Smarandache $\text{TM}_1$ Curves of Spacelike Biharmonic B-Slant Helices According To Bishop Frame In The Lorentzian $E(1,1)$

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Abstract

In this paper, we characterize Smarandache $\text{TM}_1$ curves of spacelike biharmonic B-slant helices according to Bishop frame in the Lorentzian group of rigid motions $E(1,1)$.

Keywords: Bienergy, Biharmonic curve, Bishop frame, Smarandache $\text{TM}_1$ curve.

1 Introduction

Let $E(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$
\begin{pmatrix}
\cosh x & \sinh x & y \\
\sinh x & \cosh x & z \\
0 & 0 & 1
\end{pmatrix}.
$$

Topologically, $E(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$
E(1,1) \to \mathbb{R}^3 : \begin{pmatrix}
\cosh x & \sinh x & y \\
\sinh x & \cosh x & z \\
0 & 0 & 1
\end{pmatrix} \to (x, y, z),
$$

It's Lie algebra has a basis consisting of

$$
X_1 = \frac{\partial}{\partial x}, X_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z}, X_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},
$$

for which
\[ [X_1, X_2] = X_3, \{X_2, X_3\} = 0, [X_1, X_3] = X_2. \]

Put

\[
x^1 = x, x^2 = \frac{1}{2}(y + z), x^3 = \frac{1}{2}(y - z).
\]

Then, we get

\[
X_i = \frac{\partial}{\partial x_i}, X_2 = \frac{1}{2} \left( e^{x^1} \frac{\partial}{\partial x^2} + e^{-x^1} \frac{\partial}{\partial x^3} \right), X_3 = \frac{1}{2} \left( e^{x^1} \frac{\partial}{\partial x^2} - e^{-x^1} \frac{\partial}{\partial x^3} \right).
\]

The bracket relations are

\[ [X_1, X_2] = X_3, \{X_2, X_3\} = 0, [X_1, X_3] = X_2. \]

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis \( \{X_1, X_2, X_3\} \). We consider left-invariant Lorentzian metric \([10]\), given by

\[
g = -(dx^1)^2 + \left( e^{-x^1} dx^2 + e^{x^1} dx^3 \right)^2 + \left( e^{-x^1} dx^2 - e^{x^1} dx^3 \right)^2,
\]

where

\[
g(X_1, X_1) = -1, g(X_2, X_2) = g(X_3, X_3) = 1.
\]

Let coframe of our frame be defined by

\[
\theta^1 = dx^1, \theta^2 = e^{-x^1} dx^2 + e^{x^1} dx^3, \theta^3 = e^{-x^1} dx^2 - e^{x^1} dx^3.
\]

In this paper, we characterize Smarandache \( TM_1 \) curves of spacelike biharmonic slant helices according to Bishop frame in the Lorentzian group of rigid motions \( E(1,1) \).

1) **Smarandache** \( TM_1 \) **Curves of Spacelike Biharmonic B-Slant Helices in the Lorentzian Group of Rigid Motions** \( E(1,1) \)

Let \( \gamma : I \rightarrow E(1,1) \) be a non geodesic spacelike curve on the \( E(1,1) \) parametrized by arc length. Let \( \{T, N, B\} \) be the Frenet frame fields tangent to the \( E(1,1) \) along \( \gamma \) defined as follows:
\( \mathbf{T} \) is the unit vector field \( \gamma' \) tangent to \( \gamma \), \( \mathbf{N} \) is the unit vector field in the direction of \( \nabla_{\gamma} \mathbf{T} \) (normal to \( \gamma' \)), and \( \mathbf{B} \) is chosen so that \( \{\mathbf{T}, \mathbf{N}, \mathbf{B}\} \) is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

\[
\begin{align*}
\nabla_{\gamma} \mathbf{T} &= \kappa \mathbf{N}, \\
\nabla_{\gamma} \mathbf{N} &= \kappa \mathbf{T} + \tau \mathbf{B}, \\
\nabla_{\gamma} \mathbf{B} &= \tau \mathbf{N},
\end{align*}
\]

where \( \kappa \) is the curvature of \( \gamma \) and \( \tau \) is its torsion and

\[
\begin{align*}
g(\mathbf{T}, \mathbf{T}) &= 1, g(\mathbf{N}, \mathbf{N}) = -1, g(\mathbf{B}, \mathbf{B}) = 1, \\
g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0.
\end{align*}
\]

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

\[
\begin{align*}
\nabla_{\gamma} \mathbf{T} &= k_1 \mathbf{M}_1 - k_2 \mathbf{M}_2, \\
\nabla_{\gamma} \mathbf{M}_1 &= k_1 \mathbf{T}, \\
\nabla_{\gamma} \mathbf{M}_2 &= k_2 \mathbf{T},
\end{align*}
\]

where

\[
\begin{align*}
g(\mathbf{T}, \mathbf{T}) &= 1, g(\mathbf{M}_1, \mathbf{M}_1) = -1, g(\mathbf{M}_1, \mathbf{M}_2) = 1, \\
g(\mathbf{T}, \mathbf{M}_1) &= g(\mathbf{T}, \mathbf{M}_2) = g(\mathbf{M}_1, \mathbf{M}_2) = 0.
\end{align*}
\]

Here, we shall call the set \( \{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\} \) as Bishop trihedra, \( k_1 \) and \( k_2 \) as Bishop curvatures and \( \tau(s) = \psi'(s) \), \( \kappa(s) = \sqrt{k_2^2 - k_1^2} \). Thus, Bishop curvatures are defined by

\[
\begin{align*}
k_1 &= \kappa(s) \sinh \psi(s), \\
k_2 &= \kappa(s) \cosh \psi(s).
\end{align*}
\]

With respect to the orthonormal basis \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) we can write

\[
\begin{align*}
\mathbf{T} &= T^1 \mathbf{e}_1 + T^2 \mathbf{e}_2 + T^3 \mathbf{e}_3, \\
\mathbf{M}_1 &= M_1^1 \mathbf{e}_1 + M_1^2 \mathbf{e}_2 + M_1^3 \mathbf{e}_3, \\
\mathbf{M}_2 &= M_2^1 \mathbf{e}_1 + M_2^2 \mathbf{e}_2 + M_2^3 \mathbf{e}_3.
\end{align*}
\]

**Definition 2.1.** Let \( \gamma : I \to \mathbb{E}(1,1) \) be a unit speed regular curve in the Lorentzian group of rigid motions \( \mathbb{E}(1,1) \). and \( \{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\} \) be its moving Bishop frame. Smarandache TM curves are defined by

\[
\gamma_{\mathbf{TM}} = \frac{1}{k_2}(\mathbf{T} + \mathbf{M}_1).
\]
**Definition 2.2.** [7], A regular spacelike curve $\gamma : I \rightarrow \mathbb{E}(1,1)$ is called a B-slant helix provided the timelike unit vector $M_i$ of the curve $\gamma$ has constant angle $\theta$ with some fixed timelike unit vector $u$, that is
\[
g(M_i(s), u) = \cosh \phi \text{ for all } s \in I.
\]

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

**Lemma 2.3.** [7], Let $\gamma : I \rightarrow \mathbb{E}(1,1)$ be a unit speed spacelike curve with non-zero natural curvatures. Then $\gamma$ is a slant helix if and only if
\[
\frac{k_1}{k_2} = \tanh \phi.
\]

**Theorem 2.4.** Let $\gamma : I \rightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic B-slant helix in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric equations of Smarandache $TM_i$ curves of spacelike biharmonic slant helix are
\[
\gamma_{TM_i}(s) = \frac{1}{k_2} (\cosh \phi - \sinh \phi) \mathbf{X}_1 + \frac{1}{k_2} \cos[D_1s + D_2](\sinh \phi - \cosh \phi) \mathbf{X}_2 \\
+ \frac{1}{k_2} \sin[D_1s + D_2](\sinh \phi - \cosh \phi) \mathbf{X}_3,
\]
where $D_1, D_2$ are constants of integration.

**Proof.** Assume that $\gamma$ is a non geodesic spacelike biharmonic B-slant helix according to Bishop frame.

From Definition 3.2, we obtain
\[
M_i = \cosh \phi \mathbf{X}_1 + \sinh \phi \cos[D_1s + D_2] \mathbf{X}_2 + \sinh \phi \sin[D_1s + D_2] \mathbf{X}_3. \tag{2.7}
\]

Using (1.1) in (2.7), we can choose
\[
M_2 = -\sin[D_1s + D_2] \mathbf{X}_2 + \cos[D_1s + D_2] \mathbf{X}_3. \tag{2.8}
\]

From above equations we get
\[
T = -\sinh \phi \mathbf{X}_1 - \cosh \phi \cos[D_1s + D_2] \mathbf{X}_2 - \cosh \phi \sin[D_1s + D_2] \mathbf{X}_3. \tag{2.9}
\]
Substituting (2.7) and (2.9) in (2.4) we have (2.6), which completes the proof.

In terms of Eqs. (1.1) and (2.6), we may give:

**Corollary 2.5.** Let \( \gamma : I \to \mathbb{E}(1,1) \) is a non geodesic spacelike biharmonic B-slant helix in the Lorentzian group of rigid motions \( \mathbb{E}(1,1) \). Then, the parametric equations of Smarandache \( TM_1 \) curves of spacelike biharmonic slant helix are

\[
x_{TM_1}^1(s) = \frac{1}{k_2}(\cosh \varphi - \sinh \varphi),
\]

\[
x_{TM_1}^2(s) = \frac{1}{2k_2} e^{\frac{1}{k_2}(\cosh \varphi - \sinh \varphi)} \cos[D_1s + D_2] \sinh \varphi - \cosh \varphi)
\]

\[
+ \frac{1}{2k_2} e^{\frac{1}{k_2}(\cosh \varphi - \sinh \varphi)} \sin[D_1s + D_2] \sinh \varphi - \cosh \varphi),
\]

\[
x_{TM_1}^3(s) = \frac{1}{2k_2} e^{\frac{1}{k_2}(\cosh \varphi - \sinh \varphi)} \cos[D_1s + D_2] \sinh \varphi - \cosh \varphi)
\]

\[
- \frac{1}{2k_2} e^{\frac{1}{k_2}(\cosh \varphi - \sinh \varphi)} \sin[D_1s + D_2] \sinh \varphi - \cosh \varphi),
\]

where \( D_1, D_2 \) are constants of integration.

**Proof.** Substituting (1.1) to (2.6), we have (2.10) as desired.

We may use Mathematica in Corollary 2.5, yields

![Figure 1](image-url)
In the light of Lemma 2.3 and Corollary 2.5, we express the following Corollary without proofs:

**Corollary 2.6.** Let $\gamma: I \to E(1,1)$ is a non geodesic spacelike biharmonic \(B\)-slant helix in the Lorentzian group of rigid motions \(E(1,1)\). Then, the parametric equations of Smarandache \(TM_1\) curves of spacelike biharmonic slant helix are

\[
x_{1_{TM_1}}(s) = \frac{1}{k_1} (\sinh \varphi - \frac{\sinh^2 \varphi}{\cosh \varphi}),
\]

\[
x_{2_{TM_1}}(s) = \frac{1}{2k_1} e^{\frac{1}{k_1}} \left[ \frac{1}{\sinh \varphi - \frac{\sinh^2 \varphi}{\cosh \varphi}} \right] \cos[D_1s + D_2 \left( \frac{\sinh^2 \varphi}{\cosh \varphi} - \sinh \varphi \right)]
\]

\[
+ \frac{1}{2k_1} e^{\frac{1}{k_1}} \sin[D_1s + D_2 \left( \frac{\sinh^2 \varphi}{\cosh \varphi} - \sinh \varphi \right)],
\]

\[
x_{3_{TM_1}}(s) = \frac{1}{2k_1} e^{\frac{1}{k_1}} \left[ \frac{1}{\sinh \varphi - \frac{\sinh^2 \varphi}{\cosh \varphi}} \right] \cos[D_1s + D_2 \left( \frac{\sinh^2 \varphi}{\cosh \varphi} - \sinh \varphi \right)]
\]

\[
- \frac{1}{2k_1} e^{\frac{1}{k_1}} \sin[D_1s + D_2 \left( \frac{\sinh^2 \varphi}{\cosh \varphi} - \sinh \varphi \right)],
\]

where \(D_1, D_2\) are constants of integration.

Also, we may use Mathematica in Corollary 2.6, yields
3 Open Problem

The authors can be research Smarandache $\mathbf{TM}_2$ curves of spacelike biharmonic $\mathbf{B}$-slant helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbf{E}(1,1)$.

References

[7] T. Körpınar, E. Turhan: Spacelike biharmonic $\mathbf{B}$-slant helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbf{E}(1,1)$, (submitted).