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Even-order Magic Squares with Special Properties

Saleem Al-ashhab

Department of Mathematics, Al-albayt University, Jordan.
e-mail: ahhab@aabu.edu.jo

Abstract

In this paper we introduce new concepts in the field of magic squares. We focus on special types of magic squares of order six. We list enumerations of the squares of this type. We prove that the nullity of some even-order magic squares is at least one.

Keywords: Magic Squares, nullspace.

1 Introduction

A magic square is a square matrix, where the sum of all entries in each row, column or both main diagonals yields the same number. This number is called the magic constant. A natural magic square of order n is a matrix such that its entries consist of all integers from one to n^2. The magic constant in this case is $\frac{n(n^2+1)}{2}$. A pandiagonal magic square is a magic square such that the sum of all entries in all broken diagonals equals the magic constant. A symmetric magic square is a natural magic square of order n such that the sum of all opposite entries equals n^2+1, i. e. the following relations hold

$$a_{ij} + a_{n+1-i,n+1-j} = n^2 + 1 \text{ for all } 1 \leq i, j \leq n$$

Example 1 *The following square is a natural symmetric magic square*

$$\begin{bmatrix} 15 & 14 & 1 & 18 & 17 \\ 19 & 16 & 3 & 21 & 6 \\ 2 & 22 & 13 & 4 & 24 \\ 20 & 5 & 23 & 10 & 7 \\ 9 & 8 & 25 & 12 & 11 \end{bmatrix}$$

The sum of all opposite entries is 26.

We emphasize here that a symmetric magic square is not a symmetric matrix in the sense that its identical to its transpose. A natural magic square can not be a symmetric matrix since all entries are distinct.

The number of natural magic squares of order five is known. Schroeppel computed this number in 1971 (see [11]). It is well-known that there are pandiagonal magic squares and symmetric squares of order five (see [6]). The number of natural magic squares of order six is til now unknown. We give here the number of a subset of such squares. One possible form of a symmetric magic square with magic sum $3s$ is

$$\begin{bmatrix} a & f & C & s-F & s-B & s-A \\ b & g & k & s-j & s-p & s-E \\ h & r & o & s-v & s-D & s-q \\ q & D & v & s-o & s-r & s-h \\ E & p & j & s-k & s-g & s-b \\ A & B & F & s-C & s-f & s-a \end{bmatrix}$$

where

$$\begin{gathered} A = j - 2b - g - h - a - k + p - q + 3s, \\ B = q - g - h - o - p - f - 2r + 3s + v, \\ C = \tfrac{9}{2}s - b - f - g - h - k - o - r - a, \\ D = h + o - q + r - v, \\ E = b + g - j + k - p, \\ F = a + b + f + g + h - j + r - \tfrac{3}{2}s - v. \end{gathered}$$

We obtain this form by solving the equations resulting from the definition of such squares. Due to the fraction in the assignment for C we see that noninteger numbers might appear. In particular, a natural symmetric magic square can not exist since the magic constant for natural magic squares is 111. The same problem occurs when we solve the equations resulting from the definition of the pandiagonal magic square. Hence, there are also no natural pandiagonal magic squares.

It is well-known that the following structure

$$\begin{bmatrix} A & B & C & 2s-A-B-C \\ E & 2s-A-B-E & A+E-C & B+C-E \\ s-C & A+B+C-s & s-A & s-B \\ s-A-E+C & s-B-C+E & s-E & A+B+E-s \end{bmatrix}$$

is the general structure of the pandiagonal magic square 4×4 (see [11]). Here, the magic constant is $2s$. We note that the sum in each pair of antipodals is s, i. e.

$$a_{ij} + a_{i+2,j+2} = s, \text{ for all } 1 \le i, j \le 2$$
$$a_{ij} + a_{i+2,j-2} = s, \text{ for all } 1 \le i \le 2, 3 \le j \le 4$$

We define next classes of magic squares of order six, which possesses similar properties for the antipodals.

2 Quasi Pandiagonal Magic Squares

We can generalize idea of the structure of 4×4 pandiagonal magic squares in order to obtain new types of magic squares of even order. For example, the 6×6 square having the structure

$$\begin{bmatrix} a & b & c & d & i & j \\ e & f & g & h & k & l \\ m & n & o & p & q & r \\ s-d & s-i & s-j & s-a & s-b & s-c \\ s-h & s-k & s-l & s-e & s-f & s-g \\ s-p & s-q & s-r & s-m & s-n & s-o \end{bmatrix}$$

with the following restrictions

$$a+b+c+d+i+j = 3s, e+f+g+h+k+l = 3s, m+n+o+p+q+r = 3s,$$

$$a+e+m = d+h+p, b+f+n = i+k+q, c+g+o = j+l+r,$$

is a magic square with magic constant $3s$. This structure is called a quasi pandiagonal magic square of order six. The previous system of linear equations has the following augmented coefficient matrix:

$$\begin{bmatrix} 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&-3\\ 0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&-3\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&-3\\ 1&0&0&-1&0&0&1&0&0&-1&0&0&1&0&0&-1&0&0&0\\ 0&1&0&0&-1&0&0&1&0&0&-1&0&0&1&0&0&-1&0&0\\ 0&0&1&0&0&-1&0&0&1&0&0&-1&0&0&1&0&0&-1&0 \end{bmatrix}$$

It has the following squared submatrix

$$\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&1&1&1\\ 0&0&0&0&-1&0\\ 0&0&1&0&0&-1\\ -1&-1&0&1&0&0 \end{bmatrix}$$

of determinant two. Hence, the general solution of these equations can be given as:

$$j = 3s - a - b - c - d - e,$$

$$l = 3s - g - h - i - j - k,$$

$$n = c + d + g + h + i + k - m - r - \frac{3}{2}s,$$

$$o = 6s - a - b - 2c - d - e - f - 2g - h - i - k + r,$$

$$p = a - d + e - h + m,$$

$$q = b + c + d + f + g + h - m - r - \frac{3}{2}s$$

Lemma 1 *Let Λ be a quasi pandiagonal magic square of order six. Then, the nullity of Λ is at least one.*
Proof. *Note that the square Λ has the form*

$$\begin{bmatrix} A & B\\ S-B & S-A \end{bmatrix}$$

where all entries of the matrix S are s. We seek the vectors in the nullspace of this matrix having the form

$$\begin{pmatrix} x \\ -x \end{pmatrix} \text{ for some } x \in \mathbb{R}^3$$

The product between the square Λ and this vector leads

$$\begin{pmatrix} Ax - Bx \\ -Bx + Ax \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and, hence, we look for vectors $x \in \mathbb{R}^3$ satisfying the following equation:

$$Ax - Bx = 0 \qquad (1)$$

This equation has nonzero vectors as solution because the matrix $A - B$ does not have full rank. This is due to the following reason: replacing the last row with the summation of all rows yields the following row in the matrix:

$$(a + e + m - d - h - p, b + f + n - i - k - q, c + g + o - j - l - r)$$

According to our requirement this is a row of zeros. Hence, the rank of $A - B$ is two and, there exists a nontrivial solution of (1). ■

Remark 1 *In general the rank of this type of squares is five. The following square is an example:*

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 4 \\ 1 & 3 & 1 & -3 & 4 & 0 \\ -2 & 1 & 4 & 1 & 0 & 2 \\ 1 & 2 & -2 & 2 & 2 & 1 \\ 5 & -2 & 2 & 1 & -1 & 1 \end{bmatrix}$$

We can define certain types of $\Theta \times \Theta$ magic squares, where Θ is an even natural number greater than three. For example, let us consider the case $\Theta = 8$. The square has the following structure:

$$\begin{bmatrix} a & b & c & d & A & B & C & D \\ e & f & g & h & E & F & G & H \\ i & j & k & l & I & J & K & L \\ m & n & o & p & M & N & O & P \\ s-A & s-B & s-C & s-D & s-a & s-b & s-c & s-d \\ s-E & s-F & s-G & s-H & s-e & s-f & s-g & s-h \\ s-I & s-J & s-K & s-L & s-i & s-j & s-k & s-l \\ s-M & s-N & s-O & s-P & s-m & s-n & s-o & s-p \end{bmatrix}$$

where

$$a+b+c+d+A+B+C+D=4s,$$

$$e+f+g+h+E+F+G+H=4s,$$

$$i+j+k+l+I+J+K+L=4s,$$

$$m+n+o+p+M+N+O+P=4s,$$

$$a+e+i+m=A+E+I+M, b+f+j+n=B+F+J+N,$$

$$c+g+k+o=C+G+K+O, d+h+l+p=D+H+L+P.$$

The previous system of linear equations has the following squared submatrix in the augmented coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

of determinant different than zero. Hence, the system of linear equations possess a nontrivial solution. The generated square is called quasi magic square. We can generalize these results.

Definition 1 *Let $\theta = \frac{\Theta}{2}$. The magic square having the following structure*

$$\begin{bmatrix} a_{11} & . & . & a_{1\theta} & a_{1(\theta+1)} & . & . & a_{1\Theta} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ a_{\theta 1} & . & . & a_{\theta\theta} & a_{\theta(\theta+1)} & . & . & a_{\theta\Theta} \\ s-a_{1(\theta+1)} & . & . & s-a_{1\Theta} & s-a_{11} & . & . & s-a_{1\theta} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ s-a_{\theta(\theta+1)} & . & . & s-a_{\theta\Theta} & s-a_{\theta 1} & . & . & s-a_{\theta\theta} \end{bmatrix}$$

is called a quasi pandiagonal magic square of order Θ.

The requirement that the matrix is a magic square yields to a solvable linear system. This is because we can following the above reasoning to find a nonsingular submatrix in the augmented coefficient matrix of the system. Hence, there exists quasi pandiagonal magic squares of order Θ, where Θ is an even natural number greater than three.

Proposition 2 *Let Λ be a quasi pandiagonal magic square of order Θ. Then, the nullity of Λ is at least one.*

Proof. Recall that the square Λ of order Θ can be rewritten as

$$\begin{bmatrix} A & B \\ S-B & S-A \end{bmatrix}$$

where A, B and S are square matrices of order θ. Now, we seek the vectors in the nullspace of this matrix having the form

$$\begin{pmatrix} x \\ -x \end{pmatrix} \text{ for some } x \in \mathbb{R}^{\theta}$$

The product between the square Λ and this vector leads

$$\begin{pmatrix} Ax - Bx \\ -Bx + Ax \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

The matrix $A - B$ does not have full rank. This is due to the following reason: replacing the last row with the summation of all rows yields the following row in the matrix:

$$(a_{11} + ... + a_{\theta 1} - a_{1(\theta+1)} - ... - a_{\theta(\theta+1)}, ..., a_{1\theta} + ... + a_{\theta\theta} - a_{1\Theta} - ... - a_{\theta\Theta})$$

Since the square is magic, we can assure that this row is a row of zeros. Hence, there exists a nontrivial solution of the equation

$Ax - Bx = 0$

Using this nontrivial solution we determine a nonzero vector in the nullspace of Λ. ■

3 Four-corner Magic Squares

Definition 2 *A four-corner magic square of order 6 is a magic square* $(a_{ij})_{\substack{i=1,\dots,6\\ j=1,\dots,6}}$ *with magic constant* $3s$ *such that* $a_{33}+a_{44}+a_{34}+a_{43}=2s$ *and*
$a_{i,j}+a_{(i+3),(j+3)}+a_{i,(j+3)}+a_{(i+3),j}=2s$ *for all* $i=1,2,3$ *and* $j=1,2,3.$

The entries of a four-corner magic square of order 6 satisfy

$$a_{14}+a_{25}+a_{36}+a_{41}+a_{52}+a_{63}=3s,\ a_{13}+a_{22}+a_{31}+a_{61}+a_{55}+a_{64}=3s$$

These two conditions represent the sum of the entries of two broken diagonals. If the magic square is pandiagonal, then we have to consider all broken diagonals. To see the validity of the first equation we know from the definition that

$$a_{11}+a_{44}+a_{14}+a_{41}=2s, a_{22}+a_{55}+a_{25}+a_{52}=2s, a_{33}+a_{66}+a_{36}+a_{63}=2s.$$

Adding up these equations and subtracting from the addition the following equation

$$a_{11}+a_{22}+a_{33}+a_{44}+a_{55}+a_{66}=3s$$

we obtain the desired equation.

A four-corner magic square of order 6 can be written as

$$\begin{bmatrix} x & f & g & t & G & M\\ z & h & n & j & q & Y\\ w & H & e & a & m & J\\ 2s-b-t-x & k & 2s-b-a-e & b & D & R\\ 2s-o-j-z & p & d & o & 2s-p-q-h & T\\ B & L & A & 3s-b-j-o-a-t & E & F \end{bmatrix}$$

where

$$\begin{gathered} A=a+b-d-g-n+s,\\ B=b+j+o-s+t-w,\\ D=g-j-k-o-p-q+s+w+x+e,\\ E=f+h+k-m+p-s,\\ F=p-b+q+s-x-e, \end{gathered}$$

$$
\begin{aligned}
G &= j-g-f+o+p+q+s-w-x-e,\\
H &= b+j+o-s+t-w,\\
J &= d-a+g+n-p-q+x,\\
L &= b+j+o-s+t-w,\\
M &= 2s-o-p-q-j-t+w+e,\\
R &= a+b-g+j+o+p+q-2s+t-w,\\
T &= h-d+j+q-s+z,\\
Y &= 3s-j-n-q-h-z.
\end{aligned}
$$

We are interested in a subclass of the four-corner magic squares.

Definition 3 *A four-corner magic squares with semi-symmetric center is a four-corner magic square of order 6 such that* $a_{33}+a_{43}=s$ *and* $a_{34}+a_{44}=s$.

A four-corner magic squares with semi-symmetric center of order 6 can be written as

$$
\begin{bmatrix}
x & f & g & t & G & M\\
z & h & n & j & q & Y\\
w & H & e & a & m & J\\
s+a-t-x & k & s-e & s-a & D & R\\
2s-o-j-z & p & d & o & 2s-p-q-h & T\\
B & L & 2s-g-n-d & 2s-o-j-t & E & F
\end{bmatrix} \text{......(3)}
$$

where the capital letters are dependent variables. We can see that the square has seventeen independent variables. This can be checked with computers by solving the system of equations included in the definition of four-corner magic squares with semi-symmetric center.

3.1 Property preserving transformations

There are seven classical transformations, which take a magic square into another magic square. They are the composition of rotations with angles $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and reflection about the main diagonals. Now, a four-corner magic squares with semi-symmetric center can be transformed into other of the same class. To illustrate this we transform the square in (3) into

$$\begin{bmatrix} x & f & g & t & G & M \\ z & 2s-p-q-h & d & o & p & Y \\ w & m & e & a & H & J \\ s+a-t-x & D & s-e & s-a & k & R \\ 2s-o-j-z & q & n & j & h & T \\ B & L & 2s-g-n-d & 2s-o-j-t & E & F \end{bmatrix}$$

In order to eliminate the effect of the previous transformations we compute all natural four-corner magic squares with semi-symmetric center for which the following conditions hold:

$$p < q, \;\; a < e < s-a, \;\; 1 \leq a \leq 17......(4)$$

The condition $p < q$ eliminates the effect of the last mentioned transformation since it does not change the center. Hence, the total number of all natural four-corner magic squares with semi-symmetric center is the computed number multiplied with sixteen.

The four-corner magic squares with semi-symmetric center possesses another property preserving transformations besides the previous one. The square in (3) can be transformed into

$$\begin{bmatrix} x & f & g & t & G & M \\ z & p+q+h-s & n & j & s-p & Y \\ w & H & e & a & m & J \\ s+a-t-x & k & s-e & s-a & D & R \\ 2s-o-j-z & s-q & d & o & s-h & T \\ B & L & 2s-g-n-d & 2s-o-j-t & E & F \end{bmatrix}$$

We see that the 2×2 center also remains unchanged. Besides, any square satisfying the conditions (4) will be transformed into another square satisfying (4). Hence, the number of natural four-corner magic squares with semi-symmetric center satisfying (4) will be even. Unfortunately, we can not eliminate the effect of the last transformation by requiring a comparison relation.

3.2 Number of squares

We used pentium IV computers core 2 duo CPU (3 GHz) to count the four-corner magic squares with semi-symmetric center. It took about three months to finish. The C code is presented in the appendix. It is based on the choice of a specific center in each run. The values of the other independent variables will be assigned using some nested loops. The value of the dependent variables will be sequentially computed. At each stage we test the computed value for being in the range from 1 to 36 and for being different from other existing values.

The number of centers is 306. We list the number for all different values of a tabulated according to e in the following tables:

$a = 1$

e	number	e	number	e	number	e	number
2	116511842	11	295406900	20	332716362		
3	132987568	12	325569546	21	332269470	29	350383960
4	201772222	13	314724780	22	334768630	30	360519204
5	195213542	14	330356970	23	343668656	31	351628294
6	247269840	15	319871530	24	353535258	32	352098982
7	249849504	16	338570808	25	351413848	33	347005202
8	279043734	17	319583422	26	354646898	34	346545330
9	282858168	18	334466160	27	356452980	35	336033008
10	314225850	19	334597154	28	358076270		

$a = 2$

e	number	e	number	e	number	e	number
3	206302610	11	331287390	19	326515794	27	356484732
4	210992494	12	321640070	20	345399018	28	361179738
5	251163058	13	350613710	21	335036204	29	374039976
6	245105736	14	332114584	22	333026268	30	344502648
7	274508468	15	336104578	23	356959948	31	353814694
8	287150618	16	331537886	24	360229516	32	363592188
9	296599162	17	350953660	25	353925968	33	362587560
10	302170466	18	311139312	26	358846606	34	347566238

$a = 3$

e	number	e	number	e	number	e	number
4	251739982	11	306028538	19	320541794	27	360898736
5	240505110	12	336328382	20	336807852	28	363278990
6	273691912	13	324869294	21	336809760	29	346910646
7	275204588	14	329469256	22	339051514	30	366423556
8	299899266	15	324711266	23	344044096	31	352598114
9	294890678	16	337436298	24	355226184	32	354459440
10	308909490	17	328147564	25	362975754	33	361775374
		18	323876138	26	353891828		

$a = 4$

e	number	e	number	e	number	e	number
5	277262012	12	329152950	19	335007752	26	353888706
6	278106484	13	341948432	20	326708492	27	366826140
7	305137672	14	327143618	21	341404426	28	354933908
8	286925428	15	341003048	22	346408234	29	372809534
9	318200762	16	329377468	23	354547734	30	354855806
10	326734318	17	322104494	24	358998042	31	365478056
11	325879172	18	323967196	25	358784510	32	354459440

$a = 5$

e	number	e	number	e	number	e	number
6	291627212	12	329152040	19	323827718	26	381023514
7	289665812	13	335041250	20	346739100	27	355241148
8	319609974	14	386116304	21	329511150	28	353360938
9	303484650	15	332832472	22	343736302	29	362792908
10	319630542	16	341768070	23	373008254	30	367872592
11	331302656	17	320454620	24	358755138	31	356279070
		18	328627682	25	368793648		

$a = 6$

e	number	e	number	e	number	e	number
7	307831520	13	335408322	19	327838034	25	362278616
8	307220566	14	323632092	20	325045442	26	344707584
9	326632592	15	343900408	21	346521984	27	357310716
10	315214080	16	317359492	22	346273390	28	352453064
11	329922646	17	329088296	23	366929874	29	364671418
12	337240842	18	324929256	24	357798606	30	354408080

$a = 7$

e	number	e	number	e	number	e	number
8	318913908	13	326680272	19	334337068	25	353600518
9	313804886	14	338270640	20	343814008	26	353855988
10	332965950	15	321017534	21	338858100	27	352947394
11	312416444	16	331596514	22	361348270	28	369485474
12	339820720	17	315920958	23	344125594	29	361325404
		18	333891060	24	350598284		

$a = 8$

e	number	e	number	e	number	e	number
9	322651726	14	336499586	19	326604342	24	344790314
10	313931550	15	327023778	20	351929926	25	345471054
11	350259448	16	321879182	21	342614872	26	371263102
12	328728052	17	349000474	22	338372718	27	368869600
13	331352496	18	332648602	23	355184854	28	358386792

$a = 9$

e	number	e	number	e	number	e	number
10	332885406	14	336616920	19	340019774	24	359800942
11	325241400	15	312280764	20	335943324	25	363693102
12	342083312	16	327204636	21	326267006	26	353751236
13	321193778	17	322272062	22	338824226	27	369395860
		18	331971266	23	342484208		

$a = 10$

e	number	e	number	e	number	e	number
11	346174266	15	320420772	19	333407464	23	351659296
12	331299312	16	314295184	20	317775590	24	348631644
13	343459744	17	329404430	21	334963576	25	360576518
14	316652498	18	315451068	22	324426154	26	346157032

$a = 11$

e	number	e	number	e	number	e	number
12	319612194	15	307807314	19	315707450	23	345843756
13	311520480	16	327835916	20	330765638	24	359195928
14	334495182	17	318252788	21	355567308	25	368711314
		18	318965356	22	339579470		

$a = 12$

e	number	e	number	e	number	e	number
13	333424860	16	310667578	19	330888746	22	335530678
14	307147362	17	328751622	20	321671738	23	355516716
15	315379026	18	315313592	21	353451262	24	346475818

$a = 13$

e	number	e	number	e	number	e	number
14	323015258	16	326470472	19	331125814	22	353459532
15	308092470	17	312747912	20	341748002	23	355448444
		18	340473770	21	331096842		

$a = 14$

e	number	e	number	e	number	e	number
15	315555354	17	372360568	19	334305526	21	347812602
16	313515758	18	322125932	20	370518284	22	326982930

$a = 15$

e	number	e	number	e	number	e	number
16	342594820	17	321828952	19	318003462	21	340531500
		18	351825242	20	341764226		

$a = 16$

e	number	e	number	e	number	e	number
17	339798226	18	342610906	19	324696620	20	328137776

$a = 17$

e	number	e	number
18	398369256	19	344164516

The total number of squares is

$$101,425,060,998 = 50,712,530,499 * 2$$

Hence, the total number of natural squares is

$$101425060998 * 16 = 1,622,800,975,968$$

4 Open Problem

The number of four-corner magic squares of order 6. We want here to estimate the number of four-corner magic squares of order 6. By computing the number of squares with semi-symmetric center we noticed that the number of different possible values for a and e is

$$2 + 4 + 8 + ... + 34 = 306$$

The average of the squares per one possible pattern for a and e is

$$\frac{101425060998*16}{306} = 5.3033 \times 10^9$$

The number of all different possible values for a, b and e by computing the number of four-corner magic squares is 3429. Hence we estimate the number of the four-corner magic squares

$$5.3033 \times 10^9 * 3429 = 1.8185 \times 10^{13}$$

We noticed that the number for some possible values for a, b and e, which do not lead to a symmetric center is less that the largest observed number (398369256). This motivates the following

Conjecture 3 *The number of four-corner magic squares of order 6 does not exceed*

$$398369256 * 16 * 3429 = 21,856,130,861,184.$$

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Appendix The C-code

```
#include <assert.h>
#include <errno.h>
#include <io.h>
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include <time.h>
const int N = 6; const int NN = N*N;const int Sum2 = NN - 1;
```

```
const int Sum4 = Sum2 + Sum2; const int Msum = Sum2 + Sum4;
struct bools {bool used[NN];};
struct bools allFree;
#define Uint unsigned int
void writeSquare(int *p, FILE *wfp)
{char squareString[120], *s = squareString; int cells = 0;
{int i; for (i = 0; i < NN; ++i) {int x = p[i] + 1;
if (x < 10) { *s++ = ' '; *s++ = '0' + x; }
else if (x < 20) { *s++ = '1'; *s++ = '0' - 10 + x; }
else if (x < 30) { *s++ = '2'; *s++ = '0' - 20 + x; }
else { *s++ = '3'; *s++ = '0' - 30 + x; }
if (++cells == N) { *s++ = '\n'; cells = 0; } else *s++ = ' ';}}
*s++ = '\n'; *s++ = '\0';fputs(squareString, wfp);}
Uint makeSquares(int a, int b, int e, int J, FILE *wfp)
{Uint count = 0, pcount = 0; bools v = allFree; int Z[NN]; Z[20]=J;
v.used[e] = true; v.used[a] = true; v.used[b] = true; v.used[J] = true;
{int t; for (t = 1; t < 2 ; ++t) if (!v.used[t]) {v.used[t] = true;
{int x; for (x = 21; x < 22; ++x) if (!v.used[x]) {Z[18] = Sum4-b-t-x;
if ((Z[18] < 0) || (Z[18] >= NN) || v.used[Z[18]] || (Z[18] == x)) continue;
v.used[x] = true; v.used[Z[18]] = true;{int j; for (j = 0; j < NN; ++j)
if (!v.used[j]) {
v.used[j] = true;{int o; for (o = 0; o < NN; ++o) if (!v.used[o]) {
Z[33]=Msum-j-b-o-a-t;
if ((Z[33] < 0) || (Z[33] >= NN) || v.used[Z[33]] || (Z[33] == o)) continue;
v.used[o] = true; v.used[Z[33]] = true;
{int z; for (z = 0; z < NN; ++z) if (!v.used[z]) {Z[24]=Sum4-j-z-o;
if ((Z[24] < 0) || (Z[24] >= NN) || v.used[Z[24]] || (Z[24] == z)) continue;
v.used[z] = true; v.used[Z[24]] = true;
{int w; for (w = 0; w < NN; ++w) if (!v.used[w])
{Z[30]=b+j+o+t-w-Sum2;
if ((Z[30] < 0) || (Z[30] >= NN) || v.used[Z[30]] || (Z[30] == w)) continue;
v.used[w] = true; v.used[Z[30]] = true;
{int p; for (p = 0; p < (NN-1); ++p) if (!v.used[p]) {v.used[p] = true;
{int q; for (q = p+1; q < NN; ++q) if (!v.used[q]) {
Z[5]=Sum4-o-p-q-j-t+w+e;
if ((Z[5] < 0) || (Z[5] >= NN) || v.used[Z[5]] || (Z[5] == q)) continue;
Z[35]=Sum2-b+p+q-x-e;if ((Z[35] < 0) || (Z[35] >= NN) || v.used[Z[35]] ||
(Z[35] == q) || (Z[35] == Z[5])) continue;
v.used[q] = true; v.used[Z[5]] = true; v.used[Z[35]] = true;
{int h; for (h = 2; h < 3; ++h) if (!v.used[h]) {Z[28]=Sum4-p-q-h;
if ((Z[28] < 0) || (Z[28] >= NN) || v.used[Z[28]] || (Z[28] == h)) continue;
v.used[h] = true; v.used[Z[28]] = true;
```

```
{int n; for (n = 0; n < NN; ++n) if (!v.used[n]) {Z[11]=Msum-j-z-n-q-h;
if ((Z[11] < 0) || (Z[11] >= NN) || v.used[Z[11]] || (Z[11] == n))
continue; v.used[n] = true; v.used[Z[11]] = true;
{int d; for (d = 0; d < NN; ++d) if (!v.used[d]) {
Z[29]=Msum-Z[24]-p-d-o-Z[28];
if ((Z[29] < 0) || (Z[29] >= NN) || v.used[Z[29]] || (Z[29] == d))
continue; v.used[d] = true; v.used[Z[29]] = true;
{int g; for (g = 0; g < NN; ++g) if (!v.used[g]) {Z[32]=a+b-d-g-n+Sum2;
if ((Z[32] < 0) || (Z[32] >= NN) || v.used[Z[32]] || (Z[32] == g))
continue; Z[17]=d-a+g+n-p-q+x;
if ((Z[17] < 0) || (Z[17] >= NN) || v.used[Z[17]] ||
(Z[17] == g) || (Z[17] == Z[32])) continue;
Z[23]=a+b-g+j+o+p+q+t-w-Sum4;
if ((Z[23] < 0) || (Z[23] >= NN) || v.used[Z[23]] ||
(Z[23] == g) || (Z[23] == Z[32]) || (Z[23] == Z[17])) continue;
v.used[g] = true; v.used[Z[32]] = true; v.used[Z[17]] = true;
v.used[Z[23]] = true;
{int f; for (f = 34; f < 35; ++f) if (!v.used[f]) {Z[4]=Msum-x-g-t-f-Z[5];
if ((Z[4] < 0) || (Z[4] >= NN) || v.used[Z[4]] ||(Z[4] == f)) continue;
v.used[f] = true; v.used[Z[4]] = true;
{int m; for (m = 0; m <NN; ++m) if (!v.used[m]) {
Z[13]=Msum-m-w-e-a-Z[17];
if ((Z[13] < 0) || (Z[13] >= NN) || v.used[Z[13]] || (Z[13] == m)) continue;
v.used[m] = true; v.used[Z[13]] = true;
{int k; for (k = 0; k < NN; ++k) if (!v.used[k]) {
v.used[k] = true; Z[22]=Msum-k-b-Z[18]-Z[23]-Z[20];
if ((Z[22] >= 0) && (Z[22] < NN) && !v.used[Z[22]]) {
v.used[Z[22]] = true; Z[34]=Msum-m-q-Z[4]-Z[22]-Z[28];
if ((Z[34] >= 0) && (Z[34] < NN) && !v.used[Z[34]]) {
v.used[Z[34]] = true; Z[31]=Msum-f-h-k-p-Z[13];
if ((Z[31] >= 0) && (Z[31] < NN) && !v.used[Z[31]]) {
Z[0]=x; Z[1]=f; Z[2]=g; Z[3]=t; Z[6]=z; Z[7]=h;
Z[8]=n; Z[9]=j; Z[10]=q; Z[12]=w; Z[14]=e; Z[15]=a;
Z[16]=m; Z[19]=k; Z[21]=b; Z[25]=p; Z[26]=d; Z[27]=o;
++count; writeSquare(Z, wfp);
if (++pcount == 1000000) {printf("count %lu\n", count);pcount = 0;
fflush(wfp);}}
v.used[Z[34]] = false;}v.used[Z[22]] = false;}v.used[k] = false;}}
v.used[m] = false; v.used[Z[13]] = false;}} v.used[f] = false;
v.used[Z[4]] = false;}}
v.used[g] = false; v.used[Z[17]] = false;v.used[Z[23]] = false;
v.used[Z[32]] = false;}} v.used[d] = false; v.used[Z[29]] = false;}}
```

```
v.used[n] = false; v.used[Z[11]] = false;}} v.used[h] = false;
v.used[Z[28]] = false;}}
v.used[q] = false; v.used[Z[5]] = false; v.used[Z[35]] = false;}}
v.used[p] = false;}} v.used[w] = false; v.used[Z[30]] = false;}}
v.used[z] = false; v.used[Z[24]] = false;}}v.used[o] = false;
v.used[Z[33]] = false;}} v.used[j] = false;}}
v.used[x] = false; v.used[Z[18]] = false;}} v.used[t] = false;}}
printf("number of squares %d\n", count);return count;}
void get_rest_of_line(int c) {
if (c != '\n') do { c = getchar(); } while (c != '\n');}
void get_abe(int *a, int *b, int *e) {
int unused = scanf("%d %d %d", a, b, e);
int c = getchar(); get_rest_of_line(c);}
void getNumPatterns(int *num) {int unused = scanf("%d", num);
int c = getchar(); get_rest_of_line(c);}
bool check_abe(int a, int b, int e) {bool rv = true;if ((a <= 0) ||
(a > NN) || (b <= 0) || (b > NN) || (e <= 0) || (e > NN)) {
printf("\aValue range is 1 to %d.\n\n", NN);rv = false;}return rv;}
bool checkNum(int num) {
bool rv = true;if (num <= 0) {printf("\aNumber must be a positive integer\n\n");
rv = false;}return rv;}
const int bufSize = 128;
void openOutput(int a, int b, int e, char *wfpName, FILE **wfp) {
const int defSize = 15;
char buf[bufSize], buf1[bufSize], defaultName[defSize];
if (a == 0) strcpy(defaultName, "s6_counts");
else sprintf(defaultName, "s6_a%ib%ie%i", a, b, e);
strcpy(buf, defaultName); strcat(buf, ".txt"); {int sub = 0;
do {if ((fopen(buf, "r") == NULL) && (errno == ENOENT)) {break;}
else {
strcpy(buf, defaultName); sprintf(buf1, "_%i", ++sub); strcat(buf, buf1);
strcat(buf, ".txt");} } while (true); if ( (*wfp = fopen(buf, "w")) != NULL)
{ printf("\n%s file is %s\n", a == 0 ? "Data" : "Squares", buf);
strcpy(wfpName, buf); } else { strcpy(buf1, "\a\nCan't open for write ");
strcat(buf1, buf); perror(buf1);}}}
int getSquares(int a0, int b0, int e0, int num, FILE *wfpc) {
char wfpsName[bufSize]; FILE *wfps = NULL;
int linecount = 0; Uint count = 0; int patterns = 0;
{int a; for (a = -a0; a < NN; ++a) { {int b; for (b = -b0; b < NN; ++b)
if (a != b) {
{int e; for (e = -e0; e < NN; ++e) if ((a < e) && (e < b)) {
```

```
int J = Sum4-e-a-b; if ((J <= a) || (J == b) || (J == e) || (J >= NN))
continue;
openOutput(a+1, b+1, e+1, wfpsName, &wfps);
if (wfps != NULL) { time_t startTime; startTime = time(NULL);
count = makeSquares(a, b, e, J, wfps);
fprintf(wfpc, "a: %2d b: %2d e: %2d number of squares: %10lu time: ",
a+1, b+1, e+1, count);
{ int elapsed_t = (int)(time(NULL) - startTime);
int hr = elapsed_t/3600; elapsed_t %= 3600;
{ int min = elapsed_t/60, sec = elapsed_t%60;
{ char *fmt = "%3d:%02d:%02d\n";
printf(fmt, hr, min, sec); fprintf(wfpc, fmt, hr, min, sec);
if (++linecount == 5) { fprintf(wfpc, "\n"); linecount = 0; }
fflush(wfpc); fclose(wfps); if (++patterns == num) return patterns;
}}}}}}} }} return patterns;}
int main() { int a, b, e; printf("\nInput start values for a b e: ");
get_abe(&a, &b, &e); if (check_abe(a, b, e)) { int num = -1;
printf("\nInput number of a b e patterns to run: ");
getNumPatterns(&num);
if (checkNum(num)) { char wfpcName[bufSize]; FILE *wfpc = NULL;
openOutput(0, 0, 0, wfpcName, &wfpc);
if (wfpc != NULL) { time_t startTime = time(NULL);
int patterns = getSquares(a, b, e, num, wfpc);
int elapsed_t = (int)(time(NULL) - startTime);
int hr = elapsed_t/3600; elapsed_t %= 3600;
{ int min = elapsed_t/60, sec = elapsed_t%60;
{ char *fmt = "\na, b, e patterns: %d elapsed time: %d:%02d:%02d\n";
printf(fmt, patterns, hr, min, sec);
fprintf(wfpc, fmt, patterns, hr, min, sec);
fclose(wfpc); } } } }} { int unused = getchar(); } return 0;}
```