Int. J. Open Problems Compt. Math., Vol. 5, No. 2, June 2012
ISSN 1998-6262;
www.i-csrs.org

Positive solutions for a higher order p-Laplacian boundary value problem with even derivatives

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Abstract

In this work, we study the existence and multiplicity of positive solutions for the higher order p-Laplacian boundary value problem with even derivatives

$$\begin{cases}(\varphi_p((-1)^{n-1}x^{(2n)}))'' = f(t, x, -x'', \ldots, (-1)^{n-2}x^{(2n-4)}),\\ \alpha x^{(2i)}(0) - \beta x^{(2i+1)}(0) = 0, \gamma x^{(2i)}(1) + \delta x^{(2i+1)}(1) = 0,\end{cases}$$

where $t \in [0,1]$***,*** $n \geq 2$***,*** $i = 0, 1, \ldots, n$***,*** $\alpha, \beta, \gamma, \delta \geq 0$ ***with*** $\rho := \alpha\gamma + \alpha\delta + \beta\gamma > 0$ ***and*** $f \in C([0,1] \times \mathbb{R}_+^{n-1}, \mathbb{R}_+)$ $(\mathbb{R}_+ := [0, \infty))$***. By virtue of some properties of concave functions and Jensen's integral inequality, we adopt fixed point index theory to establish our main results. Moreover, our nonlinear term*** f ***is allowed to grow superlinear and sublinear.***

Keywords: *p-Laplacian equation; positive solution; fixed point index; cone; Jensen's inequality.*

***MSC*:** 34B18, 47H07, 47H11, 45M20, 26D15

1 Introduction

The paper mainly concerns with the existence and multiplicity of positive solutions for the following higher order boundary value problem with p-Laplacian operator and the nonlinear term f involving even derivatives

$$\begin{cases}(\varphi_p((-1)^{n-1}x^{(2n)}))'' = f(t, x, -x'', \ldots, (-1)^{n-2}x^{(2n-4)}),\\ \alpha x^{(2i)}(0) - \beta x^{(2i+1)}(0) = 0, \gamma x^{(2i)}(1) + \delta x^{(2i+1)}(1) = 0,\end{cases} \tag{1.1}$$

where $t \in [0,1]$, $n \geq 2$, $i = 0, 1, \ldots, n$, $\alpha, \beta, \gamma, \delta \geq 0$ with $\rho := \alpha\gamma + \alpha\delta + \beta\gamma > 0$ and $f \in C([0,1] \times \mathbb{R}_+^{n-1}, \mathbb{R}_+)$. Here, by a positive solution (1.1) we mean a function $x \in C^{2n+2}[0,1]$ that solves (1.1) and satisfies $x(t) > 0$ for all $t \in (0,1)$.

We are here interested in the case where f depends explicitly on derivatives. There are a very large number of papers dealing with higher order boundary value problems when f is independent of derivatives, see for example [11–13, 20, 26, 32, 33, 37–39, 41] and references cited therein.

In [20], Graef and Yang applied Krasnosel'skii's fixed point theorem to derive intervals for λ in which the boundary value problem consisting of the equation

$$u^{(n)} + \lambda a(t)f(u) = 0, t \in (0,1), \tag{1.2}$$

and the boundary condition

$$u^{(i)}(0) = 0, i = 0, \dots, n-2, u^{(n-2)}(1) = \sum_{j=1}^{m} a_j u^{(n-2)}(t_j) \tag{1.3}$$

has a positive solution, Pang et al. [33] used a fixed point index argument to obtain existence criteria for the boundary value problem consisting of Eq. (1.2) with $\lambda = 1$ and the boundary condition

$$u^{(i)}(0) = 0, i = 0, \dots, n-2, u(1) = \sum_{j=1}^{m} a_j(t_j). \tag{1.4}$$

When f involves all even derivatives explicitly, many authors [1–5, 7, 14–16, 18, 31, 34] studied the following Lidstone boundary value problem ($n \geq 2$)

$$\begin{cases} (-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1}u^{(2n-2)}), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, i = 0, 1, 2, \dots, n-1. \end{cases} \tag{1.5}$$

In [42], Yang considered the existence and uniqueness of positive solutions for the following generalized Lidstone boundary value problem

$$\begin{cases} (-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1}u^{(2n-2)}), \\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = 0 (i = 0, 1, 2, \dots, n-1), \\ \alpha_1 u^{(2i)}(1) - \beta_1 u^{(2i+1)}(1) = 0 (i = 0, 1, 2, \dots, n-1), \end{cases} \tag{1.6}$$

where $\alpha_j \geq 0, \beta_j \geq 0$ $(j = 0, 1)$ and $\alpha_0\alpha_1 + \alpha_0\beta_1 + \alpha_1\beta_0 > 0$. The results obtained here are similar with that of second ($n = 1$) boundary value problem [27, 28]. In view of symmetry, these results demonstrate that problem (1.5) and (1.6) are essentially identical with Dirichlet problem and Sturm-Liouville problem of second order ordinary differential equations, respectively.

In recent years, due to mathematical and physical background [8, 17, 36], the existence of positive solutions for nonlinear boundary value problems with p-Laplacian operator has received wide attention. There exist a very large number of papers devoted to the existence of solutions for differential equations with p-Laplacian, see, for instance, [6, 9, 10, 21–23, 29, 30, 35, 43–46] and references therein.

However, the existence of positive solutions for p-Laplacian equation with Lidstone boundary value problems has not been extensively studied yet. To the best of our knowledge, only [24] is devoted to this direction. In [24], Guo and Ge considered the following boundary value problems

$$\begin{cases} (\Phi(y^{(2n-1)}))' = f(t, y, y'', \dots, y^{(2n-2)}), 0 \leq t \leq 1, \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, i = 0, 1, 2, \dots, n-1, \end{cases} \tag{1.7}$$

where $f \in C([0,1] \times \mathbb{R}^n, \mathbb{R})(\mathbb{R} := (-\infty, +\infty))$. Some growth conditions are imposed on f which yield the existence of at least two symmetric positive solutions by using a fixed point theorem on cones. An interesting feature in [24] is that the nonlinearity f may be sign-changing.

Motivated by the works mentioned above, in this paper, we discuss the positive solutions for (1.1). To overcome the difficulty resulting from even derivatives, we first transform (1.1) into a boundary value problem for an associated fourth-order integro-ordinary differential equation. Then, using fixed point index theory, combined with a priori estimates of positive solutions, we obtain some results on the existence and multiplicity of positive solutions for (1.1). Nevertheless, our methodology and results in this paper are new and entirely different from those in the papers cited above. We observe that if $p = 2$, then (1.1) reduces to problem

$$\begin{cases} (-1)^{n-1}x^{(2n+2)} = f(t, x, -x'', \ldots, (-1)^{n-2}x^{(2n-4)}), \\ \alpha x^{(2i)}(0) - \beta x^{(2i+1)}(0) = 0, \gamma x^{(2i)}(1) + \delta x^{(2i+1)}(1) = 0, (i = 0, 1, \ldots, n). \end{cases} \tag{1.8}$$

We will show some connections between them by repeatedly invoking Jensen's integral inequality in our proofs. Our main tool in the proofs is fixed point index theory based on a priori estimates achieved by utilizing some properties of concave functions and Jensen's integral inequality. The idea stems from [40], but we here study the higher order p-Laplacian boundary value problem with even derivatives, while the nonlinearity of [40] is independent of derivatives. Thus our main results here improve and extend the corresponding ones in [40].

This paper is organized as follows. Section 2 contains some preliminary results. Section 3 is devoted to the existence and multiplicity of positive solutions for (1.1). In Section 4, we introduce an open problem involving Lidstone problem by Eloe [19].

2 Preliminaries

The basic space used in this paper is $E := C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|u\| := \max_{t\in[0,1]} |u(t)|$. Put $P := \{u \in E : u(t) \geq 0, \forall t \in [0,1]\}$, then P is a cone on E. We denote $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel. Let

$$u := (-1)^{n-1}x^{(2n-2)}, \ (B_n u)(t) := \int_0^1 G_n(t,s)u(s)\mathrm{d}s,$$

where

$$G_1(t,s) = \frac{1}{\rho}\begin{cases} (\beta + \alpha s)(\gamma + \delta - \gamma t), & 0 \leq s \leq t \leq 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_n(t,s) = \int_0^1 G_{n-1}(t,\tau)G_1(\tau,s)\mathrm{d}\tau, \ t, s \in [0,1], \ n \in \mathbb{N}_+.$$

It is easy to see that problem (1.1) is equivalent to the following p-Laplacian boundary value problem of fourth-order integro-ordinary differential equation

$$\begin{cases} (\varphi_p(u''))'' = f\left(t, (B_{n-1}u)(t), (B_{n-2}u)(t), \ldots, (B_1 u)(t)\right), \\ \alpha u(0) - \beta u'(0) = 0, \gamma u(1) + \delta u'(1) = 0, \\ \alpha u''(0) - \beta u'''(0) = 0, \gamma u''(1) + \delta u'''(1) = 0, \end{cases} \tag{2.1}$$

which can be written in the form (see [40])

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)\varphi_q\left(\int_0^1 G_1(s,\tau)f(\tau,(B_{n-1}u)(\tau),(B_{n-2}u)(\tau),\ldots,(B_1u)(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &:= (Au)(t). \end{aligned} \tag{2.2}$$

Note that if $f \in C([0,1]\times\mathbb{R}_+^{n-1},\mathbb{R}_+)$, then $A: P \to P$ is a completely continuous operator, and the existence of positive solutions for (1.1) is equivalent to that of positive fixed points of A.

By the definition of B_n, we see $B_n(n=1,2,\ldots): E \to E$ are completely continuous linear operators and they are also positive operators, i.e. $B_n(P)\subset P$. Let $\lambda_1>0$ be the first eigenvalue and $\psi \in C^2[0,1]\cap P$ the associated eigenfunction of

$$\begin{cases} -u'' = \lambda u, \\ \alpha u(0) - \beta u'(0) = 0, \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

with $\int_0^1 \psi(t)\mathrm{d}t = 1$, which can be written in the form

$$\psi(s) = \lambda_1 \int_0^1 G_1(t,s)\psi(t)\mathrm{d}t = \lambda_1(B_1\psi)(s). \tag{2.3}$$

Therefore $\lambda_1 = 1/r(L)$. Moreover, the symmetry of $G_1(t,s)$ implies that

$$\psi(s) = \lambda_1^i \int_0^1 G_i(t,s)\psi(t)\mathrm{d}t = \lambda_1^i(B_i\psi)(s), \text{ for } i = 1,2,\ldots. \tag{2.4}$$

Lemma 2.1 Let $\kappa_1 := \max_{t,s\in[0,1]}\{G_1(t,s)\} > 0$, and $\kappa := \frac{\kappa_1^n - 1}{\kappa_1 - 1} - 1$, then

$$\mathscr{G}(t,s) := \frac{1}{\kappa}\sum_{i=1}^{n-1} G_i(t,s) \in [0,1], \forall t,s \in [0,1].$$

Proof. From the definition $G_i(i=2,3,\ldots)$, we have

$$G_i(t,s) = \int_0^1 G_{i-1}(t,\tau)G_1(\tau,s)\mathrm{d}\tau \le \kappa_1^i,$$

and thus

$$\sum_{i=1}^{n-1} G_i(t,s) \le \kappa_1 + \kappa_1^2 + \ldots + \kappa_1^{n-1} = \frac{\kappa_1^n - 1}{\kappa_1 - 1} - 1 := \kappa > 0.$$

Therefore, we claim $\mathscr{G}(t,s) \in [0,1], \forall t,s \in [0,1]$, as required. This completes the proof.

Lemma 2.2 If $u \in P$ is concave on $[0,1]$. Then there is a $\omega > 0$ such that

$$\int_0^1 u(t)\psi(t)\mathrm{d}t \ge \omega\|u\|. \tag{2.5}$$

Proof. We first prove that u is concave on $[0,1]$, where u is determined by (2.2). Indeed, by simple computation, we have

$$u'' = -\varphi_q\left(\int_0^1 G_1(t,s)f(s,(B_{n-1}u)(s),(B_{n-2}u)(s),\ldots,(B_1u)(s))\mathrm{d}s\right) \le 0,$$

and then u is concave on $[0,1]$. In what follows, we divide three cases.
Case 1. If $\|u\| = u(0)$. Then from the concavity and nonnegativity of u, we find

$$\int_0^1 u(t)\psi(t)\mathrm{d}t = \int_0^1 u(t\cdot 1 + (1-t)\cdot 0)\psi(t)\mathrm{d}t \geq u(0)\int_0^1 (1-t)\psi(t)\mathrm{d}t.$$

Case 2. If $\|u\| = u(1)$. Similar to Case 1, we have

$$\int_0^1 u(t)\psi(t)\mathrm{d}t = \int_0^1 u(t\cdot 1 + (1-t)\cdot 0)\psi(t)\mathrm{d}t \geq u(1)\int_0^1 t\psi(t)\mathrm{d}t.$$

Case 3. If there is a $t_0 \in (0,1)$ such that $u(t_0) = \|u\|$, then we arrive at

$$\begin{aligned}\int_0^1 u(t)\psi(t)\ \mathrm{d}t &= \int_0^{t_0} u(t)\psi(t)\ \mathrm{d}t + \int_{t_0}^1 u(t)\psi(t)\ \mathrm{d}t = \int_0^{t_0} u\left(\frac{t}{t_0}\cdot t_0 + \frac{t_0 - t}{t_0}\cdot 0\right)\psi(t)\ \mathrm{d}t \\ &+ \int_{t_0}^1 u\left(\frac{1-t}{1-t_0}\cdot t_0 + \frac{t-t_0}{1-t_0}\cdot 1\right)\psi(t)\ \mathrm{d}t \geq u(t_0)\left(\int_0^{t_0} t\psi(t)\ \mathrm{d}t + \int_{t_0}^1 (1-t)\psi(t)\ \mathrm{d}t\right).\end{aligned}$$

Combining the above three cases and taking

$$\omega := \min\left\{\int_0^1 (1-t)\psi(t)\mathrm{d}t, \int_0^1 t\psi(t)\mathrm{d}t, \int_0^{t_0} t\psi(t)\ \mathrm{d}t + \int_{t_0}^1 (1-t)\psi(t)\ \mathrm{d}t\right\} > 0,$$

then we have $\int_0^1 u(t)\psi(t)\mathrm{d}t \geq \omega\|u\|$, as claimed. This completes the proof.
Lemma 2.3([25]) Let $\Omega \subset E$ be a bounded open set and $A : \overline{\Omega}\cap P \to P$ is a completely continuous operator. If there exists $v_0 \in P\setminus\{0\}$ such that $v - Av \neq \lambda v_0$ for all $v \in \partial\Omega\cap P$ and $\lambda \geq 0$, then $i(A, \Omega\cap P, P) = 0$.
Lemma 2.4([25]) Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose $A : \overline{\Omega}\cap P \to P$ is a completely continuous operator. If $v \neq \lambda Av$ for all $v \in \partial\Omega\cap P$ and $0 \leq \lambda \leq 1$, then $i(A, \Omega\cap P, P) = 1$.
Lemma 2.5 (Jensen's integral inequalities) Let $\theta > 0$ and $f \in C([0,1], \mathbb{R}^+)$. Then

$$\left(\int_0^1 f(t)\mathrm{d}t\right)^\theta \leq \int_0^1 (f(t))^\theta \mathrm{d}t, \theta \geq 1, \quad \left(\int_0^1 f(t)\mathrm{d}t\right)^\theta \geq \int_0^1 (f(t))^\theta \mathrm{d}t, 0 < \theta \leq 1.$$

3 Main Results

For the reason of notational brevity, we denote by $y = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{R}_+^{n-1}$, $p_* := \min\{1, p-1\}$, $p^* := \max\{1, p-1\}$,

$$\beta_p := \left[\frac{1}{\kappa^{p_*-1}\sum_{i=3}^{n+1}\lambda_1^{-i}}\right]^{\frac{p-1}{p_*}}, \alpha_p := \left[\frac{1}{\kappa^{p^*-1}\sum_{i=3}^{n+1}\lambda_1^{-i}}\right]^{\frac{p-1}{p^*}}.$$

We now list our hypotheses.
(H1) $f \in C([0,1]\times\mathbb{R}_+^{n-1}, \mathbb{R}_+)$.

(H2) There exist $a_1 > \beta_p$ and $c > 0$ such that $f(t,y) \ge a_1(\sum_{i=1}^{n-1} y_i)^{p-1} - c$ for all $y \in \mathbb{R}_+^{n-1}$ and $t \in [0,1]$.
(H3) There exist $b_1 \in (0, \alpha_p)$ and $r > 0$ such that $f(t,y) \le b_1(\sum_{i=1}^{n-1} y_i)^{p-1}$ for all $y \in [0,r]^{n-1}$ and $t \in [0,1]$.
(H4) There exist $a_2 > \beta_p$ and $r > 0$ such that $f(t,y) \ge a_2(\sum_{i=1}^{n-1} y_i)^{p-1}$ for all $y \in [0,r]^{n-1}$ and $t \in [0,1]$.
(H5) There exist $b_2 \in (0, \alpha_p)$ and $c > 0$ such that $f(t,y) \le b_2(\sum_{i=1}^{n-1} y_i)^{p-1} + c$ for all $y \in \mathbb{R}_+^{n-1}$ and $t \in [0,1]$.
(H6) There are $\zeta > 0$ and $\varsigma \in \left(0, \kappa_1^{-\frac{p-1}{p}}\right)$ such that the inequality $f(t,y) \le \varsigma^{p-1}\zeta^{p-1}$ holds for all $y \in [0,\zeta]^{n-1}$ and $t \in [0,1]$.
Theorem 3.1 Suppose that (H1)-(H3) are satisfied. Then (1.1) has at least one positive solution.
Proof. Let

$$\mathscr{M}_1 := \{u \in P : u = Au + \lambda\psi \text{ for some } \lambda \ge 0\},$$

where $\psi(t)$ is determined by (2.3) and (2.4). We claim $\mathscr{M}_1$ is bounded. Indeed, $u \in \mathscr{M}_1$ implies u is concave and $u(t) \ge (Au)(t)$. Now, by Jensen's inequality and (H2), we find

$$\begin{aligned}
&u^{p_*}(t) \ge \left[\int_0^1 G_1(t,s)\varphi_q\left(\int_0^1 G_1(s,\tau)f(\tau,(B_{n-1}u)(\tau),(B_{n-2}u)(\tau),\ldots,(B_1u)(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right]^{p_*}\\
&\ge \int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)f^{\frac{p_*}{p-1}}(\tau,(B_{n-1}u)(\tau),(B_{n-2}u)(\tau),\ldots,(B_1u)(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\ge \int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left[a_1\left(\sum_{i=1}^{n-1}(B_iu)(\tau)\right)^{p-1} - c\right]^{\frac{p_*}{p-1}}\mathrm{d}\tau\mathrm{d}s\\
&\ge a_1^{\frac{p_*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left(\sum_{i=1}^{n-1}(B_iu)(\tau)\right)^{p_*}\mathrm{d}\tau\mathrm{d}s - c^{\frac{p_*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathrm{d}\tau\mathrm{d}s\\
&\ge a_1^{\frac{p_*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left(\kappa\int_0^1 \mathscr{G}(\tau,z)u(z)\mathrm{d}z\right)^{p_*}\mathrm{d}\tau\mathrm{d}s - c^{\frac{p_*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathrm{d}\tau\mathrm{d}s\\
&\ge \kappa^{p_*}a_1^{\frac{p_*}{p-1}}\int_0^1\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathscr{G}(s,\tau)u^{p_*}(z)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s - c^{\frac{p_*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathrm{d}\tau\mathrm{d}s\\
&= \kappa^{p_*-1}a_1^{\frac{p_*}{p-1}}\sum_{i=3}^{n+1}\int_0^1 G_i(t,s)u^{p_*}(s)\mathrm{d}s - c^{\frac{p_*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathrm{d}\tau\mathrm{d}s.
\end{aligned} \tag{3.1}$$

Multiply the both sides of the above by $\psi(t)$ and integrate over [0,1], and use (2.4) to obtain

$$\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t \ge \kappa^{p_*-1}a_1^{\frac{p_*}{p-1}}\sum_{i=3}^{n+1}\lambda_1^{-i}\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t - \lambda_1^{-2}c^{\frac{p_*}{p-1}}$$

and thus

$$\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t \le \frac{\lambda_1^{-2}c^{\frac{p_*}{p-1}}}{\kappa^{p_*-1}a_1^{\frac{p_*}{p-1}}\sum_{i=3}^{n+1}\lambda_1^{-i} - 1} := N_1.$$

Recall that every $u \in \mathscr{M}_1$ is concave and increasing on [0,1]. So is u^{p_*} with $p_* \in (0,1]$. Now Lemma 2.2 yields

$$\|u^{p_*}\| \le \omega^{-1} N_1 \tag{3.2}$$

for all $u \in \mathscr{M}_1$, which implies the boundedness of $\mathscr{M}_1$, as claimed. Taking $R > \sup\{\|u\| : u \in \mathscr{M}_1\}$, we have

$$u - Au \ne \lambda\psi,\ \forall u \in \partial B_R \cap P,\ \lambda \ge 0.$$

Now by virtue of Lemma 2.3, we obtain

$$i(A, B_R \cap P, P) = 0. \tag{3.3}$$

Let

$$\mathscr{M}_2 := \{u \in \overline{B}_r \cap P : u = \lambda Au \text{ for some } \lambda \in [0,1]\}.$$

We shall prove $\mathscr{M}_2 = \{0\}$. Indeed, if $u \in \mathscr{M}_2$, we have for any $u \in \overline{B}_r \cap P$

$$u(t) \le \int_0^1 G_1(t,s)\varphi_q\left(\int_0^1 G_1(s,\tau) f(\tau, (B_{n-1}u)(\tau), (B_{n-2}u)(\tau), \ldots, (B_1u)(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \tag{3.4}$$

Now by (H3) and Jensen's inequality, we obtain

$$\begin{aligned}
u^{p^*}(t) &\le \left[\int_0^1 G_1(t,s)\varphi_q\left(\int_0^1 G_1(s,\tau) f(\tau, (B_{n-1}u)(\tau), (B_{n-2}u)(\tau), \ldots, (B_1u)(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right]^{p^*} \\
&\le \int_0^1\int_0^1 G_1(t,s)G_1(s,\tau) f^{\frac{p^*}{p-1}}(\tau, (B_{n-1}u)(\tau), (B_{n-2}u)(\tau), \ldots, (B_1u)(\tau))\mathrm{d}\tau\mathrm{d}s \\
&\le b_1^{\frac{p^*}{p-1}} \int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left[\sum_{i=1}^{n-1}(B_iu)(\tau)\right]^{p^*}\mathrm{d}\tau\mathrm{d}s \\
&= \kappa^{p^*} b_1^{\frac{p^*}{p-1}} \int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left[\int_0^1 \mathscr{G}(\tau,z)u(z)\mathrm{d}z\right]^{p^*}\mathrm{d}\tau\mathrm{d}s \\
&\le \kappa^{p^*} b_1^{\frac{p^*}{p-1}} \int_0^1\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathscr{G}(\tau,z)u^{p^*}(z)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s \\
&= \kappa^{p^*-1} b_1^{\frac{p^*}{p-1}} \sum_{i=3}^{n+1}\int_0^1 G_i(t,s)u^{p^*}(s)\mathrm{d}s.
\end{aligned}$$

Multiply the both sides of the above by $\psi(t)$ and integrate over $[0,1]$ and use (2.4) to obtain

$$\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t \le \kappa^{p^*-1} b_1^{\frac{p^*}{p-1}} \sum_{i=3}^{n+1}\lambda_1^{-i}\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t.$$

Therefore, $\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t = 0$, whence $u(t) \equiv 0, \forall u \in \mathscr{M}_2$. As a result, $\mathscr{M}_2 = \{0\}$, as claimed. Consequently,

$$u \ne \lambda Au, \forall u \in \partial B_r \cap P, \lambda \in [0,1].$$

Now Lemma 2.4 yields

$$i(A, B_r \cap P, P) = 1. \tag{3.5}$$

Combining this with (3.3) gives

$$i(A,(B_R\backslash\overline{B}_r)\cap P,P)=0-1=-1.$$

Hence the operator A has at least one fixed point on $(B_R\setminus\overline{B}_r)\cap P$ and therefore (1.1) has at least one positive solution. This completes the proof.

Theorem 3.2 Suppose that (H1), (H4) and (H5) are satisfied. Then (1.1) has at least one positive solution.

Proof. Let

$$\mathscr{M}_3:=\{u\in\overline{B}_r\cap P: u=Au+\lambda\psi \text{ for some } \lambda\geq 0\},$$

where $\psi(t)$ is determined by (2.3) and (2.4). We claim $\mathscr{M}_3\subset\{0\}$. Indeed, if $u\in\mathscr{M}_3$, then we have $u\geq Au$ by definition. Now by (H4) and Jensen's inequality, we obtain

$$\begin{aligned}
u^{p_*}(t)&\geq\left[\int_0^1 G_1(t,s)\varphi_q\left(\int_0^1 G_1(s,\tau)f(\tau,(B_{n-1}u)(\tau),(B_{n-2}u)(\tau),\dots,(B_1u)(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right]^{p_*}\\
&\geq\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)f^{\frac{p_*}{p-1}}(\tau,(B_{n-1}u)(\tau),(B_{n-2}u)(\tau),\dots,(B_1u)(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\geq a_2^{\frac{p_*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left[\sum_{i=1}^{n-1}(B_iu)(\tau)\right]^{p_*}\mathrm{d}\tau\mathrm{d}s\\
&=\kappa^{p_*}a_2^{\frac{p_*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left[\int_0^1\mathscr{G}(\tau,z)u(z)\mathrm{d}z\right]^{p_*}\mathrm{d}\tau\mathrm{d}s\\
&\geq\kappa^{p_*}a_2^{\frac{p_*}{p-1}}\int_0^1\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathscr{G}(\tau,z)u^{p_*}(z)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s\\
&=\kappa^{p_*-1}a_2^{\frac{p_*}{p-1}}\sum_{i=3}^{n+1}\int_0^1 G_i(t,s)u^{p_*}(s)\mathrm{d}s.
\end{aligned}$$

Multiply the both sides of the above by $\psi(t)$ and integrate over $[0,1]$, and use (2.4) to obtain

$$\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t\geq\kappa^{p_*-1}a_2^{\frac{p_*}{p-1}}\sum_{i=3}^{n+1}\lambda_1^{-i}\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t,$$

so that $\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t=0$, whence $u(t)\equiv 0,\forall u\in\mathscr{M}_3$. Therefore, we claim $\mathscr{M}_3\subset\{0\}$. As a result of this, we have

$$u-Au\neq\lambda\psi,\forall u\in\partial B_r\cap P,\lambda\geq 0.$$

Now Lemma 2.3 gives

$$i(A,B_r\cap P,P)=0. \tag{3.6}$$

Let

$$\mathscr{M}_4:=\{u\in P: u=\lambda Au \text{ for some } \lambda\in[0,1]\}.$$

We assert $\mathscr{M}_4$ is bounded. Indeed, if $u \in \mathscr{M}_4$, then u is concave and $u \leq Au$ by definition. Now by (H5) and Jensen's inequality, we obtain

$$\begin{aligned}
u^{p^*}(t) &\leq \left[\int_0^1 G_1(t,s)\varphi_q\left(\int_0^1 G_1(s,\tau)f(\tau,(B_{n-1}u)(\tau),(B_{n-2}u)(\tau),\ldots,(B_1u)(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right]^{p^*} \\
&\leq \int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)f^{\frac{p^*}{p-1}}(\tau,(B_{n-1}u)(\tau),(B_{n-2}u)(\tau),\ldots,(B_1u)(\tau))\mathrm{d}\tau\mathrm{d}s \\
&\leq \int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left[b_2\left(\sum_{i=1}^{n-1}(B_iu)(\tau)\right)^{p-1}+c\right]^{\frac{p^*}{p-1}}\mathrm{d}\tau\mathrm{d}s \\
&\leq b_3^{\frac{p^*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left(\sum_{i=1}^{n-1}(B_iu)(\tau)\right)^{p^*}\mathrm{d}\tau\mathrm{d}s + c_1^{\frac{p^*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathrm{d}\tau\mathrm{d}s \\
&= \kappa^{p^*}b_3^{\frac{p^*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\left[\int_0^1\mathscr{G}(\tau,z)u(z)\mathrm{d}z\right]^{p^*}\mathrm{d}\tau\mathrm{d}s + c_1^{\frac{p^*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathrm{d}\tau\mathrm{d}s \\
&\leq \kappa^{p^*}b_3^{\frac{p^*}{p-1}}\int_0^1\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathscr{G}(\tau,z)u^{p^*}(z)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s + c_1^{\frac{p^*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathrm{d}\tau\mathrm{d}s \\
&= \kappa^{p^*-1}b_3^{\frac{p^*}{p-1}}\sum_{i=3}^{n+1}\int_0^1 G_i(t,s)u^{p^*}(s)\mathrm{d}s + c_1^{\frac{p^*}{p-1}}\int_0^1\int_0^1 G_1(t,s)G_1(s,\tau)\mathrm{d}\tau\mathrm{d}s
\end{aligned} \tag{3.7}$$

for all $u \in \mathscr{M}_4$, $b_3 \in (b_2, \alpha_p)$ and $c_1 > 0$ being chosen so that

$$(b_2z+c)^{\frac{p^*}{p-1}} \leq b_3^{\frac{p^*}{p-1}}z^{\frac{p^*}{p-1}} + c_1^{\frac{p^*}{p-1}}, \forall z \geq 0.$$

Multiply the both sides of (3.7) by $\psi(t)$ and integrate over $[0,1]$, and use (2.4) to obtain

$$\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t \leq \kappa^{p^*-1}b_3^{\frac{p^*}{p-1}}\sum_{i=3}^{n+1}\lambda_1^{-i}\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t + \lambda_1^{-2}c_1^{\frac{p^*}{p-1}}$$

and thus

$$\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t \leq \frac{\lambda_1^{-2}c_1^{\frac{p^*}{p-1}}}{1-\kappa^{p^*-1}b_3^{\frac{p^*}{p-1}}\sum_{i=3}^{n+1}\lambda_1^{-i}} := N_2.$$

This, together with Jensen's inequality and $\frac{\psi(t)}{\kappa_2} \in [0,1] (\kappa_2 := \|\psi\|)$, leads to

$$\int_0^1 u(t)\psi(t)\mathrm{d}t \leq \kappa_2\left(\int_0^1 u^{p^*}(t)\left(\frac{\psi(t)}{\kappa_2}\right)^{p^*}\mathrm{d}t\right)^{\frac{1}{p^*}} \leq \kappa_2\left(\int_0^1 u^{p^*}(t)\left(\frac{\psi(t)}{\kappa_2}\right)\mathrm{d}t\right)^{\frac{1}{p^*}} \leq \kappa_2^{1-\frac{1}{p^*}}N_2^{\frac{1}{p^*}} \tag{3.8}$$

for all $u \in \mathscr{M}_4$. From Lemma 2.2, we find

$$\|u\| \leq \omega^{-1}\kappa_2^{1-\frac{1}{p^*}}N_2^{\frac{1}{p^*}}, \forall u \in \mathscr{M}_4.$$

Now the boundedness of $\mathscr{M}_4$, as asserted. Taking $R > \sup\{\|u\| : u \in \mathscr{M}_4\}$, we have

$$u \neq \lambda Au, \forall u \in \partial B_R \cap P, \lambda \in [0,1].$$

Now Lemma 2.4 yields

$$i(A, B_R \cap P, P) = 1. \tag{3.9}$$

Combining this with (3.6) gives

$$i(A, (B_R \backslash \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and therefore (1.1) has at least one positive solution. This completes the proof.

Theorem 3.3 Suppose that (H1), (H2), (H4) and (H6) are satisfied. Then (1.1) has at least two positive solutions.

Proof. By (H6), we have

$$\begin{aligned}\|Au\| &= \max_{t\in[0,1]} \int_0^1 G_1(t,s)\varphi_q\left(\int_0^1 G_1(s,\tau)f(\tau,(B_{n-1}u)(\tau),(B_{n-2}u)(\tau),\ldots,(B_1u)(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &\leq \int_0^1 \kappa_1\varphi_q\left(\int_0^1 \kappa_1 \varsigma^{p-1}\zeta^{p-1}\mathrm{d}\tau\right)\mathrm{d}s < \zeta\end{aligned} \tag{3.10}$$

and thus $\|Au\| < \|u\|$ for all $u \in B_\zeta \cap P$, so that

$$u \neq \lambda Au, \ \forall u \in \partial B_\zeta \cap P, \lambda \in [0,1].$$

Now Lemma 2.4 yields

$$i(A, B_\zeta \cap P, P) = 1. \tag{3.11}$$

On the other hand, in view of (H2) and (H4), we may choose $R > \zeta$ and $r \in (0,\zeta)$ so that (3.3) and (3.6) hold (see the proofs of Theorem 3.1 and 3.2). Combining (3.3), (3.6) and (3.11), we obtain

$$i(A, (B_R \backslash \overline{B}_\zeta) \cap P, P) = 0 - 1 = -1, \quad i(A, (B_\zeta \backslash \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Hence A has at least two fixed points, one on$(B_R \setminus \overline{B}_\zeta) \cap P$ and the other on $(B_\zeta \setminus \overline{B}_r) \cap P$. This proves that (1.1) has at least two positive solutions. The proof is completed.

4 Open Problem

In [19], Eloe considered the nonlinear Lidstone boundary value problem (with $a(t)$ continuous and nonnegative)

$$\begin{cases}(-1)^n u^{(2n)} = \lambda a(t) f(t, u, -u'', \ldots, (-1)^{n-1}u^{(2n-2)}), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, i = 0, 1, 2, \ldots, n-1,\end{cases}$$

where $f \in C([0,1] \times \mathbb{R}_+^n, \mathbb{R}_+)$ and $\lambda > 0$ is a real parameter. He posed an open question in this way: "Can the methods employed here apply to a Lidstone BVP with nonlinear dependence on odd order derivatives of the unknown function ?" That question is completely open. The problem is that large in norm does not imply large componentwise; by exploiting the nested feature of Lidstone BVPs in [19], large in norm will, in fact, imply large in the appropriate components.

In this paper, we only answer the question partly by considering the simple case $a(t) := 1$. However, the resulting problem is the Lidstone problems with p-Laplacian operator, furthermore, some connections between (1.1) and (1.5) are established by repeatedly invoking Jensen's integral inequality. This, together with the fact that our nonlinearity f may be of distinct growth, means that our methodology and results in this paper are entirely new in the existing literature.

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