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A study on Some Properties of Legendre Polynomials and Integral Transforms

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Abstract

The main aim of this paper is to use the integral representation method to derive the properties of Legendre polynomials. Their recurrence relations, differential equations, relationship with Hermite polynomials have been obtained.

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1 Introduction

Orthogonal polynomials is an emergent field whose development has reached important results from both the theoretical and practical points of view. This paper deals with the introduction and the study of Legendre polynomials taking advantage of the Hermite polynomials recently treated in [9]. The Legendre polynomials are introduced and their connections with differential equations are established. The Legendre polynomials of the second kind are defined by the series

$$P_n(x) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \tag{1}$$

can be slightly generalized as

$$P_n(x,y)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}y^k x^{n-2k} \tag{2}$$

in addition, that the Hermite polynomials are generated by

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,y)=\exp\left(2xt-yt^2\right) \tag{3}$$

where

$$H_n(x,y)=n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k}{k!(n-2k)!}y^k(2x)^{n-2k} \tag{4}$$

by means of the integral transform

$$P_n(x,y)=\frac{2}{n!\sqrt{\pi}}\int_0^{\infty}\exp(-t^2)t^nH_n(xt,y)dt. \tag{5}$$

It is clear that

$$P_{-1}(x,y)=0,\quad P_0(x,y)=1,\quad P_1(x,y)=x \ \text{ and } \ P_n(-x,y)=(-1)^nP_n(x,y).$$

It has already been shown that most of the properties of the $P_n(x,y)$ polynomials, linked to the ordinary case by

$$P_n(x,y)=y^{\frac{n}{2}}P_n(\frac{x}{\sqrt{y}}) \ \text{ and } \ P_n(x,1)=P_n(x) \tag{6}$$

can be directly inferred from those of the Hermite polynomials and from the integral representation given in (5).

By recalling that the $H_n(x,y)$ satisfies the differential equation of the second order [9]

$$\left[y\frac{\partial^2}{\partial x^2}-2x\frac{\partial}{\partial x}+2n\right]H_n(x,y)=0 \tag{7}$$

and the recurrences properties can be derived either from (3) or (4). It is indeed easy to prove that

$$\begin{aligned}
&\frac{\partial}{\partial x}H_n(x,y)=2nH_{n-1}(x,y),\\
&\frac{\partial}{\partial y}H_n(x,y)=-n(n-1)H_{n-2}(x,y),\\
&H_{n+1}(x,y)=\left[2x-y\frac{\partial}{\partial x}\right]H_n(x,y),\\
&H_{n+1}(x,y)=2xH_n(x,y)-2nyH_{n-1}(x,y)
\end{aligned} \tag{8}$$

this once combined yield

$$\frac{\partial^2}{\partial x^2}H_n(x,y) + 4\frac{\partial}{\partial y}H_n(x,y) = 0 \tag{9}$$

and, in general, we get

$$\begin{aligned}
\frac{\partial^r}{\partial x^r}H_n(x,y) &= \frac{2^r n!}{(n-r)!}H_{n-r}(x,y), 0 \leq r \leq n \\
\frac{\partial^r}{\partial y^r}H_n(x,y) &= \frac{(-1)^r n!}{(n-2r)!}H_{n-2r}(x,y), 0 \leq r \leq [\frac{1}{2}n].
\end{aligned} \tag{10}$$

According (9), it is clear that the $H_n(x,y)$ is the natural solution of the heat partial differential equation.

We find the recurrences properties from (8) and by using the integral transform (5)

$$\begin{aligned}
&y\frac{\partial}{\partial x}P_{n-1}(x,y) = \big(x\frac{\partial}{\partial x} - n\big)P_n(x,y), \\
&(2n+1)P_n(x,y) = \frac{\partial}{\partial x}P_{n+1}(x,y) - y\frac{\partial}{\partial x}P_{n-1}(x,y), \\
&x\frac{\partial}{\partial x}P_n(x,y) = \frac{\partial}{\partial x}P_{n+1}(x,y) - (n+1)P_n(x,y), \\
&\left(x^2 - y\right)\frac{\partial}{\partial x}P_n(x,y) = nxP_n(x,y) - nyP_{n-1}(x,y), \\
&(n+1)P_{n+1}(x,y) = (2n+1)xP_n(x,y) - nyP_{n-1}(x,y).
\end{aligned} \tag{11}$$

In the following section, we will derive further consequences from the above relations and show how the method of the integral transform may be useful for the study of the properties of generalized forms of Legendre polynomials.

2 Operational identities for Legendre polynomials

The recurrences given in (11) can be exploited to define rising and lowering operators for Legendre polynomials, indeed introducing the operator $\hat{D}_x^{-1}$ denoting a kind of inverse derivative, we can write

$$\begin{aligned}
P_{n-1}(x,y) &= \frac{1}{y}\hat{D}_x^{-1}\left[x\frac{\partial}{\partial x} - n\right]P_n(x,y), \\
P_{n+1}(x,y) &= \frac{1}{n+1}\left[(2n+1)x - n\hat{D}_x^{-1}\left(x\frac{\partial}{\partial x} - n\right)\right]P_n(x,y).
\end{aligned} \tag{12}$$

These last relations allow the introduction of the rising and lowering operators

$$\begin{aligned}
\hat{M}_- &= \frac{1}{y}\hat{D}_x^{-1}\left[x\frac{\partial}{\partial x} - \hat{n}\right], \\
\hat{M}_+ &= \frac{1}{n+1}\left[(2n+1)x - n\hat{D}_x^{-1}\left(x\frac{\partial}{\partial x} - \hat{n}\right)\right].
\end{aligned} \tag{13}$$

where $\hat{n}$ is a number operator in the sense $\hat{n}P_s(x,y) = sP_s(x,y)$, which act on $P_n(x,y)$ according to the rules

$$\begin{aligned} \hat{M}_- P_n(x,y) &= P_{n-1}(x,y), \\ \hat{M}_+ P_n(x,y) &= P_{n+1}(x,y). \end{aligned} \tag{14}$$

Equation (14) can be exploited to derive the differential equation satisfied by $P_n(x,y)$, namely

$$\hat{M}_+ \hat{M}_- P_n(x,y) = P_n(x,y) \tag{15}$$

namely

$$\left[\frac{1}{y(n+1)} \hat{D}_x^{-1} \big[x\frac{\partial}{\partial x} - \hat{n}\big] \big[(2n+1)x - n\hat{D}_x^{-1}\big(x\frac{\partial}{\partial x} - \hat{n}\big)\big]\right] P_n(x,y) = P_n(x,y). \tag{16}$$

which after some algebra and the use of the obvious identity

$$\partial_x \hat{D}_x^{-1} = \hat{1} \tag{17}$$

yields the differential equation of second order in the form

$$\left[(y - x^2)\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + n(n+1)\right] P_n(x,y) = 0. \tag{18}$$

In the forthcoming section, we will touch on the problem of considering the whole family of Legendre polynomials, using the integral representation method and study their properties in terms of those of Hermite polynomials.

3 Connections between Legendre with Hermite polynomials

We will try to understand more deeply the role played by the above integral transform connecting Legendre polynomials and Hermite polynomials and whether it can be understood in terms of ordinary integral transforms.

By noting that both $P_n(x,y)$ and $H_n(x,y)$ are reduced to ordinary forms for

$$H_n(x, \frac{1}{t}) = t^{-\frac{n}{2}} H_n(x\sqrt{t}).$$

We find that

$$P_n(x) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty \exp(-t^2) t^n H_n(xt) dt. \tag{19}$$

After a suitable change of variable, the equation (3.1) yields (Putting $u = xt$)

$$P_n(x) = \frac{2}{n!x^{n+1}\sqrt{\pi}} \int_0^\infty \exp\Big(-(\frac{u}{x})^2\Big) u^{n+1} H_n(u) du \tag{20}$$

which ensures that the identity (19) can be viewed as a kind of Mellin transform in [6], associated with the function

$$F(\xi) = \exp\Big(-(\frac{\xi}{x})^2\Big) H_n(\xi). \tag{21}$$

Let us now consider the problem from an operational point of view. By recalling that the so called dilatation operator acts on a generic function $f(x)$ as

$$\exp\big(\lambda x \frac{\partial}{\partial x}\big) f(x) = f\Big(x \exp(\lambda)\Big). \tag{22}$$

Since $xt = xe^{\ln t}$ it follows that

$$\begin{aligned} f(xt) = f(xe^{\ln t}) = e^{\ln t\,(x\frac{\partial}{\partial x})} f(x) = t^{x\frac{\partial}{\partial x}} f(x), \\ H_n(xt) = H_n(xe^{\ln t}) = e^{\ln t\,(x\frac{\partial}{\partial x})} H_n(x) = t^{x\frac{\partial}{\partial x}} H_n(x). \end{aligned} \tag{23}$$

We obtain from (19)

$$P_n(x) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty \exp\big(-t^2\big) t^{\frac{n}{2}} e^{\Big[\ln t\,(x\frac{\partial}{\partial x})\Big]} H_n(x) dt \tag{24}$$

or

$$\begin{aligned} P_n(x) &= \frac{2}{n!\sqrt{\pi}} \int_0^\infty \exp\big(-t^2\big) t^n t^{x\frac{\partial}{\partial x}} H_n(x) dt \\ &= \frac{2}{n!\sqrt{\pi}} \int_0^\infty \exp\big(-t^2\big) t^{(n+x\frac{\partial}{\partial x})} H_n(x) dt. \end{aligned} \tag{25}$$

Using the well known definition of the Gamma function

$$\Gamma(\mu + \frac{1}{2}) = 2\int_0^\infty \exp(-t^2) t^{2\mu} dt. \tag{26}$$

Using (26), we can recast (27) in the operational form

$$P_n(x) = \frac{\Gamma(\frac{1}{2}(n+1+x\frac{\partial}{\partial x}))}{n!\sqrt{\pi}} H_n(x). \tag{27}$$

In this paper we have shown that the use of integral transforms and operational methods is a fairly useful tool to deal with old and new families of polynomials.

Further applications will be discussed in a forthcoming paper.

4 Open problem

One can use the same class of integral representation for some other polynomials of two variables. Hence, new results and further applications can be obtained.

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