i$ for $i = 1, ..., n$. The backward jump operators of $\mathbb{T}_i$ denoted by ρ_i, the nabla operators by ∇_i for $i = 1, .., n$.

Let a function $f : \wedge^n \to \mathbb{R}$ be given. The directional nabla derivative of the function f at the point $(t_1^0, ..., t_n^0)$ in the direction of the vector ω (along ω) is defined as the number

$$\frac{\partial f(t_1^0, ..., t_n^0)}{\nabla \omega} = F^{\nabla}(0), \tag{9}$$

provided it exists, where

$$F(\xi) = f(t_1^0 + \xi\omega_1, ..., t_n^0 + \xi\omega_n) \text{ for } \xi \in \mathbb{T}. \tag{10}$$

Suppose that the function f is ρ_j-completely $\nabla-$differentiable at the point $(t_1^0, ..., t_n^0)$. Then the directional nabla derivative of f at $(t_1^0, ..., t_n^0)$ in the direction of the vector ω exists and is expressed by the formula

$$\frac{\partial f(t_1^0, ..., t_n^0)}{\nabla\omega} = \frac{\partial f(t_1^0, ..., t_n^0)}{\nabla_j t_j}\omega_j + \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i}\omega_i, \tag{11}$$

for each $j \in \{1, ..., n\}$.

Let $n = 2$. Then for $j = 1,\ i = 2$ we have

$$\frac{\partial f(t_1^0, t_2^0)}{\nabla\omega} = \frac{\partial f(t_1^0, t_2^0)}{\nabla_1 t_1}\omega_1 + \frac{\partial f(\rho_1(t_1^0), t_2^0)}{\nabla_2 t_2}\omega_2, \tag{12}$$

and for $j = 2,\ i = 1$

$$\frac{\partial f(t_1^0, t_2^0)}{\nabla\omega} = \frac{\partial f(t_1^0, \rho_2(t_2^0))}{\nabla_1 t_1}\omega_1 + \frac{\partial f(t_1^0, t_2^0)}{\nabla_2 t_2}\omega_2. \tag{13}$$

Therefore, for $n = 2$, equality (11) reduces to (12) and (13) which are proved for $\Delta-$derivative by Bohner et. al. [8].

2.4 Tangent Vectors and Properties of Directional $\nabla-$Derivative

A tangent vector v_P to $\wedge^n$ consists of two points of $\wedge^n$:its vector part v and its point of application P.

Let P be a point of $\wedge^n$. The set $V_P(\wedge^n)$ consisting of all tanget vectors that have P as point of application is called the tangent space of $\wedge^n$ at P.

Let $x_i : \wedge^n \to \mathbb{T}_i$ be Euclidean coordinate functions on time scale for all $1 \leq i \leq n$, denoted by the set $\{x_1, x_2, ..., x_n\}$. Let $f : \wedge^n \to \mathbb{R}$ be a function described by $f(P) = (f_1(P), f_2(P), ..., f_m(P))$ at a point $P \in \wedge^n$. The function f is called ρ_j-completely $\nabla-$differentiable function at the point P provided that all f_i $(i = 1, 2, ..., m)$ functions are ρ_j-completely $\nabla-$differentiable at the point P. All this kind of functions set will be denoted by $C_{\rho_j}^{\nabla}$. If we define two algebraic operations as follows:

$$\begin{array}{rccc} \oplus : & C_{\rho_j}^{\nabla} \times C_{\rho_j}^{\nabla} & \longrightarrow & C_{\rho_j}^{\nabla} \\ & (f, g) & \longrightarrow & f \oplus g \end{array},$$

for $\forall x \in \wedge^n$,

$$(f \oplus g)(x) = f(x) + g(x),$$

and

$$\begin{array}{rccc} \odot: & \mathbb{R}\times C^{\nabla}_{\rho_j} & \longrightarrow & C^{\nabla}_{\rho_j} \\ & (\lambda, f) & \longrightarrow & \lambda f = \lambda \odot f \end{array},$$

for $\forall x \in \wedge^n$,

$$(\lambda \odot f)(x) = \lambda f(x).$$

In this case the set,

$$\{C^{\nabla}_{\rho_j}, \oplus, \mathbb{R}, +, \cdot, \odot\},$$

is a vector space. Next, if we define another operation by,

$$\begin{array}{rccc} \oslash: & C^{\nabla}_{\rho_j} \times C^{\nabla}_{\rho_j} & \longrightarrow & C^{\nabla}_{\rho_j} \\ & (f, g) & \longrightarrow & f \oslash g \end{array},$$

for $\forall x \in \wedge^n$

$$(f \oslash g)(x) = f(x)g(x).$$

Thus, the set $\{C^{\nabla}_{\rho_j}, \oplus, \mathbb{R}, +, \cdot, \odot, \oslash\}$ is an algebra over $\mathbb{R}$. Finally, we can consider a tangent vector of $\wedge^n$ as a function from $C^{\nabla}_{\rho_j}$ to $\mathbb{R}$. This result can be easily seen from Definition 2.3.

Let $a, b \in \mathbb{R}$ and $f, g \in C^{\nabla}_{\rho_j}$ and $v_P, \omega_P, z_P \in V_P(\wedge^n)$. Then, the following properties are proven for the directional $\nabla-$derivative of the function f at the point $P(t_1^0, t_2^0, ..., t_n^0)$:

$$\begin{array}{lll} (i) & \dfrac{\partial f(P)}{\nabla\left(av_P + b\omega_P\right)} & = a\dfrac{\partial f(P)}{\nabla\left(v_P\right)} + b\dfrac{\partial f(P)}{\nabla\left(\omega_P\right)}, \\ (ii) & \dfrac{\partial\left(af + bg\right)(P)}{\nabla v_P} & = a\dfrac{\partial f(P)}{\nabla v_P} + b\dfrac{\partial g(P)}{\nabla v_P}, \\ (iii) & \dfrac{\partial\left(fg\right)(P)}{\nabla v_P} & = g(P)\dfrac{\partial f(P)}{\nabla v_P} + f(\rho_1\left(t_1^0\right), t_2^0, ..., t_n^0)\dfrac{\partial g(P)}{\nabla v_P} \\ & & -{}_{\substack{i=1\\ i\neq j}}^{n}\left(g\left(P\right) - g_{\rho_i}(P)\right)\dfrac{\partial f_{\rho_i}(P)}{\nabla_i t_i}v_i \\ & & +{}_{\substack{i=1\\ i\neq j}}^{n}\left(f^{\rho}(P) - f(\rho_1\left(t_1^0\right), t_2^0, ..., t_n^0)\right)\dfrac{\partial g_{\rho_i}(P)}{\nabla_i t_i}v_i, \end{array}$$

where $f_{\rho_i}(P) = f(\rho_1\left(t_1^0\right), ..., \rho_{(i-1)}\left(t_{i-1}^0\right), t_i^0, \rho_{(i+1)}\left(t_{i+1}^0\right), ..., \rho_n(t_n^0))$ and $f^{\rho}(P) = (\rho_1\left(t_1^0\right), ..., \rho_{i-1}\left(t_{i-1}^0\right), \rho_i\left(t_i\right), \rho_{i+1}\left(t_{i+1}^0\right), ..., \rho_n(t_n^0)$.

2.5 Vector Fields and Properties of Directional $\nabla-$Derivative

A vector field W on $\wedge^n$ is a function that assigns to each point P of $\wedge^n$ a tangent vector ω_P to $\wedge^n$ at P.

Let Z be a vector field and $Z(P)$ belongs to the set of tangent vector space $V_P(\wedge^n)$ at the point P. Generally, a vector field is denoted by

$$Z = \sum_{k=1}^{n} g_k(x_1, ...x_n)\frac{\partial}{\nabla_k x_k}, \tag{14}$$

where $g_k(x_1, ...x_n)$ are real valued and have partial nabla derivative functions defined on $\wedge^n$ and $\left\{ \frac{\partial}{\nabla_1 x_1}, \frac{\partial}{\nabla_2 x_2}, ..., \frac{\partial}{\nabla_n x_n} \right\}$ are the basis for $V_P(\wedge^n)$. If for each $g_k(x_1, ...x_n)$ is ρ_j-completely $\nabla-$differentiable then we say the vector field Z is ρ_j-completely $\nabla-$differentiable.

Let $\chi(\wedge^n)$ be a set of the ρ_j-completely $\nabla-$differentiable vector fields and let a ρ_j-completely $\nabla-$differentiable function $f : \wedge^n \to \mathbb{R}$ be given. The directional $\nabla-$derivative of the function f at the point $P(t_1^0, t_2^0, ..., t_n^0)$ in the direction of the vector field W is defined as

$$\left(\frac{\partial f}{\nabla W} \right)(P) = \frac{\partial f(P)}{\nabla \omega_P}.$$

By this definition, we have defined a function $W : C_{\rho_j}^{\nabla} \to C_{\rho_j}^{\nabla}$ such that $W(P) = \omega_P$. Here, ω_P is the tangent vector, which belongs to the vector field W.

Let V and W be two vector fields. Then, The following are proved for any two functions $f,\ g$ and $h \in C_{\rho_1}^{\nabla}$:

$$(i) \quad \frac{\partial h}{\nabla (fV + gW)} = f\frac{\partial h}{\nabla V} + g\frac{\partial h}{\nabla W},$$

$$(ii) \quad \frac{\partial (af + bg)}{\nabla V} = a\frac{\partial f}{\nabla V} + b\frac{\partial g}{\nabla V},$$

$$(iii) \quad \frac{\partial (fg)}{\nabla V} = g(P)\frac{\partial f}{\nabla V} + f(\rho_1(t_1^0), t_2^0, ..., t_n^0)\frac{\partial g}{\nabla V}$$
$$-{}^{n}_{\substack{i=1\\ i\neq j}} \left(g(t_1^0, ..., t_n^0) - g(\rho_1(t_1^0), ..., t_i^0, ..., \rho_1(t_n^0)) \right) \frac{\partial f_{\rho_i}}{\nabla_i t_i} v_i$$
$$+{}^{n}_{\substack{i=1\\ i\neq j}} \left(f(\rho_1(t_1^0), ..., \rho_n(t_n^0)) - f(\rho_1(t_1^0), t_2^0, ..., t_n^0) \right) \frac{\partial g_{\rho_i}}{\nabla_i t_i} v_i,$$

where $f_{\rho_i}(P) = f(\rho_{j1}(t_1^0), ..., \rho_{j(i-1)}(t_{i-1}^0), t_i^0, \rho_{j(i+1)}(t_{i+1}^0), ..., \rho_{jn}(t_n^0))$

Let two vector fields Z, W be given. The covariant nabla differentiation with respect to W at the point $P(t_1^0, t_2^0, ..., t_n^0)$ is defined as the vector

$$\left(\frac{\partial Z}{\nabla W} \right)(P) = Y^{\nabla}(0)$$

provided it exists, where

$$Y(\xi) = Z(t_1^0 + \xi\omega_1, ..., t_n^0 + \xi\omega_n) \quad \text{for} \quad \xi \in \mathbb{T}.$$

Let two vector fields Z, W be given. The covariant nabla differentiation with respect to W at the point $P(t_1^0, t_2^0, ..., t_n^0)$ exists and is expressed by the formula

$$\frac{\partial Z(P)}{\nabla \omega_P} = \sum_{i=1}^{n} \frac{\partial g_i(P)}{\nabla \omega_P} \frac{\partial}{\partial x_i}(P),$$

where the functions $\dfrac{\partial g_i(P)}{\nabla \omega_P}$ can be found similarly as in Theorem 2.3.

Let $a, b \in \mathbb{R}$ and two vector fields X and Y be given. For any two tangent vectors v_P and ω_P, the following properties are proven:

$$(i)\quad \frac{\partial X}{\nabla\left(aV+bW\right)} = a\frac{\partial h}{\nabla V} + b\frac{\partial h}{\nabla W},$$

$$(ii)\quad \frac{\partial\left(aX+bY\right)}{\nabla V} = a\frac{\partial f}{\nabla V} + b\frac{\partial g}{\nabla V},$$

$$(iii)\quad \frac{\partial\left(fX\right)(P)}{\nabla v_P} =_{k=1}^{n}\Bigg[h_k(P)\frac{\partial f(P)}{\nabla v_P} + f(\rho_1\left(t_1^0\right), t_2^0, ..., t_n^0)\frac{\partial h_k(P)}{\nabla v_P} -_{\substack{i=1\\ i\neq j}}^{n}\Big(h_k\left(P\right) - \left(h_k\right)_{\rho_i}\left(P\right)\Big)\frac{\partial f_{\rho_i}(P)}{\nabla_i t_i}v_i +_{\substack{i=1\\ i\neq j}}^{n}\left(f^{\rho}(P) - f(\rho_1\left(t_1^0\right), t_2^0, ..., t_n^0)\right)\frac{\partial\left(h_k\right)_{\rho_i}\left(P\right)}{\nabla_i t_i}v_i\Bigg]\frac{\partial}{\nabla_k x_k}$$

$$(iv)\quad \left(\frac{\partial\left\langle Y, Z\right\rangle}{\nabla V}\right)(P) = \left\langle\frac{\partial Y(P)}{\nabla v_P}, Z\right\rangle + \left\langle Y(\rho_1\left(t_1^0\right), t_2^0, ..., t_n^0), \frac{\partial Z(P)}{\nabla v_P}\right\rangle -_{k=1}^{n}\Bigg[{}_{\substack{i=1\\ i\neq j}}^{n}\Big(g_k\left(P\right) - \left(g_k\right)_{\rho_i}\left(P\right)\Big)\frac{\partial\left(f_k\right)_{\rho_i}\left(P\right)}{\nabla_i t_i}v_i -_{\substack{i=1\\ i\neq j}}^{n}\left(f_k^{\rho}(P) - f_k(\rho_1\left(t_1^0\right), t_2^0, ..., t_n^0)\right)\frac{\partial\left(g_k\right)_{\rho_i}\left(P\right)}{\nabla_i t_i}v_i\Bigg],$$

where $P = P(t_1^0, t_2^0, ..., t_n^0)$.

3 Nabla 1−Forms

It follows from Definition 2.1, If $f : \wedge^n \to \mathbb{R}$ is ρ_j-completely ∇−differentiable at a point $t^0 = (t_1^0, ..., t_n^0) \in (\mathbb{T}_1)_k \times (\mathbb{T}_2)_k \times ... \times (\mathbb{T}_n)_k$, then in elementary calculus on on $\wedge^n$ one defines the ρ_j-completely ∇−differential of f to be

$$df(P) = \frac{\partial f(t_1^0, ..., t_n^0)}{\nabla_j t_j}dt_j +_{\substack{i=1\\ i\neq j}}^{n}\frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i}dt_i.$$

In this section we give a rigorous treatment using the notion of nabla 1−form.

A nabla 1−form ϕ on $\wedge^n$ is a real-valued function on the set of all tangent vectors to $\wedge^n$ such that ϕ is linear at each point , that is,

$$\phi(av_P + b\omega_P) = a\phi(v_P) + b\phi(\omega_P),$$

for any numbers a, b and tangent vectors v_P, ω_P at the same point of $\wedge^n$. The set of 1−forms will be denoted by $V^*\left(\wedge^n\right)$.

We emphasize that for every tangent vector v_P to $\wedge^n$, a nabla 1−form ϕ defines a real number $\phi(v_P)$; and for each point P in $\wedge^n$, the resulting function

$$\phi_P : V(\wedge^n) \to \mathbb{R},$$

is linear.

The sum of nabla $1-$forms ϕ and ψ is defined in the usual pointwise fashion

$$(\phi + \psi)\,(v_P) = \phi(v_P) + \psi(v_P) \text{ for all tangent vectors } v_P.$$

Similarly if f is a real-valued function on $\wedge^n$ and ϕ is a nabla $1-$form such that

$$(f\phi)(v_P) = f(P)\phi(v_P),$$

for all tangent vectors v_P.

There is also a natural way to evaluate a nabla $1-$form ϕ on a vector field X to obtain a real-valued function $\phi(X)$:at each point P the value of $\phi(X)$ is the number $\phi(X(P))$. thus a nabla $1-$form may also be viewed a a machine which converts vector fields into real-valued functions. If $\phi(X)$ is $\nabla-$differentiable whenever X is, we say that ϕ is $\nabla-$differentiable. As with vector field, we shall always assume that the nabla $1-$forms we deal with are differentiable.

A routine check of definitions shows that $\phi(X)$ is linear in both ϕ and X; that is,

$$\phi(fX + gY) = f\phi(X) + g\phi(Y),$$

and

$$(f\phi + g\psi)(X) = f\phi(X) + g\psi(X),$$

where f and g are real-valued functions on $\wedge^n$.

Using the notion of directional $\nabla-$derivative, we now define a most important way to convert functions into nabla $1-$forms.

If f is a ρ_j-completely $\nabla-$differentiable real-valued functions on $\wedge^n$, the differential *df* of f is the nabla $1-$form such that

$$df(v_P) = \frac{\partial f(P)}{\nabla v_P},$$

for all tangent vectors v_P.

In fact, *df* is a nabla $1-$ form, since by definition it is a real-valued function on tangent vectors, and by (i) of Theorem 2.4 is linear at each point P. Clearly, *df* knows all rates of change of f in all directions on $\wedge^n$, so it is not surprising that $\nabla-$differentials are fundamental to the calculus on $\wedge^n$.

The differentials $dt_1, dt_2, ..., dt_n$ of the Euclidean coordinate functions. Using Theorem 2.3 we find

$$dt_i(v_P) = \frac{\partial t_i(P)}{\nabla v_P} = v_i.$$

Thus, the value of dt_i on an arbitrary tangent vector v_P is the i.th coordinate v_i of its vector part, and does not depend at all on the point of application P.

Since dt_i is a nabla $1-$form, our definitions show that $\psi =_{i=1}^{n} f_i dt_i$ is also a nabla $1-$form for any functions f_i, $1 \leq i \leq n$. The value of ψ on an arbitrary tangent vector v_P is

$$\psi(v_P) = \left({}_{i=1}^{n} f_i dt_i \right)(v_P) =_{i=1}^{n} f_i(P) dt_i(v_P) =_{i=1}^{n} f_i(P) v_i.$$

The first of these examples show that the $1-$forms $dt_1, dt_2, ..., dt_n$ are the analogues for tangent vectors of the natural coordinate function $t_1, t_2, ..., t_n$ for points. Alternatively, we can view $dt_1, dt_2, ..., dt_n$ as the duals of the natural unit vector $U_1, U_2, ..., U_n$. In fact, it follows immediately from Example 3 that the function $dt_i(U_j)$ has the constant value δ_{ij}, where δ_{ij} is the Kronecker delta (0 if $i \neq j$, 1 if $i = j$).

We shall now show that every nabla $1-$form can be written in the concrete manner given in Example 3.

If ϕ is a nabla $1-$form on $\wedge^n$, then $\phi =_{i=1}^{n} f_i dt_i$, where $f_i = \phi(U_i)$. These functions $f_1, f_2, ..., f_n$ are called the Euclidean coordinate functions of ϕ.

By definition a nabla $1-$form is a function on tangent vectors; thus ϕ and ${}_{i=1}^{n} f_i dt_i$ are equal if and only if they have the same value on every tangent vector $v_P = {}_{i=1}^{n} v_i U_i(P)$. In Example 3, we saw that

$$\left({}_{i=1}^{n} f_i dt_i \right)(v_P) =_{i=1}^{n} f_i(P) v_i.$$

On the other hand,

$$\phi(v_P) = \phi\left({}_{i=1}^{n} v_i U_i(P) \right) =_{i=1}^{n} v_i \phi\left(U_i(P)\right) =_{i=1}^{n} v_i f_i(P),$$

since $f_i = \phi(U_i)$. Thus ϕ and ${}_{i=1}^{n} f_i dt_i$ do have the same value on every tangent vector.

This theorem shows that a nabla $1-$form on $\wedge^n$ is nothing more than an expression ${}_{i=1}^{n} f_i dt_i$, and such expression are now rigorously defined as functions on tangent vectors. Let us now how that the definition of differential of a function agrees with the informal definition given at the start of this section.

If f is a ρ_j-completely $\nabla-$differentiable real-valued functions on $\wedge^n$, then

$$df(P) = \frac{\partial f(t_1^0, ..., t_n^0)}{\nabla_j t_j} dt_j + {}_{\substack{i=1 \\ i \neq j}}^{n} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i} dt_i.$$

The value of $\frac{\partial f(t_1^0, ..., t_n^0)}{\nabla_j t_j} dt_j + {}_{\substack{i=1 \\ i \neq j}}^{n} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i} dt_i$ on an arbitrary tangent vector v_P is $\frac{\partial f(t_1^0, ..., t_n^0)}{\nabla_j t_j} v_j + {}_{\substack{i=1 \\ i \neq j}}^{n} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i} v_i$. By Theorem 2.3 $df(v_P) = \frac{\partial f(P)}{\nabla v_P}$ is the same. Thus the nabla $1-$form df and $\frac{\partial f(t_1^0, ..., t_n^0)}{\nabla_j t_j} dt_j + {}_{\substack{i=1 \\ i \neq j}}^{n} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i} dt_i$ are equal.

Finally we determine the effect of d on products of functions and on compositions of functions.

Let fg be the product of ρ_j-completely $\nabla-$differentiable real-valued functions f and g on $\wedge^n$. Then

$$\begin{aligned} d(fg) &= gdf + f^{\rho_j}dg + \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{\partial f_{\rho_i}}{\nabla_i t_i}\left(g_{\rho_i} - g\right)dt_i \\ &\quad + \sum_{\substack{i=1\\ i\neq j}}^{n} \left(f^{\rho} - f^{\rho_j}\right)\frac{\partial g_{\rho_i}}{\nabla_i t_i}dt_i, \end{aligned}$$

where $f^{\rho}(P) = f(\rho_1(t_1^0), ..., \rho_i(t_i^0), ..., \rho_n(t_n^0))$, $f^{\rho_j}(P) = f(t_1^0, ..., \rho_j(t_j^0), ..., t_n^0)$ and $f_{\rho_i}(P) = f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0)$.

Using Corollary 3, we obtain

$$\begin{aligned} d(fg) &= \frac{\partial (fg)(t_1^0, ..., t_n^0)}{\nabla_j t_j}dt_j + \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{\partial (fg)(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i}dt_i \\ &= \frac{\partial f(t_1^0, ..., t_n^0)}{\nabla_j t_j} g(t_1^0, ..., t_n^0)dt_j + f(t_1^0, ..., \rho_j(t_j^0), ..., t_n^0)\frac{\partial g(t_1^0, ..., t_n^0)}{\nabla_j t_j}dt_j \\ &\quad + \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i} g(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))dt_i \\ &\quad + \sum_{\substack{i=1\\ i\neq j}}^{n} f(\rho_1(t_1^0), ..., \rho_i(t_i^0), ..., \rho_n(t_n^0))\frac{\partial g(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i}dt_i \\ &= g(t_1^0, ..., t_n^0)df + f(t_1^0, ..., \rho_j(t_j^0), ..., t_n^0)dg \\ &\quad + \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{\partial f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i}\big(g(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0)) \\ &\quad - g(t_1^0, ..., t_n^0)\big)dt_i \\ &\quad + \sum_{\substack{i=1\\ i\neq j}}^{n} \big(f(\rho_1(t_1^0), ..., \rho_i(t_i^0), ..., \rho_n(t_n^0)) \\ &\quad - f(t_1^0, ..., \rho_j(t_j^0), ..., t_n^0)\big)\frac{\partial g(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0))}{\nabla_i t_i}dt_i \end{aligned}$$

If we set $f^{\rho}(P) = f(\rho_1(t_1^0), ..., \rho_i(t_i^0), ..., \rho_n(t_n^0))$, $f^{\rho_j}(P) = f(t_1^0, ..., \rho_j(t_j^0), ..., t_n^0)$ and $f_{\rho_i}(P) = f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0)$, then we obtain desired result.

Let $f : \wedge^n \to \mathbb{R}$ ρ_j-completely $\nabla-$differentiable real-valued functions and $h : \mathbb{R} \to \mathbb{R}$ differentiable function, so the composite function $h(f) : \wedge^n \to \mathbb{R}$ is also ρ_j-completely $\nabla-$differentiable. Then,

$$d(h(f)) = \left(h' \circ f\right)df + \sum_{\substack{i=1\\ i\neq j}}^{n} \left(h' \circ f_{\rho_i} - h' \circ f\right)\frac{\partial f_{\rho_i}}{\nabla_j t_j}dt_i.$$

From Corollary 3 and Theorem 2, we have

$$\begin{aligned} d(h\circ f)\,(P) &= h^{'}(f(P))\frac{\partial f(P)}{\nabla_j t_j}dt_j + {}^{n}_{\substack{i=1\\ i\neq j}} h^{'}(f_{\rho_i}(P))\frac{\partial f_{\rho_i}(P))}{\nabla_j t_j}dt_i. \\ &= h^{'}(f(P))df(P) + {}^{n}_{\substack{i=1\\ i\neq j}} \left(h^{'}(f_{\rho_i}(P)) - h^{'}(f(P))\right)\frac{\partial f_{\rho_i}(P))}{\nabla_j t_j}dt_i. \end{aligned}$$

where $f_{\rho_i}(P) = f(\rho_1(t_1^0), ..., t_i^0, ..., \rho_n(t_n^0)$. Thus the proof is complete.

4 Open Problem

In this paper, we have provided an introduction to nabla 1-forms for multivariable functions on $n-$dimensional time scales and we give a rigorous treatment using the notion of nabla 1-forms. This study will form the basis on the efforts in the field of discrete differential geometry and the time scale analysis.

In vector calculus, the Frenet–Serret formulas describe the kinematic properties of a particle which moves along a continuous, differentiable curve in three-dimensional Euclidean space , $\mathbb{R}^3$ or the geometric properties of the curve itself irrespective of any motion. More specifically, the formulas describe the derivatives of the so-called tangent, normal, and binormal unit vectors in terms of each other. This suggests the following open problem:

By using above structures, how to define Frenet–Serret formulas of a regular curve on $n-$dimensional time scales?

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