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Some conditions under which prime near-rings are commutative rings

Abdelkarim Boua

Université Moulay Ismaïl, Faculté des Sciences et Techniques
Département de Mathématiques, Groupe d'Algèbre et Applications
B. P. 509 Boutalamine, Errachidia; Maroc
karimoun2006@yahoo.fr

Abstract

Let N be a prime left near-ring, and I be a nonzero semigroup ideal of N. We prove that if N admits a derivation d satisfying any one of the following properties: $(i)\, d([x,y]) = [x,y]$, $(ii)\, d([x,y]) = [d(x),y]$, $(iii)\, [d(x),y] = [x,y]$, $(iv)\, d(x \circ y) = x \circ y$, $(v)\, d(x) \circ y = x \circ y$ and $(vi)\, d(x) \circ y = x \circ y$ for all $x, y \in I$, then N is a commutative ring. Moreover, example proving the necessity of the primeness condition is given.

Keywords: *prime near-rings, derivations, commutativity.*
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1 Introduction

A left near-ring is a set N with two operations $+$ and $\cdot$ such that $(N, +)$ is a group and $(N, \cdot)$ is a semigroup satisfying the left distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in N$. A left near-ring N is called Zero symmetric if $0 \cdot x = 0$ for all $x \in N$ (recall that left distributivity yields $x \cdot 0 = 0$). Throughout this paper, unless otherwise specified, we will use the word near-ring to mean zero symmetric left near-ring and denote xy instead of $x \cdot y$. A nonempty subset I of N will be called a semigroup ideal if $IN \subset I$ and $NI \subset I$. An additive mapping $d : N \longrightarrow N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [7], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. According to [4], a near-ring N is said

to be prime if $xNy = 0$ for $x, y \in N$ implies $x = 0$ or $y = 0$. For any $x, y \in N$, the symbol $[x, y]$ stands for the commutator $xy - yx$, while the symbol $x \circ y$ will denote $xy + yx$. The symbol $Z(N)$ will represent the multiplicative center of N, that is, $Z(N) = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. Recall that N is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in N$.
Recently, there has been a great deal of work concerning commutativity of prime and semi-prime rings with derivations satisfying certain differential identities (see for example [2], [3], [4], [5], [6]) asserting that the existence of a suitably-constrained derivation on a prime near-ring forces the near-ring to be a ring. In this paper we continue the line of investigation regarding the study of prime near-rings with derivations. More precisely, we shall prove that a prime near-ring which admits a nonzero derivation satisfying certain differential identities must be a commutative ring.

2 Main Results

In order to prove our main theorems, we shall need the following lemmas.

Lemma 2.1 ([3], Theorem 2.1) *Let N be a prime near-ring, and I be a nonzero semigroup ideal of N. If N admits a nonzero derivation d for which $d(I) \subset Z(N)$, then N is a commutative ring.*

Lemma 2.2 ([4], Lemma 1) *Let d be an arbitrary derivation on the near-ring N. Then N satisfies the following partial distributive law:*

$$(xd(y) + d(x)y)z = xd(y)z + d(x)yz \text{ for all } x, y, z \in N.$$

Lemma 2.3 ([3], Lemma 1.4(i)) *Let N be a prime near-ring, and I a nonzero semigroup ideal of N. Let d be a nonzero derivation on N. If $x, y \in N$ and $xIy = \{0\}$, then $x = 0$ or $y = 0$.*

Lemma 2.4 *Let N be a 2-torsion free prime near-ring, and I be a nonzero semigroup ideal of N. If N admits a nonzero derivation d for which $d(-I) \subset Z(N)$, then N is a commutative ring.*

Proof. By hypothesis given we have

$$d(-x) \in Z(N) \text{ for all } x \in I. \tag{1}$$

Replacing x by tx in (1), where $t \in Z(N)$, we get

$$d(t)(-x) + td(-x) \in Z(N) \text{ for all } x \in I, t \in Z(N).$$

This implies that

$$d(t)(-x) \in Z(N) \text{ for all } x \in I, t \in Z(N). \tag{2}$$

Hence

$$d(t)N[-x,u] = \{0\} \text{ for all } x \in I, t \in Z(N), u \in N. \tag{3}$$

Since N is prime and using (3), we obtain

$$d(Z(N)) = 0 \quad \text{or} \quad -x \in Z(N) \text{ for all } x \in I. \tag{4}$$

(i) If $d(Z(N)) = \{0\}$, then $0 = d^2(u(-x)) = d^2(u)(-x)+2d(u)d(-x)+ud^2(-x)$ for all $x, \in I$, $u \in N$ which, because of $d^2(-x) = 0$, forces

$$d^2(u)(-x) + 2d(u)d(-x) = 0 \quad \text{for all } x \in I, u \in N. \tag{5}$$

Applying d again, we get

$$d^3(u)(-x) + 3d^2(u)d(-x) = 0 \quad \text{for all } x \in I, u \in N. \tag{6}$$

Taking $d(u)$ instead of u in (6) gives

$$d^3(u(-x)) + 2d^2(u)d(-x) = 0$$

Combining the last equation with (6) we obtain $d^2(u)d(-x) = 0$ so that

$$d^2(u)Nd(-x) = 0 \quad \text{for all } x \in I, u \in N. \tag{7}$$

By primeness of N, equation (7) gives either $d^2(u) = 0$ or $d(-x) = 0$. Hence

$$d^2(u) = 0 \quad \text{or} \quad d(x) = 0 \text{ for all } x \in I, u \in N. \tag{8}$$

If $d(x) = 0$ for all $x \in I$, then it is easy to find that $d = 0$ which contradicts our hypothesis. If $d^2(u) = 0$ for all $u \in N$, then ([4], Lemma 3) assures that $d = 0$ which is impossible.
(ii) Assume that

$$-x \in Z(N) \quad \text{for all } x \in I.$$

Replacing x by vx, where $v \in N$, in the above equation we have $v(-x) \in Z(N)$ and thus

$$(-x)N[y,v] = 0 \text{ for all } x \in I, y, v \in N. \tag{9}$$

Since N is prime and $I \neq \{0\}$ so that $y \in Z(N)$ for all $y \in N$. Therefore, $d(y) \in Z(N)$, then $d(N) \subset Z(N)$. Using Lemma 2.1, we conclude that N is a commutative ring. This completes the proof of our theorem.

Remark 2.5 *Note that $-I$ is a semigroup right ideal, for if $x \in I$ and $w \in N$, $(-x)w = -xw \in -I$. Therefore, the result follows from Lemma* 2.1

Theorem 2.6 *Let N be a prime near-ring, and I be a nonzero semigroup ideal of N. If N admits a nonzero derivation d such that $d([x,y]) = [x,y]$ for all $x,y \in I$, then N is a commutative ring.*

Proof. Assume that

$$d([x,y]) = [x,y] \quad \text{for all } \; x,y \in I. \tag{10}$$

Replacing y by xy in (10), because of $[x,xy] = x[x,y]$, we get

$$x[x,y] = d(x[x,y]) \;\; \text{forall} \;\; x,y \in I.$$

Since $d(x[x,y]) = xd([x,y]) + d(x)[x,y]$, then according to (10) we obtain

$$x[x,y] = x[x,y] + d(x)[x,y]$$

and therefore $d(x)[x,y] = 0$. Hence

$$d(x)xy = d(x)yx \quad \text{for all } \; x,y \in I. \tag{11}$$

Substituting yz for y in (11), where $z \in N$, because of $d(x)xyz = d(x)yxz$, we obtain $d(x)y[x,z] = 0$ for all $x,y \in I$, $z \in N$ which leads to

$$d(x)I[x,z] = 0 \quad \text{for all } \; x \in I, z \in N. \tag{12}$$

Using Lemma 2.3, equation (12) reduces to

$$d(x) = 0 \quad \text{or} \;\; [x,z] = 0 \;\; \text{for all } \; x \in I, z \in N. \tag{13}$$

From (13) it follows that for each fixed $x \in I$ we have

$$d(x) = 0 \quad \text{or} \;\; x \in Z(N). \tag{14}$$

But $x \in Z(N)$ also implies that $d(x) \in Z(N)$ and equation (14) forces

$$d(x) \in Z(N) \quad \text{for all } \; x \in I. \tag{15}$$

In the light of (15), $d(I) \subset Z(N)$ and using Lemma 2.1 we conclude that N is a commutative ring. This completes the proof of our theorem.

Theorem 2.7 *Let N be a prime near-ring, and I be a nonzero semigroup ideal of N. If N admits a nonzero derivation d such that either $d([x,y]) = [d(x),y]$ for all $x,y \in I$ or $[x,y] = [d(x),y]$ for all $x,y \in I$, then N is a commutative ring.*

Proof. Assume that

$$d([x,y]) = [d(x),y] \quad \text{for all} \ \ x,y \in I. \tag{16}$$

Replacing y by xy in (16) we get

$$[d(x),xy] = d(x[x,y]) \ \ \text{for all} \ \ x,y \in I. \tag{17}$$

Since $d(x[x,y]) = d(x)[x,y] + xd([x,y])$, in light of (16), equation (17) reduces to

$$d(x)yx = xd(x)y \quad \text{for all} \ \ x,y \in I. \tag{18}$$

Substituting yz for y in (18), where $z \in N$ and using $xd(x)yz = d(x)yxz$, we obtain $d(x)y[x,z] = 0$. Hence

$$d(x)I[x,z] = 0 \quad \text{for all} \ \ x \in I, z \in N. \tag{19}$$

Since equation (19) is the same as equation (12), arguing as in the proof of Theorem 2.6, we conclude that N is a commutative ring.
Now suppose that

$$[d(x),y] = [x,y] \quad \text{for all} \ \ x,y \in I. \tag{20}$$

Replacing x by yx in (20), because of $[yx,y] = y[x,y]$, we get

$$[d(yx),y] = y[x,y] = y([d(x),y]) \ \ \text{for all} \ \ x,y \in I.$$

Since $[d(yx),y] = d(yx)y - yd(yx)$, then according to Lemma 2.2 we obtain

$$yd(x)y + d(y)xy - yd(y)x - y^2d(x) = yd(x)y - y^2d(x),$$

so that

$$d(y)xy = yd(y)x \quad \text{for all} \ \ x,y \in I. \tag{21}$$

Since equation (21) is the same as equation (12), arguing as in the first case we find that N is a commutative ring.

The conclusion of Theorems 2.6 and 2.7 no remains valid if we replace the product $[x,y]$ by $x \circ y$. In fact, we obtain the following result:

Theorem 2.8 *Let N be a 2-torsion free prime near-ring, and I be a nonzero semigroup ideal of N. Then there is no derivation d such that $d(x \circ y) = x \circ y$ for all $x,y \in I$.*

Proof. If there exists a nonzero d such that

$$d(x \circ y) = xy + yx \quad \text{for all} \ \ x,y \in I. \tag{22}$$

Then, replacing y by xy in (22), we get

$$d(x \circ (xy)) = x^2y + xyx \quad \text{for all} \;\; x, y \in I. \tag{23}$$

Since $x \circ (xy) = x(x \circ y)$, then (22) yields $d(x \circ (xy)) = x(x \circ y) + d(x)(x \circ y)$. Hence equation (23) reduces to

$$x(x \circ y) + d(x)(x \circ y) = x^2y + xyx \quad \text{for all} \;\; x, y \in I. \tag{24}$$

As $x^2y + xyx = x(x \circ y)$, then (24) assures that

$$d(x)(x \circ y) = 0 \quad \text{for all} \;\; x, y \in I,$$

which leads to

$$d(x)xy = -d(x)yx \quad \text{for all} \;\; x, y \in I. \tag{25}$$

Substituting yz for y in (25), where $z \in N$, we find that

$$-d(x)yzx = d(x)xyz = (-d(x)yx)z = d(x)y(-x)z \quad \text{for all} \;\; x, y \in I, z \in N. \tag{26}$$

Since $-d(x)yzx = d(x)yz(-x)$, then (26) becomes

$$d(x)yz(-x) = d(x)y(-x)z \quad \text{for all} \;\; x, y \in I, z \in N. \tag{27}$$

Then

$$d(x)I[-x, z] = 0 \quad \text{for all} \;\; x, y \in I, z \in N. \tag{28}$$

By Lemma 2.3, equation (28) assures that for each $x \in I$, either $-x \in Z(N)$ or $d(x) = 0$. Accordingly,

$$d(-x) = 0 \quad \text{or} \quad -x \in Z(N) \;\; \text{for all} \;\; x \in I. \tag{29}$$

In light of $d(Z(N)) \subset Z(N)$, equation (29) yields then

$$d(-x) \in Z(N) \;\; \text{for all} \;\; x \in I. \tag{30}$$

Hence $d(-I) \subset Z(N)$, by Lemma 2.4, equation (30) assures that N is a commutative ring. Use the fact that N is a 2-torsion free, the hypothesis $d(x \circ y) = x \circ y$ for all $x, y \in I$, becomes

$$d(xy) = xy \;\; \text{for all} \;\; x, y \in I.$$

So that

$$d(x)y + xd(y) = xy \;\; \text{for all} \;\; x, y \in I.$$

Replacing x by xz, then the last equation can be written as $xzd(y) = 0$ for all $x, y, z \in I$, thus $xId(y) = 0$ for all $x, y \in I$. Since $I \neq \{0\}$, then Lemma 2.3 shows that $d = 0$ on I, then it is easy to see that $d = o$ on N; a contradiction. If there exists a zero derivation d such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then we can easily see that $x = 0$ for all $x \in I$; a contradiction.

Theorem 2.9 *Let N be a 2-torsion free prime near-ring, and I be a nonzero semigroup ideal of N. Then there is no derivation d such that $d(x) \circ y = x \circ y$ for all $x, y \in I$.*

Proof. Suppose there exists a nonzero derivation d such that

$$d(x \circ y) = d(x)y + yd(x) \quad \text{for all } x, y \in I. \tag{31}$$

Then, replacing y by xy in (31), we get

$$d(x \circ (xy)) = d(x)xy + xyd(x) \quad \text{for all } x, y \in I. \tag{32}$$

Since $x \circ (xy) = x(x \circ y)$, then $d(x \circ (xy)) = d(x)(x \circ y) + xd(x \circ y)$. As $d(x \circ y) = d(x) \circ y$ by hypothesis, then $d(x \circ (xy)) = d(x)(x \circ y) + x(d(x) \circ y)$. Hence equation (32) reduces to

$$d(x)yx = -xd(x)y \quad \text{for all } x, y \in I. \tag{33}$$

Substituting yz for y in (33), where $z \in N$ we find that

$$d(x)yzx = -xd(x)yz = xd(x)y(-z) = d(x)y(-x)(-z) = -d(x)y(-x)z. \tag{34}$$

Since $-d(x)yzx = d(x)yz(-x)$, then (34) becomes

$$d(x)yz(-x) = d(x)y(-x)z \quad \text{for all } x, y \in I, z \in N. \tag{35}$$

Since equation (35) is the same as equation (27), arguing as in the proof of Theorem 2.8 we conclude that N is a commutative ring. Use the fact that N is a 2-torsion free and the hypothesis of Theorem we arrive at

$$d(x)y = xy \quad \text{for all } x, y \in I.$$

Replacing x par xz, we get $d(x)zy + xd(z)y = xzy$ for all $x, y, z \in I$, then $d(x)zy = 0$ for all $x, y, z \in I$. Since $I \neq 0$, then Lemma 2.3 assure that $d = 0$; a contradiction. If there exists a zero derivation d such that $d(x) \circ y = x \circ y$ for all $x, y \in I$, then we can easily see that $x = 0$ for all $x \in I$; a contradiction.

Theorem 2.10 *Let N be a 2-torsion free prime near-ring, and I be a nonzero semigroup ideal of N. If N admits a derivation d such that $d(x \circ y) = d(x) \circ y$ for all $x, y \in I$, then $d = 0$.*

Proof. Suppose that

$$d(x) \circ y = x \circ y \quad \text{for all } x, y \in I. \tag{36}$$

Replacing x by yx in (36) we obtain

$$d(yx) \circ y = y(x \circ y) = y(d(x) \circ y) \quad \text{for all } x, y \in I. \tag{37}$$

Since $d(yx) \circ y = d(yx)y + yd(yx)$, then according to Lemma 2.2 we obtain

$$yd(x)y + d(y)xy + yd(y)x + y^2d(x) = yd(x)y + y^2d(x),$$

this implies that

$$d(y)xy = -yd(y)x \quad \text{for all } \; x, y \in I. \tag{38}$$

As equation (38) is the same as equation (33), arguing as above we conclude that N is a commutative ring. Using the hypothesis of Theorem we have

$$xd(y) = 0 \quad \text{for all } \; x, y \in I.$$

Hence

$$xId(y) = 0 \quad \text{for all } \; x, y \in I.$$

Since $I \neq \{0\}$, then Lemma 2.3 shows that $d = 0$ on I, then it is easy to see $d = 0$ on N.

The following example demonstrate that the primeness hypothesis in Theorems 2.6, 2.7, 2.8, 2.9 and 2.10 cannot be omitted.

Example. Let S be a commutative near-ring. Set $N = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in S \right\}$. It is clear that N is not prime. Moreover, $d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, d is a nonzero derivation of N and I is a semigroup ideal of N such that: (i) $d([A, B]) = [A, B]$, (ii) $d([A, B]) = [d(A), B]$, (iii) $[d(A), B] = [A, B]$, (iv) $d(A \circ B) = A \circ B$, (v) $d(A) \circ B = d(A) \circ B$ and (vi) $d(A) \circ B = A \circ B$ for all $A, B \in I$, but N is a noncommutative ring.

3 Open Problem

In this section we introduce the following open question:
(i) Does the results remain valid for I a left semigroup ideal?
(ii) Does the results remain valid for I a right semigroup ideal?

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