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Existence of Multiple Positive Solutions for $3n^{th}$ Order Three-Point Boundary Value Problems on Time Scales

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Abstract

We establish the existence of at least three positive solutions for the $3n^{th}$ order three-point boundary value problem on time scales by using Leggett-Williams fixed point theorem. We also establish the existence of at least $2m-1$ positive solutions for an arbitrary positive integer m.

Keywords: *Boundary value problem, Cone, Green's function, Positive solution, Time scale.*

AMS Subject Classification: 39A10, 34B05.

1 Introduction

In this paper, we establish the existence of multiple positive solutions for the $3n^{th}$ order boundary value problem on time scales,

$$(-1)^n y^{\Delta^{(3n)}}(t) = f(t, y(t)), \quad t \in [t_1, \sigma(t_3)], \tag{1.1}$$

satisfying the general three-point boundary conditions,

$$\left.\begin{aligned}\alpha_{3i-2,1}y^{\Delta^{(3i-3)}}(t_1)+\alpha_{3i-2,2}y^{\Delta^{(3i-2)}}(t_1)+\alpha_{3i-2,3}y^{\Delta^{(3i-1)}}(t_1)=0,\\ \alpha_{3i-1,1}y^{\Delta^{(3i-3)}}(t_2)+\alpha_{3i-1,2}y^{\Delta^{(3i-2)}}(t_2)+\alpha_{3i-1,3}y^{\Delta^{(3i-1)}}(t_2)=0,\\ \alpha_{3i,1}y^{\Delta^{(3i-3)}}(\sigma(t_3))+\alpha_{3i,2}y^{\Delta^{(3i-2)}}(\sigma(t_3))+\alpha_{3i,3}y^{\Delta^{(3i-1)}}(\sigma(t_3))=0,\end{aligned}\right\} \tag{1.2}$$

for $1 \leq i \leq n$, where $n \geq 1$, $\alpha_{3i-2,j}, \alpha_{3i-1,j}, \alpha_{3i,j}$, for $j = 1, 2, 3$, are real constants, $t_1 < t_2 < \sigma(t_3)$. We assume that $f : [t_1, \sigma(t_3)] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $f(t, \cdot)$ does not vanish identically on any subset of $[t_1, \sigma(t_3)]$. The study of the existence of positive solutions of the higher order boundary value problems (BVPs) arises in various fields of applied mathematics and physics. BVPs describe many phenomena in the applied mathematical sciences, which can be found in the theory of nonlinear diffusion generated by nonlinear sources, in thermal ignition of gases and in chemical or biological problems. In these applied settings only positive solutions are meaningful.

Recently, there has been much attention focussed on existence of positive solutions to the BVPs on time scales due to their striking applications to almost all area of science, engineering and technology. Studying BVPs on time scales will unify the theory of differential and difference equations and provide an accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. The existence of positive solutions are studied by many authors. A few papers along these lines are Agarwal and Regan [1], Anderson [4], Anderson and Avery [6], F. M. Atici and G. Sh.Guseinov [7], Kaufmann [19] and Sun [20].

For convenience, we use the following notations. For $1 \leq i \leq n$, let us denote, $\beta_{i_j} = \alpha_{3i-3+j,1}t_j + \alpha_{3i-3+j,2}$, $\gamma_{i_j} = \alpha_{3i-3+j,1}t_j^2 + \alpha_{3i-3+j,2}(t_j + \sigma(t_j)) + 2\alpha_{3i-3+j,3}$, for $j = 1, 2$; $\beta_{i_3} = \alpha_{3i,1}\sigma(t_3) + \alpha_{3i,2}$ and $\gamma_{i_3} = \alpha_{3i,1}(\sigma(t_3))^2 + \alpha_{3i,2}(\sigma(t_3) + \sigma^2(t_3)) + 2\alpha_{3i,3}$. Also, for $1 \leq i \leq n$, we define,

$$m_{i_{jk}} = \frac{\alpha_{3i-3+j,1}\gamma_{i_k} - \alpha_{3i-3+k,1}\gamma_{i_j}}{2(\alpha_{3i-3+j,1}\beta_{i_k} - \alpha_{3i-3+k,1}\beta_{i_j})}, \quad M_{i_{jk}} = \frac{\beta_{i_j}\gamma_{i_k} - \beta_{i_k}\gamma_{i_j}}{\alpha_{3i-3+j,1}\beta_{i_k} - \alpha_{3i-3+k,1}\beta_{i_j}},$$

for $j, k = 1, 2, 3$ and let $p_i = \max\{m_{i_{12}}, m_{i_{13}}, m_{i_{23}}\}$,
$q_i = \min\left\{m_{i_{23}} + \sqrt{m_{i_{23}}^2 - M_{i_{23}}},\ m_{i_{13}} + \sqrt{m_{i_{13}}^2 - M_{i_{13}}}\right\}$,
$d_i = \alpha_{3i-2,1}(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) - \beta_{i_1}(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) + \gamma_{i_1}(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})$ and $l_{i_j} = \alpha_{3i-3+j,1}\sigma(s)\sigma^2(s) - \beta_{i_j}(\sigma(s) + \sigma^2(s)) + \gamma_{i_j}$, where $j = 1, 2, 3$. We assume the following conditions throughout this paper:

(A1) $\alpha_{3i-2,1} > 0, \alpha_{3i-1,1} > 0, \alpha_{3i,1} > 0$ and $\frac{\alpha_{3i,2}}{\alpha_{3i,1}} > \frac{\alpha_{3i-1,2}}{\alpha_{3i-1,1}} > \frac{\alpha_{3i-2,2}}{\alpha_{3i-2,1}}$, for all $1 \leq i \leq n$,

(A2) $p_i \leq t_1 < t_2 < \sigma(t_3) \leq q_i$ and $2\alpha_{3i-2,3}\alpha_{3i-2,1} > \alpha_{3i-2,2}^2$, $2\alpha_{3i-1,3}\alpha_{3i-1,1} < \alpha_{3i-1,2}^2$, $2\alpha_{3i,3}\alpha_{3i,1} > \alpha_{3i,2}^2$, for all $1 \leq i \leq n$, (A3) $m_{i_{23}}^2 > M_{i_{23}}$, $m_{i_{12}}^2 < M_{i_{12}}$, $m_{i_{13}}^2 > M_{i_{13}}$ and $d_i > 0$, for all $1 \le i \le n$,

(A4) The point $t \in [t_1, \sigma(t_3)]$ is not left dense and right scattered at the same time.

This paper is organized as follows. In Section 2, we establish certain lemmas which are needed in our main results. In Section 3, we establish the existence of at least three positive solutions of the BVP (1.1)-(1.2) by using the Leggett-Williams fixed point theorem. We also establish the existence of at least $2m-1$ positive solutions of the BVP (1.1)-(1.2) for an arbitrary positive integer m. Finally as an application, we give an example to illustrate our result.

2 The Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function.

Let $G_i(t,s)$ be the Green's function for the homogeneous BVP,

$$-y^{\Delta^3}(t) = 0, \qquad t \in [t_1, \sigma(t_3)], \tag{2.1}$$

satisfying the general three-point boundary conditions,

$$\left.\begin{aligned} \alpha_{3i-2,1}y(t_1) + \alpha_{3i-2,2}y^{\Delta}(t_1) + \alpha_{3i-2,3}y^{\Delta^2}(t_1) = 0,\\ \alpha_{3i-1,1}y(t_2) + \alpha_{3i-1,2}y^{\Delta}(t_2) + \alpha_{3i-1,3}y^{\Delta^2}(t_2) = 0,\\ \alpha_{3i,1}y(\sigma(t_3)) + \alpha_{3i,2}y^{\Delta}(\sigma(t_3)) + \alpha_{3i,3}y^{\Delta^2}(\sigma(t_3)) = 0, \end{aligned}\right\} \tag{2.2}$$

for $1 \le i \le n$.

Lemma 2.1 *For $1 \le i \le n$, the Green's function $G_i(t,s)$ for the homogeneous BVP (2.1)-(2.2) is given by*

$$G_i(t,s) = \begin{cases} \underset{t\in[t_1,t_2]}{G_i(t,s)} = \begin{cases} G_{i_1}(t,s), & t_1 < \sigma(s) < t \le t_2 < \sigma(t_3) \\ G_{i_2}(t,s), & t_1 \le t < s < t_2 < \sigma(t_3) \\ G_{i_3}(t,s), & t_1 \le t < t_2 < s < \sigma(t_3) \end{cases} \\[2ex] \underset{t\in[t_2,\sigma(t_3)]}{G_i(t,s)} = \begin{cases} G_{i_4}(t,s), & t_1 < t_2 < \sigma(s) < t \le \sigma(t_3) \\ G_{i_5}(t,s), & t_1 < t_2 \le t < s < \sigma(t_3) \\ G_{i_6}(t,s), & t_1 \le \sigma(s) < t_2 < t < \sigma(t_3) \end{cases} \end{cases} \tag{2.3}$$

where

$$G_{i_1}(t,s) = \frac{1}{2d_i}[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + t(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - t^2(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})]l_{i_1},$$

$$\begin{aligned}
G_{i_2}(t,s) =& \frac{1}{2d_i}\{[-(\beta_{i_1}\gamma_{i_3}-\beta_{i_3}\gamma_{i_1})+t(\alpha_{3i-2,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_1})-t^2(\alpha_{3i-2,1}\beta_{i_3}- \\
&\alpha_{3i,1}\beta_{i_1})]l_{i_2}+[(\beta_{i_1}\gamma_{i_2}-\beta_{i_2}\gamma_{i_1})-t(\alpha_{3i-2,1}\gamma_{i_2}-\alpha_{3i-1,1}\gamma_{i_1})+ \\
&t^2(\alpha_{3i-2,1}\beta_{i_2}-\alpha_{3i-1,1}\beta_{i_1})]l_{i_3}\}, \\
G_{i_3}(t,s) =& \frac{1}{2d_i}[(\beta_{i_1}\gamma_{i_2}-\beta_{i_2}\gamma_{i_1})-t(\alpha_{3i-2,1}\gamma_{i_2}-\alpha_{3i-1,1}\gamma_{i_1})+t^2(\alpha_{3i-2,1}\beta_{i_2}- \\
&\alpha_{3i-1,1}\beta_{i_1})]l_{i_3}, \\
G_{i_4}(t,s) =& \frac{1}{2d_i}\{[-(\beta_{i_2}\gamma_{i_3}-\beta_{i_3}\gamma_{i_2})+t(\alpha_{3i-1,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_2})-t^2(\alpha_{3i-1,1}\beta_{i_3}- \\
&\alpha_{3i,1}\beta_{i_2})]l_{i_1}+[(\beta_{i_1}\gamma_{i_3}-\beta_{i_3}\gamma_{i_1})-t(\alpha_{3i-2,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_1})+ \\
&t^2(\alpha_{3i-2,1}\beta_{i_3}-\alpha_{3i,1}\beta_{i_1})]l_{i_2}\}, \\
G_{i_5}(t,s) =& \frac{1}{2d_i}[(\beta_{i_1}\gamma_{i_2}-\beta_{i_2}\gamma_{i_1})-t(\alpha_{3i-2,1}\gamma_{i_2}-\alpha_{3i-1,1}\gamma_{i_1})+t^2(\alpha_{3i-2,1}\beta_{i_2}- \\
&\alpha_{3i-1,1}\beta_{i_1})]l_{i_3}, \\
G_{i_6}(t,s) =& \frac{1}{2d_i}[-(\beta_{i_2}\gamma_{i_3}-\beta_{i_3}\gamma_{i_2})+t(\alpha_{3i-1,1}\gamma_{i_3}-\alpha_{3i,1}\gamma_{i_2})-t^2(\alpha_{3i-1,1}\beta_{i_3}- \\
&\alpha_{3i,1}\beta_{i_2})]l_{i_1}.
\end{aligned}$$

Lemma 2.2 *Assume that the conditions* $(A1)$-$(A4)$ *are satisfied. Then, for* $1 \le i \le n$*, the Green's function* $G_i(t,s)$ *of* (2.1)-(2.2) *is positive, for all* $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Proof: For $1 \le i \le n$, the Green's function $G_i(t,s)$ is given in (2.3). By using the conditions $(A1)$-$(A4)$, we obtain

$$G_{i_1}(t,s) > 0, \text{ for all } (t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3].$$

Similarly, we can establish the positivity of the Green's function in the remaining cases. □

Theorem 2.3 *Assume that the conditions* $(A1)$-$(A4)$ *are satisfied. Then, for* $1 \le i \le n$*, the Green's function* $G_i(t,s)$ *satisfies the following inequality,*

$$m_i G_i(\sigma(s),s) \le G_i(t,s) \le G_i(\sigma(s),s), \text{ for all } (t,s) \in [t_1,\sigma(t_3)] \times [t_1,t_3], \tag{2.4}$$

where

$$0 < m_i = \min\left\{\frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)}\right\} < 1.$$

Proof: For $1 \le i \le n$, the Green's function $G_i(t,s)$ is given (2.3) in six different cases. In each case we prove the inequality as in (2.4).

Case 1. For $t_1 < \sigma(s) < t \le t_2 < \sigma(t_3)$.

$\frac{G_i(t,s)}{G_i(\sigma(s),s)} = \frac{G_{i_1}(t,s)}{G_{i_1}(\sigma(s),s)}$

$$= \frac{[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + t(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - t^2(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})]}{[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + \sigma(s)(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - (\sigma(s))^2(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})]}.$$

From $(A1)$-$(A4)$, we have $G_{i_1}(t,s) \le G_{i_1}(\sigma(s),s)$ and also

$$\frac{G_i(t,s)}{G_i(\sigma(s),s)} = \frac{G_{i_1}(t,s)}{G_{i_1}(\sigma(s),s)} \ge \frac{G_{i_1}(t,s)}{G_{i_1}(t_1,s)} \ge \frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}.$$

Therefore, $G_i(t,s) \le G_i(\sigma(s),s)$ and $G_i(t,s) \ge \frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}\, G_i(\sigma(s),s)$, for all $(t,s) \in [t_1,\sigma(t_3)] \times [t_1,t_3]$.

Case 2. For $t_1 \le t < t_2 < s < \sigma(t_3)$.

$\frac{G_i(t,s)}{G_i(\sigma(s),s)} = \frac{G_{i_3}(t,s)}{G_{i_3}(\sigma(s),s)}$

$$= \frac{[(\beta_{i_1}\gamma_{i_2} - \beta_{i_2}\gamma_{i_1}) - t(\alpha_{3i-2,1}\gamma_{i_2} - \alpha_{3i-1,1}\gamma_{i_1}) + t^2(\alpha_{3i-2,1}\beta_{i_2} - \alpha_{3i-1,1}\beta_{i_1})]}{[(\beta_{i_1}\gamma_{i_2} - \beta_{i_2}\gamma_{i_1}) - \sigma(s)(\alpha_{3i-2,1}\gamma_{i_2} - \alpha_{3i-1,1}\gamma_{i_1}) + (\sigma(s))^2(\alpha_{3i-2,1}\beta_{i_2} - \alpha_{3i-1,1}\beta_{i_1})]}.$$

From $(A1)$-$(A4)$, we have $G_{i_3}(t,s) \le G_{i_3}(\sigma(s),s)$ and also

$$\frac{G_i(t,s)}{G_i(\sigma(s),s)} = \frac{G_{i_3}(t,s)}{G_{i_3}(\sigma(s),s)} \ge \frac{G_{i_3}(t,s)}{G_{i_3}(\sigma(t_3),s)} \ge \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}.$$

Therefore, $G_i(t,s) \le G_i(\sigma(s),s)$ and $G_i(t,s) \ge \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}\, G_i(\sigma(s),s)$, for all $(t,s) \in [t_1,\sigma(t_3)] \times [t_1,t_3]$.

Case 3. For $t_1 \le t < s < t_2 < \sigma(t_3)$.
From $(A1)$-$(A4)$ and case 2, we have $G_{i_2}(t,s) \le G_{i_2}(\sigma(s),s)$ and also

$$\frac{G_i(t,s)}{G_i(\sigma(s),s)} \ge \min\left\{\frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}\right\}.$$

Therefore, $G_i(t,s) \le G_i(\sigma(s),s)$ and

$$G_i(t,s) \ge \min\left\{\frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}\right\} G_i(\sigma(s),s),$$

for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Case 4. For $t_1 < t_2 < \sigma(s) < t \le \sigma(t_3)$.
From $(A1)$-$(A4)$ and case 1, we have $G_{i_4}(t,s) \le G_{i_4}(\sigma(s), s)$ and

$$\frac{G_i(t,s)}{G_i(\sigma(s),s)} \ge \min\left\{\frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)}\right\}.$$

Therefore, $G_i(t,s) \le G_i(\sigma(s), s)$ and

$$G_i(t,s) \ge \min\left\{\frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)}\right\} G_i(\sigma(s),s),$$

for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Case 5. For $t_1 < t_2 \le t < s < \sigma(t_3)$.
From case 2, we have $G_i(t,s) \le G_i(\sigma(s), s)$ and $G_i(t,s) \ge \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)} G_i(\sigma(s), s)$, for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Case 6. For $t_1 \le \sigma(s) < t_2 < t < \sigma(t_3)$.
From case 1, we have $G_i(t,s) \le G_i(\sigma(s), s)$ and $G_i(t,s) \ge \frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)} G_i(\sigma(s), s)$, for all $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

From all above cases, for $1 \le i \le n$, we have

$$m_i G_i(\sigma(s),s) \le G_i(t,s) \le G_i(\sigma(s),s), \text{ for all } (t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3],$$

where

$$0 < m_i = \min\left\{\frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)}\right\} < 1.$$

□

Lemma 2.4 *Assume that the conditions* $(A1)$-$(A4)$ *are satisfied and* $G_i(t,s)$ *is defined as in* (2.3). *Take* $H_1(t,s) = G_1(t,s)$ *and recursively define*

$$H_j(t,s) = \int_{t_1}^{\sigma(t_3)} H_{j-1}(t,r) G_j(r,s) \Delta r, \ for\ 2 \le j \le n.$$

Then $H_n(t,s)$ *is the Green's function for the homogeneous BVP corresponding to* (1.1)-(1.2).

Lemma 2.5 *Assume that the conditions* (*A*1)*-*(*A*4) *hold. If we define*

$$K = \prod_{j=1}^{n-1} K_j \quad \text{and} \quad L = \prod_{j=1}^{n-1} m_j L_j,$$

then the Green's function $H_n(t,s)$ *in Lemma 2.4 satisfies*

$$0 \le H_n(t,s) \le K\|G_n(\cdot,s)\|, \text{ for all } (t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$$

and

$$H_n(t,s) \ge m_n L\|G_n(\cdot,s)\|, \text{ for all } (t,s) \in [t_2, \sigma(t_3)] \times [t_1, t_3],$$

where m_n *is given as in Theorem 2.3,*

$$K_j = \int_{t_1}^{\sigma(t_3)} \|G_j(\cdot,s)\|\Delta s > 0, \text{ for } 1 \le j \le n,$$

$$L_j = \int_{t_2}^{\sigma(t_3)} \|G_j(\cdot,s)\|\Delta s > 0, \text{ for } 1 \le j \le n$$

and $\|\cdot\|$ *is defined by*

$$\|x\| = \max_{t\in[t_1,\sigma(t_3)]} |x(t)|.$$

3 Existence of Multiple Positive Solutions

In this section, we establish the existence of at least three positive solutions to the BVP (1.1)-(1.2) by using the Leggett-Williams fixed point theorem. We also establish the existence of at least $2m-1$ positive solutions for the BVP (1.1)-(1.2) for an arbitrary positive integer m.

Let E be a real Banach space with cone P. A map $S : P \to [0,\infty)$ is said to be a nonnegative continuous concave functional on P, if S is continuous and

$$S(\lambda x + (1-\lambda)y) \ge \lambda S(x) + (1-\lambda)S(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Let α and β be two real numbers such that $0 < \alpha < \beta$ and S be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_\alpha = \{y \in P : \|y\| < \alpha\} \text{ and}$$

$$P(S,\alpha,\beta) = \{y \in P : \alpha \le S(y), \|y\| \le \beta\}.$$

Theorem 3.1 [Leggett-Williams fixed point theorem] *Let $T : \overline{P}_c \to \overline{P}_c$ be completely continuous and S be a nonnegative continuous concave functional on P such that $S(y) \leq \|y\|$, for all $y \in \overline{P}_c$. Suppose that there exist a, b, c and d with $0 < d < a < b \leq c$ such that*
(i) $\{y \in P(S, a, b) : S(y) > a\} \neq \emptyset$ *and* $S(Ty) > a$, *for* $y \in P(S, a, b)$,
(ii) $\|Ty\| < d$, *for* $\|y\| \leq d$,
(iii) $S(Ty) > a$, *for* $y \in P(S, a, c)$ *with* $\|Ty\| > b$.
Then T has at least three fixed points y_1, y_2, y_3 in $\overline{P}_c$ satisfying

$$\|y_1\| < d, \;\; a < S(y_2), \;\; \|y_3\| > d, \;\; S(y_3) < a.$$

Let

$$M = m_n \prod_{j=1}^{n-1} \frac{m_j L_j}{K_j}.$$

Theorem 3.2 *Assume that the conditions (A1)-(A4) are satisfied and also assume that there exist real numbers a_0, a_1 and a_2 with $0 < a_0 < a_1 < \frac{a_1}{M} < a_2$ such that*

$$f(t, y(t)) < \frac{a_0}{\prod_{j=1}^n K_j}, \quad \text{for } t \in [t_1, \sigma(t_3)] \text{ and } y \in [0, a_0], \tag{3.1}$$

$$f(t, y(t)) > \frac{a_1}{\prod_{j=1}^n m_j L_j}, \quad \text{for } t \in [t_2, \sigma(t_3)] \text{ and } y \in [a_1, \frac{a_1}{M}], \tag{3.2}$$

$$f(t, y(t)) < \frac{a_2}{\prod_{j=1}^n K_j}, \quad \text{for } t \in [t_1, \sigma(t_3)] \text{ and } y \in [0, a_2]. \tag{3.3}$$

Then the BVP (1.1)-(1.2) has at least three positive solutions.

Proof: Let the Banach Space $E = C[t_1, \sigma(t_3)]$ be equipped with the norm

$$\|y\| = \max_{t \in [t_1, \sigma(t_3)]} |y(t)|.$$

We denote

$$P = \{y \in E : y(t) \geq 0, \; t \in [t_1, \sigma(t_3)]\}.$$

Then, it is obvious that P is a cone in E. For $y \in P$, we define

$$S(y) = \min_{t \in [t_2, \sigma(t_3)]} |y(t)| \;\; \text{and}$$

$$Ty(t) = \int_{t_1}^{\sigma(t_3)} H_n(t, s) f(s, y(s)) \Delta s, \quad t \in [t_1, \sigma(t_3)].$$

It is easy to see that S is a nonnegative continuous concave functional on P with $S(y) \leq \|y\|$, for $y \in P$ and that $T : P \to P$ is completely continuous and fixed points of T are solutions of the BVP (1.1)-(1.2). First, we prove that, if there exists a positive number r such that $f(t,y(t)) < \frac{r}{\prod_{j=1}^{n} K_j}$, for $t \in [t_1, \sigma(t_3)]$ and $y \in [0,r]$, then $T : \overline{P}_r \to P_r$. Indeed, if $y \in \overline{P}_r$, then for $t \in [t_1, \sigma(t_3)]$, we have

$$\begin{aligned} Ty(t) &= \int_{t_1}^{\sigma(t_3)} H_n(t,s) f(s,y(s))\Delta s \\ &< \frac{r}{\prod_{j=1}^{n} K_j} \int_{t_1}^{\sigma(t_3)} H_n(t,s)\Delta s \\ &\leq \frac{r}{\prod_{j=1}^{n} K_j} K \int_{t_1}^{\sigma(t_3)} \|G_n(\cdot,s)\|\Delta s = r. \end{aligned}$$

Thus, $\|Ty\| < r$, that is, $Ty \in P_r$. Hence, we have shown that if (3.1) and (3.3) hold, then T maps $\overline{P}_{a_0}$ into P_{a_0} and $\overline{P}_{a_2}$ into P_{a_2}. Next, we show that $\{y \in P(S, a_1, \frac{a_1}{M}) : S(y) > a_1\} \neq \emptyset$ and $S(Ty) > a_1$, for all $y \in P(S, a_1, \frac{a_1}{M})$. In fact, the constant function

$$\frac{a_1 + \frac{a_1}{M}}{2} \in \{y \in P(S, a_1, \frac{a_1}{M}) : S(y) > a_1\}.$$

Moreover, for $y \in P(S, a_1, \frac{a_1}{M})$, we have

$$\frac{a_1}{M} \geq \|y\| \geq y(t) \geq \min_{t\in[t_2,\sigma(t_3)]} y(t) = S(y) \geq a_1,$$

for all $t \in [t_2, \sigma(t_3)]$. Thus, in view of (3.2), we see that

$$\begin{aligned} S(Ty) &= \min_{t\in[t_2,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} H_n(t,s) f(s,y(s))\Delta s \\ &\geq \min_{t\in[t_2,\sigma(t_3)]} \int_{t_2}^{\sigma(t_3)} H_n(t,s) f(s,y(s))\Delta s \\ &> \frac{a_1}{\prod_{j=1}^{n} m_j L_j} m_n L \int_{t_2}^{\sigma(t_3)} \|G_n(\cdot,s)\|\Delta s = a_1 \end{aligned}$$

as required. Finally, we show that, if $y \in P(S, a_1, a_2)$ and $\|Ty\| > \frac{a_1}{M}$, then $S(Ty) > a_1$. To see this, we suppose that $y \in P(S, a_1, a_2)$ and $\|Ty\| > \frac{a_1}{M}$, then, by Lemma 2.5, we have

$$\begin{aligned} S(Ty) &= \min_{t\in[t_2,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} H_n(t,s) f(s,y(s))\Delta s \\ &\geq \min_{t\in[t_2,\sigma(t_3)]} m_n L \int_{t_1}^{\sigma(t_3)} \|G_n(\cdot,s)\| f(s,y(s))\Delta s \\ &\geq m_n L \int_{t_2}^{\sigma(t_3)} \|G_n(\cdot,s)\| f(s,y(s))\Delta s \end{aligned}$$

for all $t \in [t_1, \sigma(t_3)]$. Thus

$$
\begin{aligned}
S(Ty) &\geq \frac{m_n L}{K} \max_{t\in[t_1,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} H_n(t,s) f(s, y(s))\Delta s \\
&= \frac{m_n L}{K} \|Ty\| \\
&> \frac{m_n L}{K} \frac{a_1}{M} = a_1.
\end{aligned}
$$

To sum up, all the hypotheses of Theorem 3.1 are satisfied. Hence T has at least three fixed points, that is, the BVP (1.1)-(1.2) has at least three positive solutions y_1, y_2 and y_3 such that

$$
\|y_1\| < a_0, \; a_1 < \min_{t\in[t_2,\sigma(t_3)]} y_2(t), \; \|y_3\| > a_0, \; \min_{t\in[t_2,\sigma(t_3)]} y_3(t) < a_1.
$$

□

Theorem 3.3 *Let m be an arbitrary positive integer. Assume that there exist numbers $a_i (i = 1, 2, \cdots, m)$ and $b_j (j = 1, 2, \cdots, m-1)$ with $0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < \cdots < a_{m-1} < b_{m-1} < \frac{b_{m-1}}{M} < a_m$ such that*

$$
f(t, y(t)) < \frac{a_i}{\prod_{j=1}^{n} K_j}, \; \textit{for } t \in [t_1, \sigma(t_3)] \textit{ and } y \in [0, a_i], i = 1, 2, \cdots, m, \tag{3.4}
$$

$$
f(t, y(t)) > \frac{b_j}{\prod_{j=1}^{n} m_j L_j}, \; \textit{for } t \in [t_2, \sigma(t_3)] \textit{ and } y \in [b_j, \frac{b_j}{M}], j = 1, 2, \cdots, m-1. \tag{3.5}
$$

Then the BVP (1.1)-(1.2) *has at least* $2m-1$ *positive solutions in* $\overline{P}_{a_m}$.

Proof: We use induction on m. First, for $m = 1$, we know from (3.4) that $T : \overline{P}_{a_1} \to P_{a_1}$, then, it follows from Schauder fixed point theorem that the BVP (1.1)-(1.2) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $m = k$. In order to prove that this conclusion holds for $m = k+1$, we suppose that there exist numbers $a_i (i = 1, 2, \cdots, k+1)$ and $b_j (j = 1, 2, \cdots, k)$ with $0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < \cdots < a_k < b_k < \frac{b_k}{M} < a_{k+1}$ such that

$$
f(t, y(t)) < \frac{a_i}{\prod_{j=1}^{n} K_j}, \text{ for } t \in [t_1, \sigma(t_3)] \text{ and } y \in [0, a_i], i = 1, 2, \cdots, k+1, \tag{3.6}
$$

$$
f(t, y(t)) > \frac{b_j}{\prod_{j=1}^{n} m_j L_j}, \text{ for } t \in [t_2, \sigma(t_3)] \text{ and } y \in [b_j, \frac{b_j}{M}], j = 1, 2, \cdots, k. \tag{3.7}
$$

By assumption, the BVP (1.1)-(1.2) has at least $2k-1$ positive solutions $u_i (i = 1, 2, \cdots, 2k-1)$ in $\overline{P}_{a_k}$. At the same time, it follows from Theorem 3.2, (3.6) and (3.7) that the BVP (1.1)-(1.2) has at least three positive solutions u, v and w in $\overline{P}_{a_{k+1}}$ such that

$$\|u\| < a_k,\ b_k < \min_{t\in[t_2,\sigma(t_3)]} v(t),\ \|w\| > a_k,\quad \min_{t\in[t_2,\sigma(t_3)]} w(t) < b_k.$$

Obviously, v and w are different from $u_i (i = 1, 2, \cdots, 2k-1)$. Therefore, the BVP (1.1)-(1.2) has at least $2k+1$ positive solutions in $\overline{P}_{a_{k+1}}$, which shows that this conclusion holds for $m = k+1$. $\square$

4 Example

Let us consider an example to illustrate the usage of the Theorem 3.2. Let $n = 2$ and $\mathbb{T} = \{0\} \cup \{\frac{1}{2^{n+1}} : n \in \mathbb{N}\} \cup [\frac{1}{2}, \frac{3}{2}]$. Now, consider the following BVP,

$$y^{\Delta^6}(t) = f(t, y),\ t \in [0, \sigma(1)] \cap \mathbb{T} \tag{4.1}$$

subject to the boundary conditions,

$$\left.\begin{aligned} \frac{1}{2}y(0) - y^{\Delta}(0) + 2y^{\Delta^2}(0) = 0,\\ 2y\Big(\frac{1}{2}\Big) - 3y^{\Delta}\Big(\frac{1}{2}\Big) + 2y^{\Delta^2}\Big(\frac{1}{2}\Big) = 0,\\ y(\sigma(1)) + \frac{1}{2}y^{\Delta}(\sigma(1)) + \frac{1}{3}y^{\Delta^2}(\sigma(1)) = 0,\\ \frac{3}{4}y^{\Delta^3}(0) - 2y^{\Delta^4}(0) + 3y^{\Delta^5}(0) = 0,\\ y^{\Delta^3}\Big(\frac{1}{2}\Big) - 2y^{\Delta^4}\Big(\frac{1}{2}\Big) + y^{\Delta^5}\Big(\frac{1}{2}\Big) = 0,\\ y^{\Delta^3}(\sigma(1)) + \frac{1}{2}y^{\Delta^4}(\sigma(1)) + \frac{1}{2}y^{\Delta^5}(\sigma(1)) = 0, \end{aligned}\right\} \tag{4.2}$$

and

$$f(t, y) = \begin{cases} \frac{sint}{100} + \frac{13}{50}y^{10}, & y \le 2,\\ \frac{sint}{100} + \frac{6656}{25}, & y \ge 2. \end{cases}$$

Then the conditions $(A1)$-$(A4)$ are satisfied. The Green's function $G_1(t,s)$ in Lemma 2.1 is

$$G_1(t,s)=\begin{cases} \underset{t\in[0,\frac{1}{2}]}{G_1(t,s)} = \begin{cases} G_{1_1}(t,s), & 0<\sigma(s)<t\le\frac{1}{2}<\sigma(1)\\ G_{1_2}(t,s), & 0\le t<s<\frac{1}{2}<\sigma(1)\\ G_{1_3}(t,s), & 0\le t<\frac{1}{2}<s<\sigma(1)\end{cases}\\ \\ \underset{t\in[\frac{1}{2},\sigma(1)]}{G_1(t,s)} = \begin{cases} G_{1_4}(t,s), & 0<\frac{1}{2}<\sigma(s)<t\le\sigma(1)\\ G_{1_5}(t,s), & 0<\frac{1}{2}\le t<s<\sigma(1)\\ G_{1_6}(t,s), & 0\le\sigma(s)<\frac{1}{2}<t<\sigma(1)\end{cases}\end{cases}$$

where

$$\begin{aligned}
G_{1_1}(t,s) =&\frac{12}{481}\Big[\frac{91}{12}+\frac{23}{6}t-5t^2\Big]\Big[\frac{1}{2}\sigma(s)\sigma^2(s)+(\sigma(s)+\sigma^2(s))+4\Big],\\
G_{1_2}(t,s) =&\frac{12}{481}\Big\{\Big[\frac{26}{3}-\frac{8}{3}t-\frac{7}{4}t^2\Big]\Big[2\sigma(s)\sigma^2(s)+2(\sigma(s)+\sigma^2(s))+\frac{3}{2}\Big]\\
&+\Big[\frac{13}{2}+\frac{29}{4}t+t^2\Big]\Big[\sigma(s)\sigma^2(s)-\frac{3}{2}(\sigma(s)+\sigma^2(s))+\frac{8}{3}\Big]\Big\},\\
G_{1_3}(t,s) =&\frac{12}{481}\Big[\frac{13}{2}+\frac{29}{4}t+t^2\Big]\Big[\sigma(s)\sigma^2(s)-\frac{3}{2}(\sigma(s)+\sigma^2(s))+\frac{8}{3}\Big],\\
G_{1_4}(t,s) =&\frac{12}{481}\Big\{\Big[\frac{91}{12}+\frac{23}{6}t-5t^2\Big]\Big[\frac{1}{2}\sigma(s)\sigma^2(s)+(\sigma(s)+\sigma^2(s))+4\Big]\\
&+\Big[-\frac{26}{3}+\frac{8}{3}t+\frac{7}{4}t^2\Big]\Big[2\sigma(s)\sigma^2(s)+2(\sigma(s)+\sigma^2(s))+\frac{3}{2}\Big]\Big\},\\
G_{1_5}(t,s) =&\frac{12}{481}\Big[\frac{13}{2}+\frac{29}{4}t+t^2\Big]\Big[\sigma(s)\sigma^2(s)-\frac{3}{2}(\sigma(s)+\sigma^2(s))+\frac{8}{3}\Big],\\
G_{1_6}(t,s) =&\frac{12}{481}\Big[\frac{91}{12}+\frac{23}{6}t-5t^2\Big]\Big[\frac{1}{2}\sigma(s)\sigma^2(s)+(\sigma(s)+\sigma^2(s))+4\Big].
\end{aligned}$$

The Green's function $G_2(t,s)$ in Lemma 2.1 is

$$G_2(t,s)=\begin{cases} \underset{t\in[0,\frac{1}{2}]}{G_2(t,s)} = \begin{cases} G_{2_1}(t,s), & 0<\sigma(s)<t\le\frac{1}{2}<\sigma(1)\\ G_{2_2}(t,s), & 0\le t<s<\frac{1}{2}<\sigma(1)\\ G_{2_3}(t,s), & 0\le t<\frac{1}{2}<s<\sigma(1)\end{cases}\\ \\ \underset{t\in[\frac{1}{2},\sigma(1)]}{G_2(t,s)} = \begin{cases} G_{2_4}(t,s), & 0<\frac{1}{2}<\sigma(s)<t\le\sigma(1)\\ G_{2_5}(t,s), & 0<\frac{1}{2}\le t<s<\sigma(1)\\ G_{2_6}(t,s), & 0\le\sigma(s)<\frac{1}{2}<t<\sigma(1)\end{cases}\end{cases}$$

where

$$G_{2_1}(t,s) =\frac{16}{635}\Big[\frac{39}{8}+\frac{11}{4}t-3t^2\Big]\Big[\frac{3}{4}\sigma(s)\sigma^2(s)+2(\sigma(s)+\sigma^2(s))+6\Big],$$

$$G_{2_2}(t,s) = \frac{16}{635}\Big\{\Big[15 - \frac{15}{4}t - \frac{25}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) + \frac{3}{2}(\sigma(s)+\sigma^2(s)) + \frac{1}{4}\Big] + \Big[\frac{17}{2} + \frac{93}{16}t + \frac{7}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + 3\Big]\Big\},$$

$$G_{2_3}(t,s) = \frac{16}{635}\Big[\frac{17}{2} + \frac{93}{16}t + \frac{7}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + 3\Big],$$

$$G_{2_4}(t,s) = \frac{16}{635}\Big\{\Big[\frac{39}{8} + \frac{11}{4}t - 3t^2\Big]\Big[\frac{3}{4}\sigma(s)\sigma^2(s) + 2(\sigma(s)+\sigma^2(s)) + 6\Big] + \Big[-15 + \frac{15}{4}t + \frac{25}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) + \frac{3}{2}(\sigma(s)+\sigma^2(s)) + \frac{1}{4}\Big]\Big\},$$

$$G_{2_5}(t,s) = \frac{16}{635}\Big[\frac{17}{2} + \frac{93}{16}t + \frac{7}{8}t^2\Big]\Big[\sigma(s)\sigma^2(s) - \frac{3}{2}(\sigma(s)+\sigma^2(s)) + 3\Big],$$

$$G_{2_6}(t,s) = \frac{16}{635}\Big[\frac{39}{8} + \frac{11}{4}t - 3t^2\Big]\Big[\frac{3}{4}\sigma(s)\sigma^2(s) + 2(\sigma(s)+\sigma^2(s)) + 6\Big].$$

From Theorem 2.3 and Lemma 2.5, we get

$$m_1 = 0.4406779661, \quad K_1 = 0.6552328771, \quad L_1 = 0.183991684,$$

$$m_2 = 0.5596707819, \quad K_2 = 0.7449516076, \quad L_2 = 0.2551181102.$$

Therefore, $K = 0.6552328771$, $L = 0.08108108108$ and $M = 0.06925585335$. Clearly f is continuous and increasing on $[0,\infty)$. If we choose $a_0 = 0.25$, $a_1 = 2$ and $a_2 = 150$ then $0 < a_0 < a_1 < \frac{a_1}{M} < a_2$ and f satisfies

(i) $f(t,y) < 0.512172512 = \frac{a_0}{\Pi_{j=1}^2 K_j}$, for $t \in [0,\sigma(1)]$ and $y \in [0, 0.25]$,

(ii) $f(t,y) > 172.757353 = \frac{a_1}{\Pi_{j=1}^2 m_j L_j}$, for $t \in [\frac{1}{2},\sigma(1)]$ and $y \in [2, 28.87842548]$,

(iii) $f(t,y) < 307.3035072 = \frac{a_2}{\Pi_{j=1}^2 K_j}$, for $t \in [0,\sigma(1)]$ and $y \in [0,150]$.

Then all the conditions of Theorem 3.2 are satisfied. Thus, by Theorem 3.2, the BVP (4.1)-(4.2) has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\| y_1 \| < 0.25, \; 2 < \min_{t\in[\frac{1}{2},\sigma(1)]} y_2(t), \quad \| y_3 \| > 0.25, \quad \min_{t\in[\frac{1}{2},\sigma(1)]} y_3(t) < 2.$$

Remark: If f involves the derivatives of y in the equation (1.1), we can establish the existence of positive solutions by defining a suitable cone in Banach space with suitable norm or by applying the method used by Xu and Yang [22].

5 Open Problem

In this paper, we established the existence of at least three positive solutions for $3n^{th}$ order three-point boundary value problem on time scales by using Leggett-Williams fixed point theorem. It will be interesting to obtain multiple positive solutions for the $3n^{th}$ order general boundary value problem on time scales,

$$(-1)^n y^{\Delta^{(3n)}}(t) = f(t, y(t), y^{\Delta}(t), \cdots, y^{\Delta^{(3n-1)}}(t))$$

satisfying the general three-point boundary conditions,

$$\sum_{k=1}^{3}[\alpha_{3i-3+j,k}y^{\Delta^{(3i-4+k)}}(t_1)+\beta_{3i-3+j,k}y^{\Delta^{(3i-4+k)}}(t_2)+\gamma_{3i-3+j,k}y^{\Delta^{(3i-4+k)}}(\sigma(t_3))] = 0,$$

for $j = 1, 2, 3$, and $1 \leq i \leq n$, where $n \geq 1$.

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