*Int. J. Open Problems Compt. Math., Vol. 5, No. 2, June 2012*
*ISSN 1998-6262;* 
*www.i-csrs.org*

# Smarandache $\mathbf{TM}_1$ Curves of Spacelike Biharmonic B-Slant Helices According To Bishop Frame In The Lorentzian $\mathsf{E}(1,1)$

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**Abstract**

***In this paper, we characterize Smarandache*** $\mathbf{TM}_1$ ***curves of spacelike biharmonic B-slant helices according to Bishop frame in the Lorentzian group of rigid motions*** $\mathsf{E}(1,1)$.

**Keywords**: *Bienergy, Biharmonic curve, Bishop frame, Smarandache* $\mathbf{TM}_1$ *curve.*

## 1 Introduction

Let $\mathsf{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathsf{E}(1,1)$ is diffeomorphic to $\mathsf{R}^3$ under the map

$$\mathsf{E}(1,1) \to \mathsf{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \to (x, y, z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z}, \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1,\mathbf{X}_2]=\mathbf{X}_3,[\mathbf{X}_2,\mathbf{X}_3]=0,[\mathbf{X}_1,\mathbf{X}_3]=\mathbf{X}_2.$$

Put

$$x^1=x,x^2=\frac{1}{2}(y+z),x^3=\frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{X}_1=\frac{\partial}{\partial x^1},\mathbf{X}_2=\frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2}+e^{-x^1}\frac{\partial}{\partial x^3}\right),\mathbf{X}_3=\frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2}-e^{-x^1}\frac{\partial}{\partial x^3}\right). \tag{1.1}$$

The bracket relations are

$$[\mathbf{X}_1,\mathbf{X}_2]=\mathbf{X}_3,[\mathbf{X}_2,\mathbf{X}_3]=0,[\mathbf{X}_1,\mathbf{X}_3]=\mathbf{X}_2.$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [10], given by

$$g=-(dx^1)^2+\left(e^{-x^1}dx^2+e^{x^1}dx^3\right)^2+\left(e^{-x^1}dx^2-e^{x^1}dx^3\right)^2,$$

where

$$g(\mathbf{X}_1,\mathbf{X}_1)=-1,g(\mathbf{X}_2,\mathbf{X}_2)=g(\mathbf{X}_3,\mathbf{X}_3)=1.$$

Let coframe of our frame be defined by

$$\boldsymbol{\theta}^1=dx^1,\boldsymbol{\theta}^2=e^{-x^1}dx^2+e^{x^1}dx^3,\boldsymbol{\theta}^3=e^{-x^1}dx^2-e^{x^1}dx^3.$$

In this paper, we characterize Smarandache $\mathbf{TM}_1$ curves of spacelike biharmonic slant helices according to Bishop frame in the Lorentzian group of rigid motions $\mathrm{E}(1,1)$.

## 2 Smarandache $\mathbf{TM}_1$ Curves of Spacelike Biharmonic B-Slant Helices in the Lorentzian Group of Rigid Motions $\mathrm{E}(1,1)$

Let $\gamma: I\to \mathrm{E}(1,1)$ be a non geodesic spacelike curve on the $\mathrm{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the $\mathrm{E}(1,1)$ along $\gamma$ defined as follows:

$\mathbf{T}$ is the unit vector field $\gamma'$ tangent to $\gamma$, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to $\gamma$), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N},\end{aligned} \tag{2.1}$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ is its torsion and

$$g(\mathbf{T},\mathbf{T})=1,\ g(\mathbf{N},\mathbf{N})=-1,\ g(\mathbf{B},\mathbf{B})=1,$$
$$g(\mathbf{T},\mathbf{N})=g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1-k_2\mathbf{M}_2,\\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= k_1\mathbf{T},\\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= k_2\mathbf{T},\end{aligned} \tag{2.2}$$

where

$$g(\mathbf{T},\mathbf{T})=1,\ g(\mathbf{M}_1,\mathbf{M}_1)=-1,\ g(\mathbf{M}_2,\mathbf{M}_2)=1,$$
$$g(\mathbf{T},\mathbf{M}_1)=g(\mathbf{T},\mathbf{M}_2)=g(\mathbf{M}_1,\mathbf{M}_2)=0.$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_1\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures and $\tau(s)=\psi'(s)$, $\kappa(s)=\sqrt{\left|k_2^2-k_1^2\right|}$. Thus, Bishop curvatures are defined by

$$k_1=\kappa(s)\sinh\psi(s),$$
$$k_2=\kappa(s)\cosh\psi(s).$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned}\mathbf{T} &= T^1\mathbf{e}_1+T^2\mathbf{e}_2+T^3\mathbf{e}_3,\\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1+M_1^2\mathbf{e}_2+M_1^3\mathbf{e}_3,\\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1+M_2^2\mathbf{e}_2+M_2^3\mathbf{e}_3.\end{aligned} \tag{2.3}$$

**Definition 2.1.** *Let $\gamma: I \to \mathrm{E}(1,1)$ be a unit speed regular curve in the Lorentzian group of rigid motions $\mathrm{E}(1,1)$. and $\{\mathbf{T},\mathbf{M_1},\mathbf{M}_2\}$ be its moving Bishop frame. Smarandache $\mathbf{TM}_1$ curves are defined by*

$$\gamma_{\mathbf{TM}_1}=\frac{1}{k_2}(\mathbf{T}+\mathbf{M}_1). \tag{2.4}$$

**Definition 2.2.** **[**7], *A regular spacelike curve* $\gamma : I \to \mathsf{E}(1,1)$ *is called a B-slant helix provided the timelike unit vector* $\mathbf{M}_1$ *of the curve* $\gamma$ *has constant angle* $\theta$ *with some fixed timelike unit vector* $u$, *that is*

$$g(\mathbf{M}_1(s), u) = \cosh \wp \,\text{for all} s \in I. \tag{2.4}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

**Lemma 2.3.** **[**7], *Let* $\gamma : I \to \mathsf{E}(1,1)$ *be a unit speed spacelike curve with non-zero natural curvatures. Then* $\gamma$ *is a slant helix if and only if*

$$\frac{k_1}{k_2} = \tanh \wp. \tag{2.5}$$

**Theorem 2.4.** *Let* $\gamma : I \to \mathsf{E}(1,1)$ *is a non geodesic spacelike biharmonic B-slant helix in the Lorentzian group of rigid motions* $\mathsf{E}(1,1)$. *Then, the parametric equations of Smarandache* $\mathbf{TM}_1$ *curves of spacelike biharmonic slant helix are*

$$\gamma_{\mathbf{TM}_1}(s) = \frac{1}{k_2}(\cosh \wp - \sinh \wp)\mathbf{X}_1 + \frac{1}{k_2}\cos[D_1 s + D_2](\sinh \wp - \cosh \wp)\mathbf{X}_2 \tag{2.6}$$
$$+\frac{1}{k_2}\sin[D_1 s + D_2](\sinh \wp - \cosh \wp)\mathbf{X}_3,$$

*where* $D_1, D_2$ *are constants of integration.*

**Proof.** Assume that $\gamma$ is a non geodesic spacelike biharmonic B-slant helix according to Bishop frame.

From Definition 3.2, we obtain

$$\mathbf{M}_1 = \cosh \wp \mathbf{X}_1 + \sinh \wp \cos[D_1 s + D_2]\mathbf{X}_2 + \sinh \wp \sin[D_1 s + D_2]\mathbf{X}_3. \tag{2.7}$$

Using (1.1) in (2.7), we can choose

$$\mathbf{M}_2 = -\sin[D_1 s + D_2]\mathbf{X}_2 + \cos[D_1 s + D_2]\mathbf{X}_3. \tag{2.8}$$

From above equations we get

$$\mathbf{T} = -\sinh \wp \mathbf{X}_1 - \cosh \wp \cos[D_1 s + D_2]\mathbf{X}_2 - \cosh \wp \sin[D_1 s + D_2]\mathbf{X}_3. \tag{2.9}$$

Substituting (2.7) and (2.9) in (2.4) we have (2.6), which completes the proof.

In terms of Eqs. (1.1) and (2.6), we may give:

**Corollary 2.5.** *Let* $\gamma : I \to \mathrm{E}(1,1)$ *is a non geodesic spacelike biharmonic B-slant helix in the Lorentzian group of rigid motions* $\mathrm{E}(1,1)$. *Then, the parametric equations of Smarandache* $\mathbf{TM}_1$ *curves of spacelike biharmonic slant helix are*

$$x^1_{\mathbf{TM}_1}(s) = \frac{1}{k_2}(\cosh \wp - \sinh \wp),$$

$$x^2_{\mathbf{TM}_1}(s) = \frac{1}{2k_2} e^{\frac{1}{k_2}(\cosh\wp - \sinh\wp)} \cos[D_1 s + D_2](\sinh \wp - \cosh \wp)$$

$$+\frac{1}{2k_2} e^{\frac{1}{k_2}(\cosh\wp - \sinh\wp)} \sin[D_1 s + D_2](\sinh \wp - \cosh \wp), \qquad (2.10)$$

$$x^3_{\mathbf{TM}_1}(s) = \frac{1}{2k_2} e^{-\frac{1}{k_2}(\cosh\wp - \sinh\wp)} \cos[D_1 s + D_2](\sinh \wp - \cosh \wp)$$

$$-\frac{1}{2k_2} e^{-\frac{1}{k_2}(\cosh\wp - \sinh\wp)} \sin[D_1 s + D_2](\sinh \wp - \cosh \wp),$$

*where* $D_1, D_2$ *are constants of integration.*

**Proof.** Substituting (1.1) to (2.6), we have (2.10) as desired.

We may use Mathematica in Corollary 2.5, yields

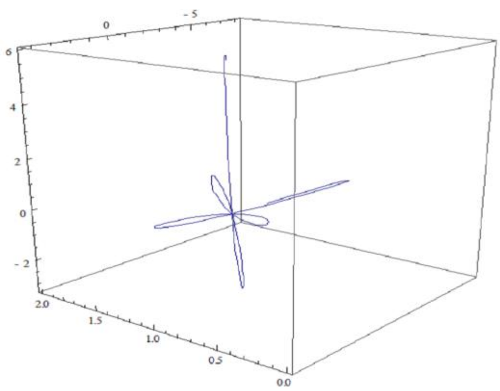

Figure1.

In the light of Lemma 2.3 and Corollary 2.5, we express the following Corollary without proofs:

**Corollary 2.6.** *Let* $\gamma : I \to \mathrm{E}(1,1)$ *is a non geodesic spacelike biharmonic B-slant helix in the Lorentzian group of rigid motions* $\mathrm{E}(1,1)$. *Then, the parametric equations of Smarandache* $\mathbf{TM}_1$ *curves of spacelike biharmonic slant helix are*

$$x^1_{\mathbf{TM}_1}(s) = \frac{1}{k_1}(\sinh\wp - \frac{\sinh^2\wp}{\cosh\wp}),$$

$$x^2_{\mathbf{TM}_1}(s) = \frac{1}{2k_1} e^{\frac{1}{k_1}(\sinh\wp - \frac{\sinh^2\wp}{\cosh\wp})} \cos[D_1 s + D_2](\frac{\sinh^2\wp}{\cosh\wp} - \sinh\wp)$$

$$+\frac{1}{2k_1} e^{\frac{1}{k_1}(\sinh\wp - \frac{\sinh^2\wp}{\cosh\wp})} \sin[D_1 s + D_2](\frac{\sinh^2\wp}{\cosh\wp} - \sinh\wp),$$

$$x^3_{\mathbf{TM}_1}(s) = \frac{1}{2k_1} e^{-\frac{1}{k_1}(\sinh\wp - \frac{\sinh^2\wp}{\cosh\wp})} \cos[D_1 s + D_2](\frac{\sinh^2\wp}{\cosh\wp} - \sinh\wp)$$

$$-\frac{1}{2k_1} e^{-\frac{1}{k_1}(\sinh\wp - \frac{\sinh^2\wp}{\cosh\wp})} \sin[D_1 s + D_2](\frac{\sinh^2\wp}{\cosh\wp} - \sinh\wp),$$

*where* $D_1, D_2$ *are constants of integration.*

Also, we may use Mathematica in Corollary 2.6, yields

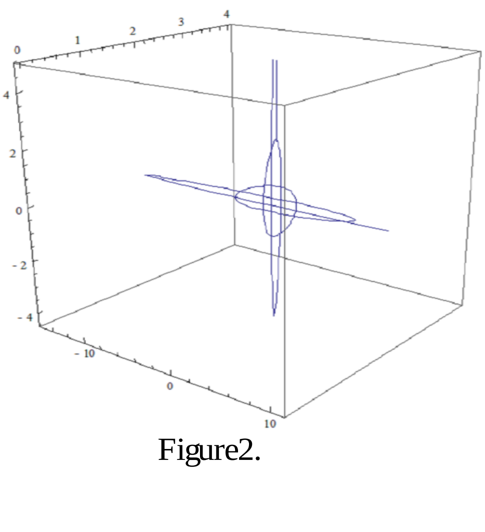


Figure2.

# 3 Open Problem

The authors can be resarch Smarandache $\mathbf{TM}_2$ curves of spacelike biharmonic B-slant helices according to Bishop frame in the Lorentzian group of rigid motions $\mathsf{E}(1,1)$.

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