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Newton-Raphson Type Methods

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Abstract

The most famous iteration scheme for solving algebraic equations is Newton-Raphson method. It is derived by the first order Taylor expansion and gives a recurrence formula for the iterations that approximate an exact root of the equation. Methods have appeared recently, based on certain recurrent polynomial equations which are called in this paper resolvent equations and from which the variations of iterations are obtained, instead of a recurrence formula for iterations. Thus, in 2003 J. H. He proposed a method based on a quadratic resolvent equation, deduced by the second-order Taylor expansion. Unfortunately, his method is not so effective, because the resolvent equation has no real roots for any initial data. Using third-order Taylor expansion, in 2008 D. Wei, J. Wu and M. Mei obtained an iteration scheme based on solving a cubic resolvent equation, which has always a real root and therefore the method has no restriction regarding the choice of the initial data. The trouble with the resolvent polynomial equations obtained by both He and Wei-Wu-Mei is the constant term that has a complicated expression. In this paper we give two types of methods for finding the approximate values (iterations) of a real root of an algebraic equation, based on the determination of the resolvent polynomial equation of general order, one of its roots being the variation from an iteration to the next. The two types of methods will be obtained in accordance with the type of conditions that will be considered. The resolvent equation of the first type methods is the Taylor polynomial of general order in the considered iteration. Particularly, the first-order case gives the initial Newton-Raphston method. The resolvent equations of the second type methods will differ from those of first type only by

the constant term. In the second-order and third-order cases, He and respective Wei-Wu-Mei methods are obtained. A simple algebraic calculation, based on a telescopic procedure, will produce an improvement of the constant term of the resolvent equations of the second type. We give several examples and two applications to some extremum problems, namely to calculation of the minimum distance from the origin to the graphs of two functions inverse to each other, the exponential and the natural logarithm. The algebraic equations to which these problems are reduced will be solved by the methods given in the present paper.

Keywords: *algebraic equations, Newton-Raphson iterative method, Taylor expansion.*
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1 Introduction

We consider the algebraic equation

$$f(x) = 0, \tag{1}$$

where $f(x)$ is a function of a real variable with derivatives up to a necessary order.

The iterative methods of Newton-Raphson type consist in determining a sequence of iterations x_n, $n = 0, 1, 2, \ldots$, that approximate an exact real root of the equation (1), starting with conveniently chosen initial datum. In the new methods of this type, the variations of iterations

$$t_n = x_{n+1} - x_n,\ n = 0, 1, 2, \ldots, \tag{2}$$

will be obtained, starting with the initial data x_0 and x_1, hence with the initial variation $t_0 = x_1 - x_0$, from a recurrent polynomial equation, deduced by the Taylor expansion of the function $f(x)$ and named *resolvent equation.* From these variations, the iterations are determined from the relation

$$x_{n+1} = x_n + t_n,\ n = 1, 2, \ldots, \tag{3}$$

starting with x_1.

We consider the value of the function f at x_{n+1} given by the Taylor expansion of order $m = 1, 2, 3 \ldots$, at the point x_n, with the remainder $R_{m,n}$,

$$f(x_{n+1}) = P_{m,n}(t_n) + R_{m,n},\ n = 0, 1, 2, \ldots, \tag{4}$$

where

$$P_{m,n}(t_n) = \sum_{k=0}^{m} \frac{f^{(k)}(x_n)}{k!} t_n^k,\ n = 0, 1, 2, \ldots, \tag{5}$$

is the Taylor polynomial of order m in x_n with the variable t_n.

2 Newton-Raphson Type Methods of First Kind

2.1 Methods of General Order

For an arbitrary natural number $n \geq 1$, we require to have approximately

$$f(x_{n+1}) = 0,\ R_{m,n} = 0. \tag{6}$$

Then the Taylor expansion (4) takes the form

$$P_{m,n}(t_n) = \sum_{k=0}^{m} \frac{f^{(k)}(x_n)}{k!} t_n^k = 0,\ n = 1, 2, \ldots, \tag{7}$$

this being the resolvent equation of the method. Starting with x_0 and x_1, we have $t_0 = x_1 - x_0$ and the *resolvent equation* (7) gives recurrently the variation t_n, which is one of the roots of the equation (7), conveniently chosen. Then the iterations x_n are obtained by relation (3).

Remark 2.1 (*i*) *The formula* (7) *gives the simplest form of the resolvent equation, from which t_n can be determined. It is the Taylor polynomial in x_n.*

(*ii*) *Because in the conditions* (6) *and* (13) *below we have approximate equality, we obtain approximate values of the variations t_n and iterations x_n.*

2.2 First Order Method: Initial Newton-Raphson Method

For $m = 1$, the resolvent equation (7) takes for $n = 1, 2, \ldots,$ the form $f'(x_n)t_n + f(x_n) = 0$, with the root $t_n = -\dfrac{f(x_n)}{f'(x_n)}$, hence according to (3), the iterations are given by the well-known recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 1, 2, \ldots, \tag{8}$$

starting with x_1. Here we consider $x_0 = 0$.

Remark 2.2 *Newton used a variant of this formula to solve the polynomial equation presented in example* 2.4 *below, in his work Method of Fluxions, written in* 1671 *but published only in* 1736. *In turn, J. Raphson gave the formula in his book Analysis Aequationum Universalis, published in* 1690.

2.3 Second and Third Order Methods

For $m = 2$, the resolvent equation (7) is the quadratic equation

$$\frac{1}{2} f''(x_n)t_n^2 + f'(x_n)t_n + f(x_n) = 0,\ n = 1, 2, \ldots, \tag{9}$$

and for $m = 3$ is the cubic equation

$$\frac{1}{6} f'''(x_n)t_n^3 + \frac{1}{2} f''(x_n)t_n^2 + f'(x_n)t_n + f(x_n) = 0,\ n = 1, 2, \ldots \cdot \tag{10}$$

2.4 Example: Newton's Polynomial Equation

The polynomial equation $f(x) = x^3 - 2x - 5 = 0$, was initially solved by Newton using his recurrence relation (8), which gives $x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} = \dfrac{2x_n^3 + 5}{3x_n^2 - 2}$, $n = 1, 2, \ldots \cdot$ Taking $x_1 = 2$, we obtain $x_2 = 2.1$ and $x_3 = 2.0946$.

The second order resolvent equation is $3x_n t_n^2 + (3x_n^2 - 2)t_n + x_n^3 - 2x_n - 5 = 0$, $n = 1, 2, \ldots \cdot$ Taking $x_0 = 0$, $x_1 = 2$ and $n = 1$, we obtain the equation $6t_1^2 + 10t_1 - 1 = 0$, with the roots -1.7613 which do not agree and $t_1 = 0.0946$. Hence we obtain the iteration $x_2 = x_1 + t_1 = 2,0946$.

2.5 Calculation of a Radical

We calculate by the first type method the radical $\sqrt[3]{2}$, so we solve the equation $f(x) = x^3 - 2 = 0$. The quadratic resolvent equation is $3x_n t_n^2 + 3x_n^2 t_n + x_n^3 - 2 = 0$, $n = 1, 2, \ldots \cdot$ Taking $x_0 = x_1 = 1$, hence $t_0 = 0$, we obtain the equation $3t_1^2 + 3t_1 - 1 = 0$, with the roots -1.2638, inconvenient and $t_1 = 0.2638$, hence $x_2 = x_1 + t_1 = 1.2638$ and $3.7914t_2^2 + 4.7916t_2 + 0.0185 = 0$, with the roots -1.2599, inconvenient and $t_2 = -0.0039$, hence $x_3 = x_2 + t_2 = 1.2599$.

2.6 Example

We consider the algebraic equation $f(x) = x - e^{-x} = 0$. For $m = 1$, the recurrent relation (8) is $x_{n+1} = x_n - \dfrac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \dfrac{1 + x_n}{1 + e^{x_n}}$, $n = 1, 2, \ldots \cdot$ Taking $x_1 = 0.5$, we obtain $x_2 = 0.5663$ and $x_3 = 0.5671$. In the methods based on resolvent equation we will take $x_0 = x_1 = 0$, hence $t_0 = 0$.

For $m = 2$, the resolvent equation (9) takes the form

$$-\frac{1}{2}e^{-x_n}t_n^2 + (1 + e^{-x_n})t_n + x_n - e^{-x_n} = 0,\ n = 1,,2,\ldots \cdot \tag{11}$$

For $n = 1$, the resolvent equation (11) reduces to $-0.5t_1^2 + 2t_1 - 1 = 0$ from which one obtains $t_1 = 0.5858 = x_2$ and for $n = 2$, to equation $-0.2783t_2^2 + 1.5567t_2 + 0.0291 = 0$, which gives $t_2 = -0.0186$ and $x_3 = x_2 + t_2 = 0.5672$.

For $m = 3$, the resolvent equation (10) takes the form

$$\frac{1}{6}e^{-x_n}t_n^3 - \frac{1}{2}e^{-x_n}t_n^2 + (1 + e^{-x_n})t_n + x_n - e^{-x_n} = 0,\ n = 1, 2, \ldots \cdot \tag{12}$$

From (12) we obtain the equation $0.1667t_1^3 - 0.5t_1^2 + 2t_1 - 1 = 0$ from which one obtains $t_1 = 0.5647 = x_2$ and $0.0948t_2^3 - 0.2843t_2^2 + 1.5685t_2 - 0.0038 = 0$, which gives $t_2 = 0.0024$, hence $x_3 = 0.5671$.

3 Newton-Raphson Type Methods of Second Kind

3.1 Methods of General Order

We now suppose $m \geq 2$. For an arbitrary natural number $n \geq 1$, we require to have approximately

$$f(x_{n+1}) = 0,\ R_{m,n} = R_{m,n-1}. \tag{13}$$

Then the Taylor expansion (4) takes the form

$$P_{m,n}(t_n) + R_{m,n} = 0. \tag{14}$$

According to (13) and (4), the last relation for n substituted by $n-1$, the remainder $R_{m,n}$ is given by formula

$$R_{m,n} = R_{m,n-1} = f(x_n) - P_{m,n-1}(t_{n-1})$$

and the equation (14) becomes

$$P_{m,n}(t_n) + f(x_n) - P_{m,n-1}(t_{n-1}) = 0,\ n = 1, 2, \dots \cdot \tag{15}$$

According to the notation (5), the equation (15) has the form

$$\sum_{k=0}^{m} \frac{f^{(k)}(x_n)}{k!} t_n^k + f(x_n) - \sum_{k=0}^{m} \frac{f^{(k)}(x_{n-1})}{k!} t_{n-1}^k = 0,\ n = 1, 2, \dots, \tag{16}$$

which is the resolvent equation from which t_n can be recurrently determined, starting with the value $t_0 = x_1 - x_0$, given by the initial data x_0 and x_1.

Remark 3.1 *The resolvent equation* (16) *was given by J. H. He,* [3]*, for* $m = 2$ *and by D. Wei, J. Wu and M. Mei,* [4]*, for* $m = 3$*. The method of resolvent polynomial equation introduced by Ji-Huan He in the above mentioned paper must be added to other methods of solving various types of equations proposed by the same author, such as the homotopy perturbation technique,* [2] *and the variational iteration method,* [3].

3.2 Improved Methods

Now we give a new form of the resolvent equation (16), more convenient for its writing. Replacing in the form (15) of the resolvent equation (16) the index n by $n-1$, $n-2, \dots, 2$, 1, we obtain the relations

$$\begin{aligned}
&P_{m,n-1}(t_{n-1}) + f(x_{n-1}) - P_{m,n-2}(t_{n-2}) = 0,\\
&P_{m,n-2}(t_{n-2}) + f(x_{n-2}) - P_{m,n-3}(t_{n-3}) = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&P_{m,2}(t_2) + f(x_2) - P_{m,1}(t_1) = 0,\\
&P_{m,1}(t_1) + f(x_1) - P_{m,0}(t_0) = 0.
\end{aligned}$$

Adding these relations to (15), we obtain the following improved form of this equation,

$$P_{m,n}(t_n) + \sum_{j=1}^{n} f(x_j) - P_{m,0}(t_0) = 0,\ n = 1, 2, \ldots, \tag{17}$$

hence, according to the notation (5), one obtains the resolvent equation

$$\sum_{k=0}^{m} \frac{f^{(k)}(x_n)}{k!} t_n^k + \sum_{j=1}^{n} f(x_j) - \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!} t_0^k = 0,\ n = 1, 2, \ldots \cdot \tag{18}$$

3.3 Simplified Methods

If we choose $x_0 = x_1$, then $t_0 = 0$ and the equation (18) reduces to the equation

$$\sum_{k=0}^{m} \frac{f^{(k)}(x_1)}{k!} t_1^k = 0, \tag{19}$$

that is the same as (7), for $n = 1$ and

$$\sum_{k=0}^{m} \frac{f^{(k)}(x_n)}{k!} t_n^k + \sum_{j=2}^{n} f(x_j) = 0, \tag{20}$$

that is different from (7), for $n \geq 2$.

Remark 3.2 (i) *For $m = 2$, respectively $m = 3$, we call the methods obtained in this Section and based on the resolvent equations (19) and (20), the improved He, respectively improved Wei-Wu-Mei method.*

(ii) *In both methods of first and second type, the resolvent equations, namely the equations (7), (15), (17), (19) and (20), differ only by the constant term. They can not be used if they have only complex roots. See example* 3.5 *below.*

3.4 Example

We solve here by second type methods, namely by improved He and Wei-Wu-Mei methods, the same equation $f(x) = x - e^{-x} = 0$, solved in example 2.6 by methods of first type. We start with $x_0 = x_1 = 0$, hence $t_0 = 0$. As a result of (19), the resolvent equation for t_1 is the same as in example 2.6. Therefore, for $m = 2$, we have $t_1 = 0.5858 = x_2$. For $n \geq 2$ the resolvent equation is

$$-\frac{1}{2} e^{-x_n} t_n^2 + (1 + e^{-x_n}) t_n + x_n - e^{-x_n} + \sum_{j=2}^{n} (x_j - e^{-x_j}) = 0.$$

For $n = 2$ this equation is $-0.2783t_2^2 + 1.5567t_2 + 0.0582 = 0$, resulting $t_2 = -0.0371$, hence $x_3 = 0.5487$.

For $n = 3$, the equation is $-0.2889t_3^2 + 1.5777t_3 - 0.0289 = 0$, resulting $t_3 = 0.0184$, hence $x_4 = 0.5671$.

For $m = 3$, we have $t_1 = 0.5647 = x_2$, according to the example 2.6. For $n \geq 2$ the resolvent equation is

$$\frac{1}{6}e^{-x_n}t_n^3 - \frac{1}{2}e^{-x_n}t_n^2 + (1 + e^{-x_n})t_n + x_n - e^{-x_n} + \sum_{j=2}^{n}(x_j - e^{-x_j}) = 0.$$

For $n = 2$ this equation is $0.0948t_2^3 - 0.2843t_2^2 + 1.5685t_2 - 0.0076 = 0$, resulting $t_2 = 0.0048$, hence $x_3 = 0.5695$.

For $n = 3$ the equation is $0.0943t_3^3 - 0.2829t_3^2 + 1.5658t_3 + 0.0037 = 0$, resulting $t_3 = -0.0024$, hence $x_4 = 0,5671$.

3.5 Example

We give an example, taken from [6], in which He's method does not apply. Here we use first type and improved second type methods, i.e. the resolvent equations (7) and (20), while in [6] the usual He and Wei-Wu-Mei methods were used, i.e. the resolvent equation (16), for $m = 2$ and $m = 3$.

Consider equation $f(x) = x^3 - e^{-x} = 0$. Newton-Raphson formula (8) gives the recurrence relation $x_{n+1} = x_n - \dfrac{x_n^3 - e^{-x_n}}{3x_n^2 + e^{-x_n}}$, $n = 1, 2, \ldots$.

Taking $x_1 = 1$, we obtain $x_2 = 0.8123$, $x_3 = 0.7743$ and $x_4 = 0.7729$.

For the methods based on the resolvent equation, we take $x_0 = x_1 = t_0 = 0$.

For $m = 2$, both the first and second type methods give the same quadratic equation $t_1^2 - 2t_1 + 2 = 0$, which has complex roots, therefore these methods are not applicable.

For $m = 3$, both the first and second type methods give the same cubic equation $1.1667t_1^3 - 0.5t_1^2 + t_1 - 1 = 0$, from which one obtains $t_1 = 0.7673 = x_2$. The first method gives the cubic equation $1.0774t_2^3 + 2.3019t_2^2 + 2.2305t_2 - 0.0125 = 0$, from which one obtains $t_2 = 0.0056$, hence $x_3 = 0.7729$. By the second method, the improved Wei-Wu-Mei method gives the following recurrence process

$$1.0774t_2^3 + 2.0698t_2^2 + 2.2305t_2 - 0.025 = 0, \quad t_2 = 0.0111,\ x_3 = 0.7784$$

and

$$1.0765t_3^3 + 2.1056t_3^2 + 2.2769t_3 + 0.0125 = 0,\ t_3 = -0.0055,\ x_4 = 0.7729.$$

4 Applications

In the applications below, the algebraic equations will be solved approximately by the Newton-Raphson method and also by the new methods, based on the resolvent equations, given in the present work.

4.1. *Determine the point on the graph of the function $y = e^x$ that is closest to the origin of the coordinate axes in plane and the distance from the origin to graph.*

We should minimize the distance $d(x) = \sqrt{x^2 + e^{2x}}$ from origin to an arbitrary point on the graph, hence its square, namely the function $d^2(x) = x^2+e^{2x}$. With this end in view, it is necessary to have $(d^2(x))' = 2x + 2e^{2x} = 0$. Therefore we have to solve the equation $f(x) = x + e^{2x} = 0$, which is done by the above methods. For those based on the resolvent equations, the initial data $x_0 = x_1 = 0$, $t_0 = 0$ are used.

The Newton-Raphson method is based on the recurrence relation

$$x_{n+1} = \frac{(2x_n - 1)e^{2x_n}}{1 + 2e^{2x_n}},\ n = 1, 2, \dots .$$

Taking $x_1 = 0$, we have $x_2 = -0.3333$, $x_3 = -0.4222$, and $x_4 = -0.4263$.

By the first type method of second-order, the resolvent equations are $2t_1^2 + 3t_1 + 1 = 0$, from which one obtains $t_1 = -0.5 = x_2$ and $0.7358t_2^2 + 1.7358t_2 - 0.1321 = 0$, from which one obtains $t_2 = 0.0738$, hence $x_3 = -0.4262$.

By the improved He's method we have the same first resolvent equation, with the root $t_1 = -0.5 = x_2$, then $0.7358t_2^2 + 1.7358t_2 - 0.2642 = 0$, with the root $t_2 = 0.1435$, hence $x_3 = x_2+t_2 = -0.3565$, $0.9804t_3^2+1.9803t_3+0.1353 = 0$, with the root $t_3 = -0.0708$, hence $x_4 = x_3 + t_3 = -0.4273$ and $0.8509t_4^2 + 1.8509t_4 - 0.0021 = 0$ with the root $t_4 = 0.0011$, hence $x_5 = x_4 + t_4 = -0.4262$.

By the first type method of third-order, we obtain the resolvent equations $1.3333t_1^3 + 2t_1^2 + 3t_1 + 1 = 0$, with the root $t_1 = -0.417 = x_2$ and $0.5791t_2^3 + 0.8686t_2^2 + 1.8686t_2 + 0.0173 = 0$, hence $t_2 = -0.0093$ and $x_3 = -0.4263$.

By the improved Wei, Wu and Mei method we have the same first resolvent equation, with the root $t_1 = -0.417 = x_2$, then $0.5791t_2^3+0.8686t_2^2+1.8686t_2+0.0346 = 0$, from which one obtains $t_2 = -0.0187$, hence $x_3 = x_2+t_2 = -0.4357$ and $0.5578t_3^3 + 0.8367t_3^2 + 1.8367t_3 - 0.0174 = 0$, from which one obtains $t_3 = 0.0094$, hence $x_4 = x_3 + t_3 = -0.4263$.

It follows that the point on the graph, which is the closest to the origin of coordinates is

$$(x, e^x) = (\,-0.426, 0.653)$$

the minimum distance being $d(x) = \sqrt{x^2 + e^{2x}} = 0.78$.

4.2. *Determine the point on the graph of the function $y = \ln x$ that is closest to the origin of the coordinate axes in plane and the distance from the origin to graph.*

We should minimize the distance $d(x) = \sqrt{x^2 + \ln^2 x}$ from the origin to an arbitrary point on the graph, hence its square, namely the function $d^2(x) = x^2 + \ln^2 x$. For this, it is necessary to have $(d^2(x))' = 2x + \frac{2}{x}\ln x = \frac{2}{x}(x^2 + \ln x) = 0$. Therefore we should solve the equation $f(x) = x^2 + \ln x = 0$, which is done by the above methods. For the methods based on resolvent equations the initial data $x_0 = x_1 = 0.5,\ t_0 = 0$ are used.

The Newton-Raphson method is based on recurrence relation $x_{n+1} = x_n - \frac{x_n^3 + x_n \ln x_n}{2x_n^2 + 1}$. Taking $x_1 = 0.5$, we have $x_2 = 0.6477$ and $x_3 = 0.6529$.

By the first type method of second-order, we have the resolvent equations $-t_1^2 + 3t_1 - 0.4431 = 0$, with the roots 2.8442 and $t_1 = 0.1558$, hence $x_2 = 0.6558$, and $-0.1626t_2^2 + 2.8365t_2 + 0.0082 = 0$, from which one obtains $t_2 = -0.0029$ and $x_3 = 0.6529$.

By the improved He's method we obtain the same first resolvent equation from which one obtains $t_1 = 0.1558$, hence $x_2 = x_1 + t_1 = 0.6558$, and the second resolvent equation $-0.1626t_2^2 + 2.8365t_2 + 0.0164 = 0$, with the roots 17.4504 and $t_2 = -0.0058$, hence $x_3 = x_2 + t_2 = 0.65$.

By the first type method of third-order, we have the resolvent equations $2.6667t_1^3 - t_1^2 + 3t_1 - 0.4431 = 0$, with real root $t_1 = 0.1523$, hence $x_2 = x_1 + t_1 = 0.6523$, then $0.0417t_2^3 - 0.1626t_2^2 + 2.8365t_2 + 0.0082 = 0$, from which $t_2 = -0.0029$, hence $x_3 = 0.6494$ and $0.0913t_3^3 - 0.1856t_3^2 + 2.8387t_3 - 0.01 = 0$, from which $t_3 = 0.0035$, hence $x_4 = 0.6529$.

By the improved Wei, Wu and Mei method we obtain by the same first resolvent equation the root $t_1 = 0.1523$, hence $x_2 = x_1 + t_1 = 0.6523$ and $1.201t_2^3 - 0.1751t_2^2 + 2.8376t_2 - 0.0036 = 0$, with real root $t_2 = 0.0013$, hence $x_3 = x_2 + t_2 = 0.6536$.

It follows that the point on the graph, which is the closest to the origin, is

$$(x, \ln x) = (0.653, -0.426),$$

the minimum distance being $d(x) = \sqrt{x^2 + \ln^2 x} = 0.78$.

Remark 4.1 *The points obtained in the two applications are symmetrical about the bisetrix of the first quadrant and the obtained minimum distances are the same, because the two considered functions are inverse to each other.*

5 Conclusions and open problems

Following the direction initiated by J. H. He of finding new methods of Newton-Raphson type for solving algebraic equations, which are based on resolvent polynomial equations, in this paper we present methods whose resolvent equations are simpler than those previously given by He and Wei-Wu-Mei. Taking into consideration the importance of finding such new methods and determining their theoretical foundation, we propose the following open problems related to the methods presented in this paper:

5.1. Demonstrate the convergence of the methods of Newton-Raphson type based on resolvent equations and determine their speed of convergence. Compare different methods presented in this paper in terms of their speed of convergence.

5.2. Expand the methods based on resolvent equations to systems of algebraic equations.

5.3 Establish criteria to indicate the real root of the resolvent polynomial equation that can be taken as variation of iterations of the exact real root of an algebraic equation.

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