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The Existence of Intrinsic Set Properties Implies Cantor's Theorem. The concept of Cardinal Revisited.

Juan-Esteban Palomar Tarancón

Dep. Math. Institut Jaume I, Burriana - (Castellón) Spain
e-mail: jepalomar.tarancon@gmail.com

Abstract

The author introduces the concept of intrinsic set property, by means of which the well-known Cantor's Theorem can be deduced. As a natural consequence of this fact, it is proved that Cantor's Theorem need not imply the existence of a tower of different-size infinities, because the impossibility of defining a bijection between any infinite countable set and its power can be a consequence of the existence of any intrinsic property which does not depend on size.

Keywords: Cardinal, actual infinity, computability [1].

1 Introduction

Roughly speaking, what we term intrinsic set properties are those predicates, being satisfied by some sets, which are preserved under bijections. Thus, if an intrinsic set property **P** is satisfied by a set X, then it is also satisfied by any other set Y having the same cardinal as X. Recall that two sets have the same cardinal if and only if there is a bijection between them. Thus, cardinality is the most natural intrinsic set property, because of it is a concept that cannot be separated from the bijection notion.

Other intrinsic property of any set is its size. Indeed, size is preserved under bijections. It is worth mentioning that, although size and cardinality are frequently regarded as synonymous, in this paper cannot be identified.

[1] *Mathematical Subject classification: 03E10,03E65*

Cardinals are defined by means of bijective maps; therefore the concept of cardinal is comparative. In fact, cardinals are equivalence-classes, and the corresponding equivalence relation $\mathfrak{C}$ is cardinality, which is defined as follows. For every couple of sets X and Y, the pair (X, Y) lies in $\mathfrak{C}$ if and only if there is a bijection $f : X \to Y$. Usually this relation is denoted writing $\#(X) = \#(Y)$. However, assuming the existence of at least one couple of sets X and Y such that the existence of a bijection between them is an undecidable question, then the cardinals of both X and Y cannot be compared, but we can suppose that, although there were no possibility of discerning whether or not the equality $\#(X) = \#(Y)$ holds, each of these sets has a certain size. Of course, for ordinary finite sets cardinality and size can be always identified, but in our context we are assuming that there are sets the cardinals of which cannot be compared. Nevertheless, even when it is a non-accessible attribute, size is an intrinsic set property which cannot be disregarded without forgetting the identity of the considered set.

To analyze these concepts deeply, consider the following statements.

A1: *"There is a bijection between X and Y".*

A2: *"Both sets X and Y have the same size".*

The first predicate is equivalent to the relation $\#(X) = \#(Y)$, since bijection-existence and cardinal-equality are synonymous. With these statements, the well-known Hume's principle can be stated as follows.

A3: There is a bijection between X and Y if and only if both sets are equal in size.

This equivalence can be split into the following implications.

A3a: If there is a bijection between X and Y, then both sets are equal in size.

A3b: If both sets X and Y are equal in size, then there is a bijection between them.

The later is equivalent to the following one.

A3bb: If there is no bijections between X and Y, then both sets are different in size.

Statement **A3a** is evident; but, in general, both Statement **A3b** and Statement **A3bb** cannot be assumed without proving them. On the one hand, suppose that each member of a set $X = \{x_1, x_2 \dots x_n\}$ is defined by means of an infinite collection of predicates and there is no algorithm handling them. Under these circumstances it is not clear that a bijection can be defined between X and $I_n = \{1, 2 \dots n\}$, since to define a bijection it is required to discern whether or not two members of X are the same. If there is no algorithm by means of which they can be distinguished, it is not clear that a bijection can be defined by some finite procedure. Even the existence of a bijection $\gamma : X \to I_n$ can be an undecidable question. In fact, the possibility of defining a bijection only is guaranteed whenever the members of the involved sets can be determined or approached arbitrarily, by means of some finite procedure or algorithm. If a general bijection theory dealing only with bijection existence is consistent, then, according to Gödel's theorem [6], such a theory cannot be complete, that is to say, it must contain at least one undecidable proposition. Such an undecidability can involve the existence of some bijection, and consequently the cardinals of the corresponding sets cannot be compared. If this is the case, the nonexistence of a bijection need not imply size-inequality, that is to say, Statement **A3bb** need not be true.

To illustrate this fact, consider a real number r lying in the unit interval $[0, 1]$, and suppose that the decimal notation of r is $0.c_1c_2 \dots c_n \dots$. If no previous constrain is imposed, then, in order to construct r, the figures c_n can be chosen at random; however to obtain a rational number lying in the unit interval the method is not admissible. Thus, the possibility of constructing a real number by this method is a property that no rational one can satisfy. It is not difficult to see, that the randomness nature is preserved under bijections; consequently, no bijection can translate random constructions from $\mathbb{R} \cap [0, 1]$ into $\mathbb{Q} \cap [0, 1]$, and proofs and inferences can be based upon this construction method. By contrast, for other purposes any result can be biased depending on the assumed construction method, for instance see [9].

On the other hand, consider that bijections are structure-free set isomorphisms. An isomorphism between two structured sets X and Y can only be defined provided that both X and Y satisfy the same structure properties. For instance, if $G_4 = (\mathbb{Z}_4, +)$ is the cyclic group of order 4, and $P_4 = (\mathbb{Z}_2, +) \times (\mathbb{Z}_2, +)$ is the direct product of two instances of $(\mathbb{Z}_2, +)$, then both G_4 and P_4 consist of 4 members, but there is no isomorphism between them. However, if the algebraic structure of both groups is forgotten, then there is a bijection, that is, a set-isomorphism between the corresponding underlying sets. Accordingly, bijection existence is possible when structures are forgotten. However, set-size is an unforgettable property, and this is why bijection existence depends on size, and set-size is an intrinsic set property.

The aim of this paper consists of proving the existence of another intrinsic or unforgettable set property, namely, $\mathcal{T}$-representability, and consequently the nonexistence of a bijection between two arbitrary sets need not be always a consequence of size-difference. At least, to assume Statement **A3b** a proof must be required, and this is an open problem. After these considerations the following definition is adequate.

Definition 1.1 *A property* $\mathbf{P}$ *of a set* X *is forgettable, provided that there is at least one set* Y *which does not satisfy* $\mathbf{P}$ *together with a bijection* $f : X \to Y$*. From now on, let us say every non-forgettable property* $\mathbf{P}$ *of any set* X *to be intrinsic.*

Lemma 1.2 *If a set property* $\mathbf{P}$ *is preserved under bijections, then it is intrinsic.*

Proof. If $\mathbf{P}$ is a property of a class of sets $\mathbf{C}$ which is preserved under bijections, then for every set Y and any member X of $\mathbf{C}$, the existence of a bijection $f : X \to Y$ implies that Y satisfies $\mathbf{P}$; consequently, $\mathbf{P}$ cannot be forgettable.

Lemma 1.3 *If* $\{\mathbf{P}_i | i \in I\}$ *is a class of intrinsic set-properties, then their conjunction*

$$\mathbf{P} = \wedge_{i \in I} \mathbf{P}_i$$

is again an intrinsic one.

Proof. It is a straightforward consequence of Definition 1.1.

In Theorem 2.3 it is shown the existence of an intrinsic set property, namely, $\mathcal{T}$-representability, being satisfied by $\mathbb{N}$ that its power $\mathcal{P}(\mathbb{N})$ does not satisfy. Likewise, in Corollary 2.4 Cantor' Theorem is derived from $\mathcal{T}$-representability; consequently the existence of a tower of different infinities need not be interpreted as the existence of different-size infinities. Perhaps the only difference consists of $\mathcal{T}$-representability. Discerning which is the correct case is an open problem which is proposed in section 3.

2 Intrinsic Properties

Henceforth, say a set $\mathcal{T} = \{T_i \mid i \in I\}$ of Turing machines, indexed by a no-void subset I of $\mathbb{N}$, to be regular provided that the following axioms hold.

Axiom 1: *All members of* $\mathcal{T}$ *have the same finite tape-alphabet* Γ*, the same blank* $\varnothing \in \Gamma$*, the same initial state* q_0 *and the same one-element final state set* $F = \{\mathbf{Halt}\}$*. In addition, the tape of each member of* $\mathcal{T}$ *is infinite.*

Axiom 2: *The tape alphabet* Γ *contains the* n *figures* $B_n = \{0, 1, 2 \ldots n\}$ *of a base-*n *numeral system.*

Notation. To denote a tape-configuration $a_1 a_2 a_3 \ldots a_n$ together with the underlying state q of any Turing machine T, write $(a_1 a_2 a_3 \ldots a_n; q)$. Recall that, in general, blanks are disregarded, hence only those blanks lying between two non-blank symbols will be considered. In addition, denote as Γ^* the free-monoid generated by tape-alphabet Γ; and likewise, denote as $\Gamma^{\mathbb{N}}$ the collection of all sequences in Γ.

Definition. With the same notations as in the preceding axioms, given a regular set of Turing machines $\mathcal{T}$, say a nonempty set X to be $\mathcal{T}$-representable, provided that there are two maps $\varphi : X \to \Gamma^*$ and $\tau : X \to \mathcal{T}$ together with an injective one $\Lambda : X \to \Gamma^{\mathbb{N}}$ satisfying the following condition.

Condition 1: For each $x \in X$, if $\varphi(x) = a_{x,1}a_{x,2}\ldots a_{x,k}$, then for every integer $j \in \mathbb{N}$, the base-n expression of which is $c_1c_2\ldots c_m$, and from the initial tape-configuration $(c_1c_2\ldots c_m\varnothing a_{x,1}a_{x,2}\ldots a_{x,k}; q_0)$, after a finite step sequence the Turing machine $\tau(x) = T_{i_x}$ gets the final state **Halt** together with a tape-configuration $(e_1e_2\ldots e_l\varnothing b_{x,1}b_{x,2}\ldots b_{x,s}; \textbf{Halt})$ such that, if

$$\Lambda(x) = r_1r_2r_3\cdots \in \Gamma^{\mathbb{N}}$$

then $\forall m \leq j : b_{x,m} = r_m$.

It is worth pointing out, that $\mathcal{T}$-representability for a set X implies that there is some algorithm, which is performed by a member T of $\mathcal{T}$, by means of which, for every positive integer j, the member r_j of the sequence $\Lambda(x) = r_1r_2r_3\ldots$ can be computed. For instance, the set $X = \{\sqrt{n} \mid n \in \mathbb{N}\}$ is $\mathcal{T}$-representable, whenever there is $T \in \mathcal{T}$ the corresponding algorithm of which computes the digits of the square root of any integer, in any base-m numeral system. In this case, the injective mapping $\varphi : X \to \Gamma^*$ sends each member $\sqrt{x}$ of X into the corresponding expression $s_1s_2s_3\ldots$ denoting x in the base-m numeral system, while $\Lambda(\sqrt{n}) = r_1r_2r_3\ldots$ is the numeric expression for $\sqrt{n}$ in the base-m numeral system.

Analogously, the set $\mathbb{A}$ of all algebraic numbers is $\mathcal{T}$-representable with respect to any regular Turing-machine set $\mathcal{T}$ such that some member T of which can perform the following actions. On the one hand, T can compute arbitrary numeric approaches for every solution x of any algebraic equation. On the other hand, for each positive integer j, T can compute an approach for x containing at least j-digits. Indeed, at the initial state, the tape-configuration of T contains the expression for the integer j together with the equation coefficients.

Lemma 2.1 *Let $\mathcal{T}$ be a regular set of Turing machines. Every nonempty subset Y of each $\mathcal{T}$-representable one X is again $\mathcal{T}$-representable.*

Proof. It is a straightforward consequence of the preceding definition.

Lemma 2.2 *For every regular set $\mathcal{T}$ of Turing machines, $\mathcal{T}$-representability is preserved under bijections; consequently, by virtue of Lemma 1.2, it is an intrinsic property.*

Proof. Let X be a $\mathcal{T}$-representable set and $f : X \to Y$ a bijection. By hypothesis, there are two maps $\varphi : X \to \Gamma^*$, and $\tau : X \to \mathcal{T}$ together with an injective one $\Lambda : X \to \Gamma^{\mathbb{N}}$ satisfying Condition 1. It is not difficult to see, that the three maps $\varphi \circ f^{-1} : Y \to \Gamma^*$, $\Lambda \circ f^{-1} : Y \to \Gamma^{\mathbb{N}}$ and $\tau \circ f^{-1} : Y \to \mathcal{T}$ satisfy Condition 1 too.

Remark. Lemma 2.1 implies that the concept of $\mathcal{T}$-representability does not depend on set-size, since it is a property inherited by all proper nonempty subsets. Likewise, the preceding lemma shows that it is an intrinsic property. Thus, it is an instance of intrinsic property which does not depend on size.

Theorem 2.3 *The following statements are true.*

1. *There is a regular set of Turing machines $\mathcal{T}$ such that the set $\mathbb{N}$ of all natural numbers is $\mathcal{T}$-representable.*
2. *There is a regular set of Turing machines $\mathcal{T}$ such that the set $\mathbb{Q}$ of all rational numbers is $\mathcal{T}$-representable.*
3. *If a set X is $\mathcal{T}$-representable, then there is an injective map $\gamma : X \to \mathbb{N}$, for every regular set of Turing machines $\mathcal{T}$.*
4. *There is no regular Turing-machine set $\mathcal{T}$, such that the powerset $\mathcal{P}(\mathbb{N})$ is $\mathcal{T}$-representable.*
5. *There is no regular Turing-machine set $\mathcal{T}$, such that the unit interval $[0, 1]$ is $\mathcal{T}$-representable.*

Proof.

1. Let $B = \{c_1 c_2 \dots c_n\}$ be the digits of a base-n numeral system. For every $m \in \mathbb{N}$, let $\varphi(m) = c_1 c_2 \dots c_k \in \Gamma^*$ be the expression for m in the numeral system B, and let $\Lambda(m) = c_1 c_2 \dots c_k \varnothing \varnothing \cdots \in \Gamma^{\mathbb{N}}$. If T is a Turing machine such that from the initial state q_0 changes to the final one **Halt**, remaining its tape unaltered, then Condition 1 is satisfied. Thus, Statement 1) holds for any regular Turing-machine set $\mathcal{T}$ containing T.

2. Let $B_2 = \{0, 1\} \subseteq \Gamma$ the figures of the binary numeral system, and Λ the map sending each rational number $\frac{n}{m}$ into its binary expression
$$\Lambda\left(\frac{n}{m}\right) = r_1 r_2 \dots r_n . r_{n+1} r_{n+2} \cdots$$
Let
$$\varphi\left(\frac{n}{m}\right) = a_{m,1} a_{m,2} \dots a_{m,k} \varnothing a_{n,k+1} a_{n,k+2} \dots a_{n,k+j}$$
where the sequences $a_{m,1} a_{m,2} \dots a_{m,k}$ and $a_{n,k+1} a_{n,k+2} \dots a_{n,k+j}$ are the binary expressions for n and m respectively. Let $T = \tau\left(\frac{n}{m}\right)$ be a Turing machine performing the ordinary division algorithm working with binary digits. Assume that the algorithm T can determine the j-th figure of

every quotient, for every positive integer j. With these assumptions, from the initial configuration

$$(c_1c_2\dots c_s\varnothing a_{m,1}a_{m,2}\dots a_{m,k}\varnothing a_{n,k+1}a_{n,k+2}\dots a_{n,k+j};q_0)$$

where $c_1c_2\dots c_2$ is the binary expression for j, by means of the division algorithm performed by T, it is possible to obtain the j-th digit of $\Lambda\left(\frac{n}{m}\right)$, for every positive integer $j\in\mathbb{N}$, and Condition 1 can be satisfied. Accordingly, for every regular Turing machine set $\mathcal{T}$ containing such an algorithm T, the set $\mathbb{Q}$ of rational numbers is $\mathcal{T}$-representable.

3. Since it is assumed X to be $\mathcal{T}$-representable, then, by definition, for every $x\in X$ there are a Turing machine $\tau(x)=T_{i_x}\in\mathcal{T}$ and a word $\varphi(x)=a_{x,1}a_{x,2}\dots a_{x,k}\in\Gamma^*$ satisfying Condition 1. Likewise, since $\mathcal{T}$ is indexed by a subset I of $\mathbb{N}$, there is an injective map $\xi:\mathcal{T}\to\{p_i\mid i\in\mathbb{N}\}$; where $\{p_i\mid i\in\mathbb{N}\}$ is the collection of all primes.

   With these assumptions, the injective map $\gamma:X\to\mathbb{N}$ can be defined as follows.

$$\gamma(x)=p_{i_x}^{\beta(a_{x,1})\beta(a_{x,2})\dots\beta(a_{x,k})}\in\mathbb{N}$$

   where $p_{i_x}=\xi(T_{i_x})$ and $\beta:\Gamma\to C$ is any bijection onto the figure set C of a base-n numeral system. Recall, that since Γ is assumed to be finite, then there is such a bijection, provided that $\#(C)=\#(\Gamma)$.

   It remains to be shown that γ is injective. To this end, consider that for each couple x and y of members of X, the equality

$$\gamma(x)=p_{i_x}^{\beta(a_{x,1})\beta(a_{x,2})\dots\beta(a_{x,k})}=p_{i_y}^{\beta(a_{y,1})\beta(a_{y,2})\dots\beta(a_{y,k})}=\gamma(y) \tag{1}$$

   implies $p_{i_x}=p_{i_y}$, and because of ξ is injective, then $T_{i_x}=T_{i_y}$; accordingly $\tau(x)=\tau(y)$. In addition, since both p_{i_x} and p_{i_y} are primes, then equation (1) implies that both exponents must be the same; therefore $\forall j\leq k:\beta(a_{x,j})=\beta(a_{y,j})$, and because β is bijective, then it follows that $\forall j\leq k: a_{x,j}=a_{y,j}$; hence $\varphi(x)=\varphi(y)$. Since $T_{i_x}=T_{i_y}$, by virtue of Condition 1, if $\Lambda(x)=a_1a_2a_3\dots$ and $\Lambda(y)=b_1b_2b_3\dots$, then for every $j\in\mathbb{N}$: $a_j=b_j$; consequently $\Lambda(x)=\Lambda(y)$. Now, taking into account, that by definition, the map Λ is injective; it follows that $x=y$, and γ is also injective.

4. By virtue of the preceding statement, the existence of a regular Turing machine set $\mathcal{T}$, such that the powerset $\mathcal{P}(\mathbb{N})$ is $\mathcal{T}$-representable, implies the existence of an injective map $\gamma:\mathcal{P}(\mathbb{N})\to\mathbb{N}$. Thus, if $\lambda:\mathrm{img}(\gamma)\to\mathbb{N}$ is the canonical inclusion, since γ is injective, then the map $\gamma^*:\mathcal{P}(\mathbb{N})\to\mathrm{img}(\gamma)$ determined by $\gamma=\lambda\circ\gamma^*$ is bijective, consequently there is a bijection between $\mathcal{P}(\mathbb{N})$ and a subset of $\mathbb{N}$ which contradicts Cantor's Theorem. Recall that since λ is the canonical inclusion, then it is injective; hence γ^* is uniquely specified by $\gamma=\lambda\circ\gamma^*$.

5. First, we prove that there is a bijection $\delta : [0,1] \to \mathcal{P}(\mathbb{N})$. For every $x \in [0,1]$, let $0, c_{x,1}c_{x,2} \ldots c_{x,n} \ldots$ be the expression for x in the binary numeral system; hence $\forall x_j \in \mathbb{N} : c_{x,j} \in \{0,1\}$. Now, letting $\delta(x) = \{n \in \mathbb{N} \mid c_{x,n} = 1\}$ the mapping δ is bijective; hence, by virtue of Lemma 2.2, if $[0,1]$ is $\mathcal{T}$-representable, then so is $\mathcal{P}(\mathbb{N})$, which contradicts the former statement.

Remark. Both Statement 3) and Statement 4) in the preceding theorem state some relationship between countability and $\mathcal{T}$-representability, and such a relationship can be understood. To this end, consider any infinite subset X of the unit interval $[0,1]$. If X can be defined by means of a finite set of conditions, in general, its members can be constructed through some procedure, which involves some effective algorithm. However, it is possible to build members of $[0,1]$ at random. For instance, one can write the binary expression for a member x of $[0,1]$ tossing a coin in an endless process, that is to say, the n-th digit of the binary expression for x is 0 or 1 depending on the obtained result. For any member $x = 0.c_1c_2c_3\ldots$ of an infinite subset of $[0,1]$ being built at random, in general, it is not possible to define an algorithm satisfying Condition 1 in order to determine its digits, because, there is no predictable pattern in the corresponding digit sequence $0.c_1c_2c_3\ldots$. By contrast, Condition 1 implies that every member x of X is determined by a finite set of initial data $a_1a_2\ldots a_n$; hence the expression for x in terms of tape-alphabet symbols cannot be obtained at random; consequently Condition 1 cannot be satisfied by $[0,1]$. In addition, to define a bijection by means of a finite procedure it is required to discern whether or not two members of $[0,1]$ are the same by means of a finite method, because under bijections different points must have different images. Notice, that for each couple of rational numbers $x_1 = 0.c_1c_2\ldots c_n\ldots$ and $x_2 = 0.d_1d_2\ldots d_n\ldots$ there is a positive integer N_{x_1,x_2} such that the predicate $\forall n \leq N_{x_1,x_2} : c_n = d_n$ implies $x_1 = x_2$. By contrast, for all members of $[0,1]$ this criterion does not work properly, since the possibility of defining the figures $c_1, c_2, c_3\ldots$ at random excludes the existence of any predictable pattern.

Corollary 2.4 *Cantor's Theorem can be derived from the conjunction of Statement 1) and Statement 4) in the preceding theorem.*

Proof. Let us assume that both Statement 1) Statement 4) in the preceding theorem are true for some regular Turing machine set $\mathcal{T}$. If there were a bijection $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, by virtue of Lemma 2.2 together with Statement 1) in the preceding theorem, $\mathcal{P}(\mathbb{N})$ would be $\mathcal{T}$-representable, which contradicts Statement 4); consequently there is no bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$, and Cantor's Theorem holds.

3 Open Problem

From Aristotle to Gauss and Pointcaré, the existence of actual infinity is frequently rejected. For instance consider the following quotation from Henri Poincaré (1854-1912).

> *Actual infinity does not exist. What we call infinite is only the endless possibility of creating new objects no matter how many exist already.*

However, Cantor's Theorem introduces a tower of different infinities. Nevertheless, the axiomatic set theorists state that all existing definitions of "actual infinity" are unsatisfactory and have no role to play in serious logical and mathematical analysis, for instance see [1] or [11]. In fact, there is some opposition against the negation of the uniqueness of infinity, or more accurately, against the assumption that an infinity could be surpassed by another. To clarify these ideas, Corollary 2.4 provides a version of Cantor's Theorem deduced from the existence of an intrinsic property, namely, $\mathcal{T}$-representability. If Cantor's Theorem is induced by any intrinsic property, this fact need not imply the existence of two infinities being different in size, but it implies one of the following statements.

1. *Both* $\mathbb{N}$ *and* $\mathcal{P}(\mathbb{N})$ *have the same size, and for only one of them there is a regular Turing-machine set* $\mathcal{T}$ *with respect to which is* $\mathcal{T}$*-representable.*

2. $\mathbb{N}$ *and* $\mathcal{P}(\mathbb{N})$ *have different size, and for only one of them there is a regular Turing-machine set* $\mathcal{T}$ *with respect to which is* $\mathcal{T}$*-representable.*

Thus, Corollary 2.4 implies that the existence of two infinities being different in size is not an unavoidable consequence of Cantor's Theorem, but in order to solve this question it must be proved which of the preceding statements holds, and this is an open problem.

Finally, since there are other cardinals apart from $\#(\mathbb{N})$ and $\#\left(\mathcal{P}(\mathbb{N})\right)$, for instance, the Woodin or Shelah cardinals [10], it must be proved whether or not the existence of each of which can be a consequence of a specific intrinsic set property, and this is another open problem.

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