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Diffusion Processes With Generalized Beta Density Functions

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Abstract

We consider two time-inhomogeneous (one-dimensional) diffusion processes having a particular generalized beta density function. The initial state of the processes also has a generalized beta distribution. We show that the classical beta distribution is stationary for two transformations of the original diffusion processes. Finally, we find stationary density functions for other diffusion processes.

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1 Introduction

The main diffusion processes, such as the Wiener and Ornstein-Uhlenbeck processes, are Gaussian processes, so that they can take on any value in $\mathbb{R}$. Similarly, the geometric Brownian motion, which is very important in mathematical finance, takes its values in an infinite interval, namely $(0, \infty)$.

Suppose that we want to find a model for a certain random variable $X(t)$, which varies with t, but which remains in a bounded domain for a fixed value of t. For instance, suppose that $X(t)$ always belongs to the interval $(0, 1)$. We could of course consider a diffusion process like the Wiener process (or Brownian motion), if we assume that the boundaries at 0 and 1 are reflecting. Then, the process $\{X(t), t \geq 0\}$ will indeed evolve between 0 and 1. However, it would be nice to have a diffusion process for which its first-order density function

$$f(x;t) := \lim_{dx \downarrow 0} \frac{P[X(t) \in (x, x+dx)]}{dx}$$

is that of a variable that is intrinsically bounded, rather than being a truncated Gaussian random variable, for example. One such distribution is the beta distribution. Remember that we say that the random variable X has a beta distribution with parameters α and β (both positive) if

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \quad \text{for } 0 < x < 1, \tag{1}$$

where

$$B(\alpha,\beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(and $\Gamma(\cdot)$ denotes the gamma function). Moreover, if $a < b$ and if we let

$$Y := a + (b-a)X,$$

then Y has a *generalized* beta distribution on (a, b):

$$f_Y(y) = \frac{1}{(b-a)^{\alpha+\beta-1}} \frac{(y-a)^{\alpha-1}(b-y)^{\beta-1}}{B(\alpha,\beta)} \quad \text{for } a < y < b.$$

In the next section, we will find two time-inhomogeneous diffusion processes $\{X(t), t \geq 0\}$ for which $X(t)$ has a (t-dependent) generalized beta distribution with parameters $\alpha = \beta = 2$. Next, we will see that, in each case, a simple transformation $Y(t)$ of $X(t)$ leads to a diffusion process having a stationary beta distribution with parameters $\alpha = \beta = 2$, if we assume that $Y(0)$ also has this distribution. Then, in Section 3, we will find stationary density functions for the Wiener and Ornstein-Uhlenbeck processes, as well as for the geometric Brownian motion.

2 Processes having generalized beta density functions

The first time-inhomogeneous diffusion process that we consider is defined by its infinitesimal mean

$$m(x;t) = \frac{x}{1+t} + \frac{1}{x-(1+t)} \tag{2}$$

and infinitesimal variance

$$v(x;t) \equiv 1. \tag{3}$$

The density function of $X(t)$ satisfies the Kolmogorov forward equation (see Lefebvre (2007, p. 64), for instance)

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left[m(x;t)f(x;t)\right] - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[v(x;t)f(x;t)\right] = 0. \tag{4}$$

The boundaries at $x = 0$ and $x = t + 1$ are assumed to be reflecting. Hence, this equation is considered in the interval $(0, t + 1)$.

Remark. Notice that the functions m and v are continuous for all $t \geq 0$ and $x \in (0, t + 1)$.

One can check the validity of the following proposition.

Proposition 2.1 *The function $f(x;t)$ that satisfies Eq. (4), subject to the initial condition*

$$f(x;0) = \frac{x(1-x)}{B(2,2)} = 6x(1-x) \quad \textit{for } 0 < x < 1, \tag{5}$$

is

$$f(x;t) = \frac{6}{(1+t)^3}\, x(1+t-x) \quad \textit{for } 0 < x < t+1.$$

Remark. The proposition means that if $X(0)$ has a beta distribution with parameters $\alpha = \beta = 2$, then we find that $X(t)$ has a generalized beta distribution (also with parameters $\alpha = \beta = 2$), such that $a = 0$ and $b = t + 1$.

Next, assume that

$$m(x;t) = \frac{x}{x-(1+t)} \quad \text{and} \quad v(x;t) = 1 + t \tag{6}$$

for $t \geq 0$ and $x \in (t, t + 1)$ (that is, the boundaries at $x = t$ and $x = t + 1$ are reflecting). As above, the functions m and v are continuous in the intervals considered. We must solve the partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left[\frac{x}{x-(1+t)}\, f(x;t)\right] - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[(1+t)f(x;t)\right] = 0. \tag{7}$$

Proposition 2.2 *The function*

$$f(x;t) = 6(x-t)(1+t-x) \quad \textit{for } t < x < t+1$$

solves Eq. (7), subject to the initial condition (5). Thus, $X(t)$ has a generalized beta distribution with parameters $\alpha = \beta = 2$, and $a = t$ and $b = t + 1$.

The two diffusion processes considered above take on their values in a bounded interval, which depends on t. In the case of the process with infinitesimal parameters (2) and (3), let

$$Y(t) := \frac{X(t)}{t+1} \quad \forall\, t \geq 0.$$

The stochastic process $\{Y(t), t \geq 0\}$ will always remain in the interval $(0,1)$. Moreover, we calculate

$$f(y;t) = 6y(1-y) \quad \text{for } 0 < y < 1,$$

so that $Y(t)$ has a beta distribution for all $t \geq 0$. Hence, the probability density function $f(y;t)$ is stationary for the stochastic process $\{Y(t), t \geq 0\}$.

Let us compute the infinitesimal parameters of $\{Y(t), t \geq 0\}$. We have [see Lefebvre (2007, p. 181)]:

$$\begin{aligned} m(y;t) &:= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} E\left[Y(t+\epsilon) - Y(t) \mid Y(t) = y\right] \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} E\left[\frac{X(t+\epsilon)}{1+t+\epsilon} - \frac{X(t)}{1+t} \,\middle|\, X(t) = (1+t)y\right]. \end{aligned}$$

We find, after some calculation, that

$$m(y;t) = \frac{1}{(1+t)^2}\left(\frac{1}{y-1}\right). \tag{8}$$

Next, we may write that

$$\begin{aligned} v(y;t) &:= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} E\left[\{Y(t+\epsilon) - Y(t)\}^2 \mid Y(t) = y\right] \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} E\left[\left\{\frac{X(t+\epsilon)}{1+t+\epsilon} - \frac{X(t)}{1+t}\right\}^2 \,\middle|\, X(t) = (1+t)y\right]. \end{aligned}$$

We obtain that

$$v(y;t) = \frac{1}{(1+t)^2}. \tag{9}$$

We can now state the following proposition.

Proposition 2.3 *Let $\{Y(t), t \geq 0\}$ be the diffusion process having infinitesimal mean (8) and variance (9). If the initial state $Y(0)$ has a beta distribution with parameters $\alpha = \beta = 2$, and if the boundaries at 0 and 1 are reflecting, then so has $Y(t)$ for all $t > 0$.*

In the case of the diffusion process $\{X(t), t \geq 0\}$ having the infinitesimal parameters given in (6), we define

$$Z(t) = X(t) - t \quad \text{for } t \geq 0.$$

Like $Y(t)$, the random variable $Z(t)$ takes on its values in the interval $(0,1)$. Furthermore, we also obtain that

$$f(z;t) = 6z(1-z) \quad \text{for } 0 < z < 1.$$

Proceeding as above, we can calculate the infinitesimal parameters of $\{Z(t), t \geq 0\}$.

Proposition 2.4 *Suppose that the diffusion process $\{Z(t), t \geq 0\}$ has infinitesimal parameters*

$$m(z;t) = \frac{t+1}{z-1} \quad \text{and} \quad v(z;t) = 1+t.$$

Then, given that $Z(0)$ has a beta distribution with parameters $\alpha = \beta = 2$ and that the boundaries at 0 and 1 are reflecting, $Z(t)$ has the same beta distribution $\forall\ t > 0$.

In the next section, we will consider the problem of obtaining stationary density functions for important diffusion processes.

3 Stationary probability density functions

If $m(x;t) = m(x)$ and $v(x;t) = v(x)$, and the function $f(x;t)$ can be written as

$$f(x;t) = f(x) \quad \forall\ t \geq 0,$$

then the Kolmogorov forward equation (4) reduces to

$$\frac{d}{dx}[m(x)f(x)] - \frac{1}{2}\frac{d^2}{dx^2}[v(x)f(x)] = 0,$$

which implies that

$$m(x)f(x) - \frac{1}{2}\frac{d}{dx}[v(x)f(x)] = c, \tag{10}$$

where c is a constant. We will find interesting solutions to this ordinary differential equation (o.d.e.) for important diffusion processes.

I) Assume that $m(x) \equiv 0$ and $v(x) \equiv 1$, so that $\{X(t), t \geq 0\}$ is a standard Brownian motion. It follows that we have:

$$f'(x) = c \quad \Longrightarrow \quad f(x) = c_1 x + c_0,$$

where c_0 and c_1 are constants. Let us choose $c_0 = 0$ and $c_1 = 2$. Then, $f(x)$ is a valid density function on the interval $(0,1)$.

Proposition 3.1 *If $\{X(t),\, t \geq 0\}$ is a standard Brownian motion, considered between reflecting boundaries at $x = 0$ and $x = 1$, for which $X(0)$ is a random variable such that*

$$f_{X(0)}(x) = 2x \quad \textit{for } 0 < x < 1,$$

then this probability density function (p.d.f.) is also the p.d.f. of $X(t)$ for $t > 0$.

Remarks. i) This result is also valid if $v(x) \equiv v_0 > 0$.
ii) By choosing $c_1 = 0$ and $c_0 = 1$ instead, we obtain that $X(t)$ has a uniform distribution on the interval $(0, 1)$ for all $t \geq 0$.

II) When $m(x) \equiv m_0 \neq 0$ and $v(x) \equiv v_0 > 0$, the diffusion process $\{X(t), t \geq 0\}$ is a Wiener process with drift m_0 and diffusion coefficient v_0. We must solve

$$m_0 f(x) - \frac{v_0}{2} f'(x) = c.$$

We find at once that

$$f(x) = \frac{c}{m_0} + c_1 \exp\left\{\frac{2m_0 x}{v_0}\right\}.$$

As in the previous case, we see that by choosing $c_1 = 0$ and $c = m_0$, we get a uniform distribution on the interval $(0, 1)$ for all random variables $X(t)$. If we assume that $m_0 < 0$, and if we take $c = 0$ and $c_1 = -2m_0/v_0$, we obtain the following proposition.

Proposition 3.2 *When $\{X(t), t \geq 0\}$ is a Wiener process with infinitesimal parameters $m_0 < 0$ and $v_0 > 0$ and there is a reflecting boundary at $x = 0$, if $X(0) \sim Exp(-2m_0/v_0)$, that is, if $X(0)$ has an exponential distribution with parameter $-2m_0/v_0$, then $X(t) \sim Exp(-2m_0/v_0)$ for any $t > 0$ as well.*

III) Next, let us consider the Ornstein-Uhlenbeck process. That is, $\{X(t), t \geq 0\}$ is the diffusion process for which $m(x) = -\alpha x$, where $\alpha > 0$, and $v(x) \equiv v_0 > 0$. The o.d.e. that we must solve is

$$-\alpha x f(x) - \frac{v_0}{2} f'(x) = c.$$

The general solution to this differential equation involves the error function and does not lead to a classical distribution for $X(t)$, apart from the Gaussian distribution. Actually, it is well known that the distribution of the stationary Ornstein-Uhlenbeck process is Gaussian.

Proposition 3.3 *Suppose that $X(0)$ has the probability density function*

$$f_{X(0)}(x) = \frac{\sqrt{\alpha}}{\sqrt{\pi v_0}} \exp\left\{-\frac{\alpha}{v_0} x^2\right\} \quad \textit{for } x \in \mathbb{R},$$

where $X(0)$ is the initial state of the Ornstein-Uhlenbeck process $\{X(t), t \geq 0\}$ having infinitesimal mean $-\alpha x$ and infinitesimal variance v_0. Then,

$$f_{X(t)}(x) = f_{X(0)}(x) \quad \textit{for all } t > 0.$$

Remark. The function $f_{X(0)}(x)$ is the p.d.f. of a Gaussian random variable with mean $\mu = 0$ and variance $\sigma^2 = v_0/(2\alpha)$. By choosing a different constant in front of the exponential function, the random variable $X(t)$ could have a Gaussian distribution, conditioned to be positive or to remain in the interval $(0, 1)$ (if we assume that the boundary at $x = 0$ (or the boundaries at $x = 0$ and $x = 1$) is (are) reflecting), etc.

IV) Finally, if $\{X(t), t \geq 0\}$ is a geometric Brownian motion, we can write that

$$m(x) = \alpha x \quad \text{and} \quad v(x) = x^2, \tag{11}$$

where $\alpha \in \mathbb{R}$.

Remark. Actually, $m(x) = (\mu + \frac{1}{2}\sigma^2)x$ and $v(x) = \sigma^2 x^2$ for a geometric Brownian motion, where $\mu \in \mathbb{R}$ and $\sigma > 0$. However, to solve the differential equation (10), we may assume, without loss of generality, that $\sigma = 1$ and $\alpha = \mu + \frac{1}{2}$.

The general solution of

$$\alpha x f(x) - \frac{d}{dx}\left[x^2 f(x)\right] = c$$

is

$$f(x) = \begin{cases} \dfrac{2c}{x(2\alpha - 1)} + c_0 x^{2(\alpha-1)} & \text{if } \alpha \neq 1/2, \\[2ex] \dfrac{-2c\ln(x) + c_0}{x} & \text{if } \alpha = 1/2. \end{cases}$$

Proposition 3.4 *Consider a geometric Brownian motion for which α in (11) is greater than 1/2. If $X(0)$ has the following probability density function:*

$$f_{X(0)}(x) = (2\alpha - 1)x^{2(\alpha-1)} \quad \textit{for } 0 < x < 1,$$

then $X(t)$ has the same p.d.f. as $X(0)$ for all $t > 0$ (assuming that the boundary at $x = 1$ is reflecting).

Remarks. i) Remember that the origin is a *natural* boundary for the geometric Brownian motion.

ii) With $\alpha = 3/2$, we obtain that $f_{X(0)}(x) = 2x$, for $0 < x < 1$, as in the case of the standard Brownian motion.

To conclude, it is worth mentioning yet another interesting case. Consider the time-homogeneous diffusion process $\{X(t), t \geq 0\}$ characterized by

$$m(x) = \frac{\alpha}{2}(1-x) + \frac{\beta}{2}x \quad \text{and} \quad v(x) = x(1-x)$$

for $0 < x < 1$, where α and β are positive constants. This process has applications in population genetics [see Tavaré and Zeitouni (2004), and also Karlin and Taylor (1981, section 15.2)].

We find that as t tends to infinity, the p.d.f. $f(x;t)$ of $X(t)$ tends to that of a random variable having a beta distribution with parameters α and β. Therefore, if we assume that $X(0)$ also has a beta distribution with parameters α and β, then we can assert that the p.d.f. defined in (1) is stationary for the diffusion process considered. However, we cannot state that $X(t)$ has a generalized beta distribution for any finite (and positive) t.

4 Conclusion and Open Problems

In this work, we have shown that two particular time-inhomogeneous diffusion processes, considered between two reflecting boundaries, have a generalized beta distribution (with parameters $\alpha = \beta = 2$) which depends on t. Next, by considering simple transformations of the original processes, we have obtained two diffusion processes for which the beta density function (with the same parameters) is stationary. Finally, in Section 3, we have found various interesting stationary density functions for important diffusion processes in (generally) bounded intervals.

This type of work could be carried out in two or more dimensions. In Section 3, other one-dimensional diffusion processes, such as the Bessel process, could also be considered. Finally, it would be nice to find a diffusion process having a t-dependent generalized beta distribution for which $X(0)$ is deterministic; that is, $f_{X(0)}(x)$ would be a Dirac delta function.

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