Int. J. Open Problems Compt. Math., Vol. 5, No. 2, June 2012
ISSN 1998-6262; Copyright ©ICSRS Publication, 2012
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Inclined Curves of Null Curves in $\mathbb{L}^5$ and Their Characterizations

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Abstract

In this paper, we give the necesssary and sufficient conditions to be inclined curves of null curves in the Lorentzian space $\mathbb{L}^5$ and we obtain some characterizations of these curves with respect to Cartan frame by using Harmonic curvature.

MSC: 53B30, 53C50.
Key words: *Null curves, Lorentzian space, Inclined curves*.

1 Introduction

A general helix or an inclined curve in $\mathbb{E}^3$ defined as the tangent lines make a constant angle with a fixed direction called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 184 $[14]$ says that: A curve is a general helix if and only if $\frac{k_1(s)}{k_2(s)}$=constant , where k_1 and k_2 denote the first curvature and the second curvature $(torsion)$, respectively. E. Özdamar and H. H. Hacısalihoğlu extended this consept to the space $\mathbb{E}^n$ obtained the necessary and sufficient conditions, $[11]$.

In $[9]$, A. Magden characterized the inclined curves in $\mathbb{E}^4$ and he obtain that

$$\left(\frac{k_1}{k_2}\right)^2 + \left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]^2$$

is constant along curve, where k_3 is the third curvature.

Recently, in [1], A. T. Ali characterized the inclined curves in $\mathbb{E}^5$ and he obtain that

$$\left(\frac{k_1}{k_2}\right)^2+\left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]^2+\frac{1}{k_4^2}\left[\frac{k_1k_3}{k_2}+\left(\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right)'\right]^2$$

is constant along the curve, where k_4 is the fourth curvature.

Moreover, in the geometry of null curves difficulties arise because the arc-length vanishes, such that it is impossible to normalize the tangent vector in the usual method. A process of developing is to introduce a new paramater called the pseudo-arc which normalizes the derivate of the tangent vector. Several researcher generalize the results of Bonnor, since for a null curve in $\mathbb{L}^n$ they introduce a Frenet frame with the minimum number of functions (which they call the Cartan frame), and then they study the null helices in these spaces, that is, null curves with constant curvature, [7].

A general helix or inclined curve of the null curves in 3-dimensional $\mathbb{L}^3$ is a curve where the velocity vector L make a constant angle with a fixed direction called the axis of the general helix of the null curves, [4].

In this paper, we have characterized the inclined curves of the null curves in Lorentzian 5-space $\mathbb{L}^5$ by using Cartan's frame.

2 Preliminaries

Real $(m+2)$- dimensional vector space $\mathbb{R}^{m+2}$ with the metric tensor

$$\langle X,Y\rangle=\sum_{i=0}^{q-1}x^iy^i+\sum_{a=q}^{m+2}x^ay^a,\forall X,Y\in\mathbb{R}^{m+2}$$

is called semi-Euclidean space and denoted by $\mathbb{R}^{m+2}$, where q is called the index of $\langle,\rangle$, [13].

Let M be a real $(m+2)$-dimensional smooth manifold and $\langle,\rangle$ be a symmetric tensor field of type $(0,2)$ on M. Thus $\langle,\rangle$ assigns smootly, to each point x of M, a symmetric bilinear form $\langle,\rangle_x$ on the tangent space T_xM, where we suppose $\langle,\rangle_x$ is non-degenerate on T_xM and the index of $\langle,\rangle_x$ is the same for all $x\in M$. Then each T_xM becomes an $(m+2)$-dimensional semi-Euclidean space. The tensor field $\langle,\rangle$ satisfying the above conditions is called a semi-Riemannian metric. M endowed with $\langle,\rangle$ is called a semi-Riemannian manifold denoted by $(M,\langle,\rangle)$, [6].

Definition 2.1. A tangent vector u to M is spacelike, if $\langle u,u\rangle>0$ or $u=0$, timelike, if $\langle u,u\rangle<0$, null, if $\langle u,u\rangle=0$ and $u\neq 0$, [5].

Suppose q index of M, that is, q is common value of index of $\langle,\rangle$ for any

$x \in M$. In case $q = 0$ and $q = 1$, $dim\ \ m \geq 2$, $(M, \langle,\rangle)$ is called Riemannian manifold and a Lorentzian manifold, respectively. In case $0 < q < dim\ m$, we say that $(M, \langle,\rangle)$ is a proper semi-Riemannian manifold, [5].

Definition 2.2. The smooth curve α is said to be a null curve if the tangent vector to α at any point is a null vector, [6].

Definition 2.3. Let $\alpha: I \to \mathbb{L}^5$ be a null curves in $\mathbb{L}^5$ and $\{L, N, W_1, W_2, W_3\}$ be the Cartan frame along α, where L, N are null vector fields, W_1, W_2 and W_3 are space-like vector fields. Then there exist only one Cartan frame satisfying the following equations;

$$\begin{bmatrix} L' \\ N' \\ W_1' \\ W_2' \\ W_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_1 & k_2 & 0 \\ -k_1 & -1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 & k_3 \\ 0 & 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} L \\ N \\ W_1 \\ W_2 \\ W_3 \end{bmatrix}, \qquad (2.1)$$

where the functions $\{k_1, k_2, k_3\}$ are called the Cartan curvatures of α [7].

3 Inclined Curves of Null Curves in the $\mathbb{L}^5$ and Their Characterizations

In this section we give characterizations of inclined curves of null curves in $\mathbb{L}^5$ by using harmonic curvature functions of curve.

Definition 3.1. A null curve α in 5-dimensional Lorentzian space is called inclined curves if there exist a constant vector field $U \neq 0$ such that $\langle L, U\rangle = \lambda \neq 0$ is constant.

Theorem 3.1. Let $\alpha: I \to \mathbb{L}^5$ be a null curve in $\mathbb{L}^5$. Let the functions $\{G_1^*, G_2^*, G_3^*, G_4^*\}$ are given by

$$G_1^* = -k_1,\ G_2^* = 1,\ G_3^* = 0,\ \ G_4^* = -\frac{k_1'}{k_2} G_2^*,\ \ G_5^* = \frac{1}{k_3}\left(G_4^{*'} + k_2 G_2^*\right). \quad (3.1)$$

Then α is an inclined curves if and only if the function

$$2G_1^* + (G_4^*)^2 + (G_5^*)^2 = C$$

is constant. Furtheremore there exist a angle θ that makes L with the fixed direction U that determinens α.

Proof. Let α be a null curve in $\mathbb{L}^5$. Suppose that, α is an inclined curve. Let

U be fixed direction which makes a constant agle θ with L, (Assume that $\langle U, U \rangle = 1$). Consider differentiable functions $a_i,\ 1 \leq i \leq 5$,

$$U = a_1(s)L(s) + a_2(s)N(s) + a_3(s)W_1(s) + a_4(s)W_2(s) + a_5(s)W_3(s), s \in I, \quad (3.2)$$

here, if we consider $\langle L, N \rangle = \langle N, L \rangle = 1$, we can obtain

$$a_1 = \langle N, U \rangle, a_2 = \langle L, U \rangle, a_3 = \langle W_1, U \rangle, a_4 = \langle W_2, U \rangle, a_5 = \langle W_3, U \rangle.$$

Then the function $a_2(s) = \langle L, U \rangle$ is constant and it agrees with $cos\theta$, that is,

$$a_2(s) = \langle L, U \rangle = \cos\theta = const. \quad (3.3)$$

along the null curve. By the (3.3) with respect to s the using the Cartan formula (2.1), we have $a_2^{'}(s) = 0$.

Then $a_3 = 0$ and therefore U is in the subspcase $Sp\{L, N, W_2, W_3\}$. Because the vector field U is constant, a differention in (3.2) together (2.1) gives the following ordinary differential equation system,

$$\left.\begin{aligned} a_1^{'} - k_2 a_4 &= 0\,, \\ k_1 a_2 + a_1 &= 0\,, \\ a_4^{'} + k_2 a_2 - k_3 a_5 &= 0\,, \\ a_5^{'} + k_3 a_4 &= 0\,. \end{aligned}\right\} \quad (3.4)$$

Let use define the functions $G_i^* = G_i^*(s)$ as follows

$$a_i(s) = G_i^*(s) a_2(s).$$

We point out that $a_2 \neq 0$ because $G_i^*(s) = \frac{a_i(s)}{a_2(s)}$. Thus, the equation in (3.4) lead to,.

$$\left.\begin{aligned} G_1^* &= -k_1\,, \\ G_2^* &= 1\,, \\ G_3^* &= 0\,, \\ G_4^{*} &= -\frac{k_1^{'}}{k_2} G_2^*\,, \\ G_5^{*} &= \frac{1}{k_3}\left(G_4^{*'} + k_2 G_2^*\right)\,. \end{aligned}\right\} \quad (3.5)$$

We do the change of variables,

$$t = \int_0^s k_3(u)du\,, \frac{dt}{ds} = k_3(s),$$

from equation (3.4), we have

$$G_4^{*'}(t) = -k_2(t) + k_3(t) G_5^{*}.$$

As a consequence, if α is an inclined curve, the last equation of (3.5) yields,

$$G_5^{*''}(t) + G_5^{*} = \frac{k_2(t)}{k_3(t)}. \quad (3.6)$$

The general solutions of this equation is

$$G_5^{*}(t) = \left[\left(A - \int \frac{k_2(t)}{k_3(t)} \sin t dt\right) \cos t + \left(B + \int \frac{k_2(t)}{k_3(t)} \cos t dt\right) \sin t\right] a_2, \quad (3.7)$$

where A and B are arbitrary constants. Using equation (3.7) takes following form

$$G_5^*(s) = \begin{bmatrix} (A - \int [k_2(s)\sin\int k_3(s)ds]ds)\cos\int k_3(s)ds \\ +(B + \int [k_2(s)\cos\int k_3(s)ds]ds)\sin\int k_3(s)ds \end{bmatrix}. \tag{3.8}$$

From (3.4), the function G_4^* is given by,

$$G_4^*(s) = \begin{bmatrix} (A - \int [k_2(s)\sin\int k_3(s)ds]ds)\sin\int k_3(s)ds \\ -(B + \int [k_2(s)\cos\int k_3(s)ds]ds)\cos\int k_3(s)ds \end{bmatrix}. \tag{3.9}$$

Now, equation (3.9) with the latter two equations (3.5) can be obtained the following condition.

$$-\frac{k_1'}{k_2} = (A - \int [k_2(s)\sin\int k_3(s)ds]ds)\sin\int k_3(s)ds \tag{3.10}$$
$$-\left(B + \int \left[k_2(s)\cos\int k_3(s)ds\right]ds\right)\cos\int k_3(s)ds.$$

$$\frac{1}{k_3}\left(\left(-\frac{k_1'}{k_2}\right)' + k_2\right) = (A - \int [k_2(s)\sin\int k_3(s)ds]ds)\cos\int k_3(s)\mathrm{d}s \tag{3.11}$$
$$+\left(B + \int \left[k_2(s)\cos\int k_3(s)ds\right]ds\right)\sin\int k_3(s)ds.$$

If we integrate the above equation we have,

$$2G_1^* = C - (A - \int [k_2(s)\sin\int k_3(s)ds]ds)^2$$
$$-(B + \int [k_2(s)\cos\int k_3(s)ds]ds)^2 \quad , \tag{3.12}$$

where C is a constant of integration. From equations (3.8), (3.9), (3.12) we get

$$2G_1^* + {G_4^*}^2 + {G_5^*}^2 = C, \tag{3.13}$$

where using (2.1) and the fact that U is a unit vector field.

Conversely, suppose that the condition (3.14) is satisfied for a curve α. Define the unit vector U by

$$U = \cos\theta\left[G_1^* L + N + \textstyle\sum_{\mathrm{i}=2}^{3} G_{i+2}^* W_i\right]. \tag{3.14}$$

Considering equation (3.13) and a differentation of U gives that $\frac{dU}{ds} = 0$, which it means that U is a constant vector. On the other hand, the scalar product between the null vector L and U is

$$\langle L, U\rangle = const.$$

Then α is an inclined curve.

Theorem 3.2. Let $\alpha: I \to \mathbb{L}^5$ be a null curve in Lorentzian space $\mathbb{L}^5$. Hence if α is an inclined curve, the following condition is satisfied,

$$k_3 g(s) = \left({G_4^*}' + k_2\right), \frac{1}{k_3}\frac{d}{ds}g(s) = -G_4^*(s), \tag{3.15}$$

where g is a C^2 function.

Proof. Assume that α is an inclined curve, a differention of (3.13) gives

$$G_1^{*\prime} + G_4^* G_4^{*\prime} + G_5^* G_5^{*\prime} = 0. \tag{3.16}$$

After some computations the equation (3.16) takes the following form.

$$k_3 G_1^{*\prime} + k_2 G_5^{*\prime} = 0. \tag{3.17}$$

If we write $g = g(s)$ as follows

$$g(s) = G_5^*(s).$$

Therefore equations (3.17) writes as

$$g'(s) = -k_3 G_4^*.$$

Conversely, if (3.15) holds, we define a unit constant vector U by

$$U = \cos\theta\left[G_1^* L + N + G_4^* W_2 + g(s) W_3\right].$$

We have that $\langle L, U\rangle = \cos\theta$, is a constant, therefore a null curve α is inclined curve with respect to the Cartan frame.

Theorem 3.3. Let $\alpha: I \to \mathbb{L}^5$ be a null curve in Lorentzian space $\mathbb{L}^5$. Then a null curve α is an inclined curve with respect to the Cartan frame if and only if the following condition is satisfied

$$\begin{aligned} G_4^* = {} & (A - \int [k_2(s)\sin\int k_3(s)ds]ds)\sin\int k_3(s)ds \\ & -(B + \int [k_2(s)\cos\int k_3(s)ds]ds)\cos\int k_3(s)ds. \end{aligned} \tag{3.18}$$

for some constants A and B.

Proof Let now space that α is inclined curve. From Theorem 3.2, let $m(s)$ and $n(s)$ are defined by

$$\phi = \phi(s) = \int^{s} k_3(u)du \tag{3.19}$$

$$m(s) = g(s)\cos\phi + G_4^*\sin\phi + \int (k_2\sin\phi)ds, \tag{3.20}$$

$$n(s) = g(s)\sin\phi - G_4^*\cos\phi - \int (k_2\cos\phi)ds. \tag{3.21}$$

From differantiating (3.20) and (3.21) with respect to s and taking into account of (3.19) and (3.15), we obtain $\frac{dm}{ds} = 0$ and $\frac{dn}{ds} = 0$. Thus, there exist constants A and B such that $m(s) = A$, $n(s) = B$. Substituting this expression into (3.20) and solving the resulting equations for G_4^*, we get

$$G_4^* = (A + \int (k_2\sin\phi)ds)\sin\phi - (B + \int (k_2\cos\phi)ds)\cos\phi.$$

Conversely, assume that (3.18) holds. From a direct differentiation of (3.18) gives

$$G_4^{*\prime} = k_3 g(s) - k_2.$$

Furthermore computation leads to

$$g'(s) = -k_3 G_4^*.$$

4 Conclusion

In this study given by a null curve is an inclined curves if and only if

$$2G_1^* + (G_4^*)^2 + (G_5^*)^2 = C = \text{constant}.$$

5 Open Problem

In this work we give characterizations of inclined curves of null curves in $\mathbb{L}^5$ by using harmonic curvature functions of the null curves. It is hoped that generalization of the inclined curves of null curves in $\mathbb{L}^{m+2}$ by using harmonic curvature functions of null curve is given in the next work.

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