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Polar Bergman Polynomials on Domain with Corners

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Abstract

In this paper we present a new class named polar of monic orthogonal polynomials with respect to the area measure supported on G where G is a bounded simply-connected domain in the complex plane $\mathbb{C}$.We analyze some open questions and discuss some ideas properties related to solving asymptotic behavior of polar Bergman polynomials over domains with corners and asymptotic behavior of modified Bergman polynomials by affine transforms in variable and polar modified Bergman polynomials by affine transforms in variable. We show that uniform asymptotic of Bergman polynomials over domains with corners and by Pritsker's theorem imply uniform asymptotic for all their derivatives.

Keywords: *Bergman orthogonal polynomials, polar orthogonal polynomials, asymptotic behavior, Faber polynomials.*

1 Introduction

We consider G be a bounded simply-connected domain in the complex plane $\mathbb{C}$ whose boundary $E = \partial G$ is a **Jordan** curve. Let σ be a positive finite area measure defined on the **Borel** set $B(\mathbb{C}) \cap G$ with compact support G .The L_2 extremal pol--ynomials $(L_n)_{n=0,1,2....}$ associated to the measure σ and the support G are defined as the solutions of the extremal problems in the space $L^2(E, d\sigma)$. Let $m_{n,2}(\sigma)$ ($n =$ 0,1,2…..) denotes the extremal constants associated with σ and G. $m_{n,2}(\sigma)$ satisfy

$$m_{n,2}(\sigma) = \min_{q_1, q_2, \ldots q_n} \left\{ \|Q_n\|_{L^2(E,d\sigma)}, Q_n = z^n + q_1 z^{n-1} + q_2 z^{n-2} + ..q_n \right\} \qquad (1)$$

Let us remark that the polynomials $(L_n)_{n=0,1,2....}$ coincide exactly with the monic or- -thogonal polynomials, and satisfy

$$L_n(z) = z^n + \text{lower degree terms.}$$

And

$$\int_G L_n(z)\overline{L_m(z)}d\sigma = \delta_{m,n} \|L_n\|^2_{L^2(E,d\sigma)}, \qquad n, m = 0,1,2.... \qquad (2)$$

Where $d\sigma_z$ stands for the area measure. In this case $\|L_n\|_{L^2(E,d\sigma)} = \sqrt{\int_G |L_n(z)|^2 d\sigma}$.

Let $(\varphi_n(z))_{n=0,1,2....}$ be the sequence of corresponding orthonormal **Bergman** polyn- -omials such that $\varphi_n(z) = \lambda_n z^n +$ lower degree terms $(\lambda_n \succ 0)$. And

$$\int_G \varphi_n(z)\overline{\varphi_m(z)}d\sigma = \delta_{m,n} \qquad , n, m = 0,1,2..... \qquad (3)$$

Let Ω= $\overline{\mathrm{C}} - \overline{G}$ and let E be a **Jordan** closed curve, Ω=Ext (E), $H = \{w, |w| \succ 1\}$.Let $w = \phi(z)$ be a function which maps Ω conformally on H in a such manner that $Lim_{z\to\infty} \frac{\phi(z)}{z}$ >0 and $\phi(\infty) = \infty$.In fact this limit is equal to γ, where γ^{-1} is the log- -arithmic capacity of E. Let $\psi : H \to \Omega$ be the inverse function of ϕ.Let us rem- -ark that ϕ can be represented by the **Laurent** expansion at infinity

$$\phi(z) = \frac{1}{cap(E)} z + \nu_0 + \sum_{i=1}^{\infty} \nu_i z^i$$

The two functions $\phi(z)$ and $\psi(w)$ have a continuous extension to E and on the un- -it circle respectively (Caratheodory's theorem**)**.Their derivatives ϕ' and ψ' have no zeros in Ω and H and have limit values on E and on the unit circle almost every- -where (with respect to the **Lebesgue** measure).The functions $\phi'(z)$ and $\psi'(w)$ are defined and integrable on E and on the unit circle.
We introduce a set of monic polynomials associated to this monic **Bergman** poly- -nomials on domain with corners. This class of **Bergman** polynomials is studied in ([1]) by **D.Khavinson** and **N.Stylianopoulos**.We study their relevant recurren- -ce relations (polar polynomials) and show that they satisfy mixed recurrence rela- -tions which are similar to those of the **Bergman** polynomials case. We also indi - -cate some questions concerning the asymptotic behavior of **Bergman** polynomia-

-ls over domains with corners,in short **BPDC** and discuss some theorems speaking the asymptotic behavior of polar **BPDC** , and giving some compatible equations between them. It was shown that some of polar's characterizations can be represe- -nted in terms of the corresponding **BPDC**.We investigate this results for showing some relations concerning the asymptotic behavior of modified **Bergman** polyno- -mials by affine transforms in variable and Polar modified **Bergman** polynomials by affine transforms in variable and by comparison equations we conclude the ne- -ws's asymptotic behavior for derivatives of polar **BPDC**. We also gave some alg- -ebraic tools for working with this polar **BPDC**.

The following theorem concerning the finite recurrence relation of **Bergman** pol- -nomials over domains with corners is the main basic result of these works.

Theorem 1.1 ([1]) *.The **Bergman** polynomials* $\left(\varphi_n(z)\right)_{n=0,1,2....}$ *satisfy an* $(s+1)$ *-terms recurrence relation, for any* $n \geq s-1$.

$$z\varphi_n(z) = a_{n+1,n}\varphi_{n+1}(z) + a_{n,n}\varphi_n(z) + ...a_{n-s+1,n}\varphi_{n-s+1}(z) \qquad (4)$$

*Assume that E is piecewise analytic without cusps. If the **Bergman** polynomials* $\left(\varphi_n(z)\right)_{n=0,1,2....}$ *satisfy an* $(s+1)$*-terms recurrence relation, with some* $s \geq 2$. *Then s=2 and E is an ellipse.*

***Proof** .For more details see ([1]).*

The modified **Bergman** polynomials by affine transforms in variable is one of re- -marquable orthogonal polynomials associated **Bergman** Polynomials.

Definition 1.2. *If* we denote for brevity $l_n(z) = a^{-n}L_n(az+b)$, $n = 0,1,2...$ *These polynomials are called monic modified **Bergman** polynomials by affine transform -s in variable.*

2 Problem Formulations

2.1 BPDC and Recurrence relations

Proposition 2.1. *The **modified*** **BPODC** *satisfy*

$$zl_n(z) = b_{n+1,n}l_{n+1}(z) + b_{n,n}l_n(z) + b_{n,n-1}l_{n-1}(z) \qquad (5)$$

Where

$$b_{n+1,n} = \frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n} \quad , \; b_{n,n} = \frac{a_{n,n}-b}{a} \quad , \quad b_{n-1,n} = \frac{\lambda_{n-1}}{a^2\lambda_n}a_{n-1,n}$$

Proof. From the know three terms $\varphi_n{}'$ s recurrence relations we have for a su- -ch that $a \neq 0$.

$$zL_n(z) = \frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n}L_{n+1}(z) + a_{n,n}L_n(z) + \frac{\lambda_{n-1}}{\lambda_n}a_{n-1,n}L_{n-1}(z)$$

Thus we get

$$(az+b)L_n(az+b)=\frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n}L_{n+1}(az+b)+a_{n,n}L_n(az+b)+\frac{\lambda_{n-1}}{\lambda_n}a_{n-1,n}L_{n-1}(az+b)$$

After cancellations we readily conclude that (5). And this completes the proof.

Proposition 2.2 *Notice that*

$$\sum_{n=0}^{\infty}\frac{|a|^{2n}}{\|L_n\|^2}l_n(z)\overline{l_n(z_0)}=\frac{1}{\pi}\phi'(az+b)\phi'(az_0+b) \qquad (6)$$

Moreover If $|a|=1$ *we conclude that*

$$\sum_{n=0}^{\infty}\frac{l_n(z)\overline{l_n(z_0)}}{\|L_n\|^2}=\frac{1}{\pi}\phi'(e^{i\nu}z+b)\phi'(e^{i\nu}z_0+b) \qquad (7)$$

Where $\lambda_n=\|L_n\|^{-2}_{L^2(E,d\sigma)}$.

Proof. The well known bilinear series formula with taking into account that $\phi'(z_0)\succ 0$ is

$$\sum_{n=0}^{\infty}\frac{L_n(z)\overline{L_n(z_0)}}{\|L_n\|^2}=\frac{1}{\pi}\phi'(z)\phi'(z_0),\quad |z|\prec 1, and, |z_0|\prec 1.$$

Applying this equation we readily conclude that (6), (7).

Example 2.3. *For a system of polynomials orthonormal on the unit disc the bi--linear serie of* $\varphi_n(z)=\sqrt{\frac{n+1}{\pi}}z^n$, $n=0,1,2....$ *is follows*

$$\sum_{n=0}^{\infty}\frac{L_n(z)\overline{L_n(z_0)}}{\|L_n\|^2}=\frac{1}{\pi}\frac{1}{(1-\overline{z_0}z)^2} \qquad ,|z|\prec 1, and, |z_0|\prec 1 \qquad (8)$$

Let us now put $l_n(z)=a^{-n}L_n(az+b)$ *for* $n=0,1,2.....$ *accordingly we get for* z *such that:* $|z|\prec 1, and, |z_0|\prec 1.$

$$\sum_{n=0}^{\infty}(n+1)(az+b)^n\left(\overline{a}\overline{z_0}+\overline{b}\right)^n=\frac{1}{\left(1-(\overline{a}\overline{z_0}+\overline{b})(az+b)\right)^2}$$

2.2 Polar BPODC and Recurrence relations

Definition 2.4. *For a fixed complex number* ξ *we are going to define* $P_n(z)$ *as a monic polynomials such that*

$$(z-\xi)P_n(z)=(n+1)\int_{\xi}^{z}L_n(t)dt \qquad (9)$$

Obviously the following calculus shows that the pole of $P_n(z)$ *do not have to be irregular.*

$$\lim_{z\to\xi} P_n(z) = (n+1)\lim_{z\to\xi}\frac{\int_\xi^z L_n(t)dt}{z-\xi} = (n+1)L_n(\xi).$$

More information on the history and applications of this concept may be found in ([5]).

The L_n orthogonality can be written as

$$\int_G L_n(z)\bar{z}^{\nu}\, d\sigma = \delta_{n,\nu}\|L_n\|^2_{L^2(E,d\sigma)}, \qquad \nu = 0,1,2....n-1.$$

Analogously to (7) we derive for $\nu = 0,1,2....n-1.$

$$\int_G \left(P_n(z)+(z-\xi)P_n'(z)\right)\overline{L_\nu(z)}d\sigma = \delta_{n,\nu}\|L_n\|^2_{L^2(E,d\sigma)},$$

Proposition 2.5. *The recurrence relation of Polar can be written in the form*

$$zP_n(z) = d_{n+1,n}P_{n+1}(z) + d_{n,n}P_n(z) + d_{n-1,n}P_{n-1}(z)$$
$$+(z-\xi)\left(d_{n+1,n}P_{n+1}'(z) + (z-d_{n,n})P_n'(z) + d_{n-1,n}P_{n-1}'(z)\right) \qquad (10)$$

Where

$$d_{n+1,n} = \frac{n+1}{n+2}\frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n} \;,\; d_{n,n} = a_{n,n} \;,\; d_{n-1,n} = \frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}a_{n-1,n}$$

Proof. Integrating both sides of the three term's recurrence relation we get for $z \neq \xi$

$$(n+1)(z-\xi)^{-1}\int_\xi^z tL_n(t)dt =$$

$$\frac{n+1}{n+2}\frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n}P_{n+1}(z) + a_{n,n}P_n(z) + \frac{n+1}{n+2}\frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n}P_{n+1}(z) + \frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}a_{n-1,n}P_{n-1}(z)$$

In other hand .Let us remark

$$(n+1)(z-\xi)^{-1}\int_\xi^z tL_n(t)dt = (z-\xi)^{-1}\int_\xi^z t\{(t-\xi)P_n(t)\}'dt =$$

$$(z-\xi)^{-1}\left(z(z-\xi)P_n(z) - \int_\xi^z (t-\xi)P_n(t)dt\right) =$$

Therefore

$$-(z-\xi)^{-1}\int_{\xi}^{z}(t-\xi)P_n(t)dt=$$

$$\frac{n+1}{n+2}\frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n}P_{n+1}(z)-(z-a_{n,n})P_n(z)+\frac{n+1}{n+2}\frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n}P_{n+1}(z)+\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}a_{n-1,n}P_{n-1}(z)$$

Differentiating the two sides of this equality we get, $-(z-\xi)P_n(z)=$

$$=\frac{n+1}{n+2}\frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n}P_{n+1}(z)-(z-a_{n,n})P_n(z)+\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}a_{n-1,n}P_{n-1}(z)+$$

$$(z-\xi)\left(\frac{n+1}{n+2}\frac{\lambda_{n+1}}{\lambda_n}a_{n+1,n}P'_{n+1}(z)-(z-a_{n,n})P'_n(z)-P_n(z)+\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}a_{n-1,n}P'_{n-1}(z)\right)$$

Which leads to (10).

Example 2.6. The **Bergman** Polynomials on **Cassini** Ovals ([2]).

Let us consider the curve $z\rightarrow\{|z-1||z+1|=\alpha\}$ if $\alpha\leq 1$ the curve define two bo--unded domains, let us denote either of them by B_z the function $\frac{1}{\alpha}(z^2-1)$ maps B_z one to one on to the disc |w|<1.The orthonormal **Bergman** Polynomials on **C--assini** Ovals in short **BPCO** can be written in the form:

$$\varphi_{2n+1}(z)=2\frac{\sqrt{n+1}}{\sqrt{\pi}\alpha^{n+1}}z(z^2-1)^n \qquad n=0,1,2......$$

is also a closed orthonormalised system in the class $H^2(B_z,d\sigma_z)$.

The corresponding monic orthogonal polynomials can be written in the form

$$L_{2n+1}(z)=2z(z^2-1)^n \qquad n=0,1,2......$$

They satisfy for $\nu=0,1,2......2n.$

$$\int_{B_z}L_{2n+1}(z)\bar{z}^{\nu}d\sigma=\delta_{2n+1,\nu}\frac{\pi\alpha^{2n+2}}{(n+1)},$$

Define $(P_{2n+1}(z))_{n=0}^{\infty}$ as a monic polynomials polar **BPCO** .In the next we prove the following assertions

$$(z-\xi)P_{2n+1}(z)=(z^2-1)^{n+1}-(\xi^2-1)^{n+1} \tag{11}$$

$$P_{2n+1}(1)=(\xi-1)^n(\xi+1)^{n+1}\ ,\quad \text{and}\quad P_{2n+1}(-1)=(\xi-1)^{n+1}(\xi+1)^n. \tag{12}$$

Clearly we have

$$(z-\xi)P_{2n+1}(z)=2(n+1)\int_{\xi}^{z}t(t^2-1)^n dt=(z^2-1)^{n+1}-(\xi^2-1)^{n+1}.$$

Thus, the results (11), (12) are proved .Discovering that the pole of $P_n(z)$ do not have to be irregular.

$$\lim_{z\to\xi} P_{2n+1}(z) = (n+1)2\xi(\xi^2-1)^n .$$

Next we use two excellent important theorems due to **Pritsker.I.E,** ([3]).

Theorem 2.7. *([3])*. Let us put *$\partial\Omega = E$ is an analytic curve so that φ is confo- -rmal map from Ω to G with properties: Suppose that $\partial\Omega = E$ is an analytic curve so that φ is conformal map from Ω to G with properties:*

$$\varphi\ (\infty)=\infty\ ,\ \lim_{z\to\infty}\varphi(z) = \frac{1}{cap(E)} = \nu \succ 0$$

*can be used (by **Carathéodory**'s theorem ([10],[8]),$|\varphi(\xi)|=1$, $\forall\xi\in E$.In addition $\varphi(E)=T=\{w,|w|=1\}$ conformally.Suppose that : $\psi: G\to\Omega$, its inverse map .Let us put*

$$(\varphi(z))^n = \left(\frac{1}{cap(E)}z + \nu_0 + \sum_{i=1}^{\infty}\nu_i z^i\right)^n = F_n(z) + E_n\left(\frac{1}{z}\right) \qquad (13)$$

Where $E_n\left(\frac{1}{z}\right) \to 0$ *as* $n\to\infty$. *Then*

$$F_n^{(k)}(z) = \frac{d^k}{dz^k}[\varphi(z)]^n + O(r^n)\ \ ,\ r\in\{0,1\}.\ as\ n\to\infty,\ z\in\tilde{\Omega}\supseteq\overline{\Omega} \qquad (14)$$

And as $n\to\infty$.

$$F_n^{(k)}(z_0) = \frac{\alpha k!\, n^{\alpha k} w_0^n}{(a_\alpha w_0^\alpha)^k\,\Gamma(1+\alpha k)} + o(n^{\alpha k}) \qquad (15)$$

Where z_0 *is fixed point,* $\varphi(z_0) = w_0$. *And*

$$\psi(w) = \psi(w_0) + a_\alpha(w-w_0)^\alpha + \ldots\ldots \qquad a_\alpha \neq 0$$

Theorem 2.8. *([3]). Let Γ be a rectifiable **Jordan** curve and let g(z) belongs to* $H^\infty(\Omega)$. *For* $z_0\in\Gamma$*,assume that Γ has un exterior angle of opening $\alpha\pi$ at the point* $z_0, 0\le\alpha\le 2\pi$ *,formed by* C^ω *arcs ,where $\omega(x)$ satisfies the **Alper** condition:*

$$\int_0^\varepsilon \frac{\omega(x)}{x}\log\left(\frac{1}{x}\right)dx \prec \infty$$

*Suppose that the boundary values of g (z) satisfy a **Dini** condition at the point* z_0

$$|g(z)-g(z_0)| \le \lambda(|z-z_0|)\ \ ,\ \ |z-z_0| \prec \delta\ \ , z\in\Gamma.$$

With $\delta\succ 0$ *and a modulus of continuity $\lambda(x)$ such that*

$$\int_0^\delta \frac{\lambda(x)}{x}dx \prec \infty.$$

*Then, the generalized **Faber** polynomials satisfy*

$$Q_n(z_0) = \alpha g(z_0)\varphi^n(z_0) + o(1) \qquad as\ \ n\to\infty.$$

Under condition of convexity and $\alpha\pi$ is the largest exterior angle at the boundary Γ, and for the case $g(z)=1$ then:

$$Max\left|F_n(z)\right| \to \alpha \qquad as \quad n \to \infty. \tag{16}$$

Moreover under more general conditions we have

$$\left|F_n(z_0)\right| \to \alpha \qquad as \quad n \to \infty. \tag{17}$$

Proof. For more details concerning the proofs of two theorems see *([3],[6]).*

3 Main Results

3.1 Asymptotic behavior of BPODC and their derivatives.

Theorem 3.1 ([1]). *Assume that E is piecewise analytic without cusps. Then, for any $n \in N$*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = \nu \quad , \quad \lim_{n\to\infty} \sqrt{\frac{\pi}{n+1}}\frac{\lambda_{n+1}}{\nu^{n+1}} = 1 \tag{18}$$

Where γ^{-1}=Cap (E).In addition uniformly for compacts subsets of $\overline{\Omega}$

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = \varphi(z)$$

Also uniformly for compacts subsets of $\overline{\Omega}$

$$\lim_{n\to\infty} \sqrt{\frac{\pi}{n+1}}\varphi_n(z) = \varphi'(z)\lim_{n\to\infty} F_n(z) \tag{19}$$

Proof. For more details ([1]). Under the same conditions as in theorem 2.7 we can only need to show that (19).

Corollary 3.2. *Under same conditions as in theorem 2.7 follows asymptotic formulas holds*

$$\lim_{n\to\infty} \nu^{n+1} L_n(z) = \varphi'(z)\lim_{n\to\infty} F_n(z). \tag{20}$$

Uniformly for compacts subsets of $\overline{\Omega}$.Hence uniformly for compacts subsets of $\overline{\Omega}$

$$\lim_{n\to\infty} \frac{L_{n+1}(z)}{L_n(z)} = \nu^{-1}\varphi(z). \tag{21}$$

Moreover uniformly for compacts subsets of $\overline{\Omega}$

$$\lim_{n\to\infty} \nu^{n+1} L_n^{(k)}(z) = \sum_{j=0}^{k} C_j^k \varphi^{(k+1-j)}(z)\lim_{n\to\infty} F_n^{(j)}(z) \tag{22}$$

Under more general conditions we have

$$\lim_{n\to\infty} \nu^{n+1} L_n(z_0) = \alpha\left|\varphi'(z_0)\right| \tag{23}$$

Where z_0 *and α are defined by* ***Pritsker****'s theorem ([3])*.

Proof. Notice that

$$\lim_{n\to\infty}\frac{\varphi_{n+1}(z)}{\varphi_n(z)}=\varphi(z)\Rightarrow \lim_{n\to\infty}\frac{L_{n+1}(z)}{L_n(z)}=v^{-1}\varphi(z)$$

By (18) and (19) we get (20) and (21) uniformly for compacts subsets of $\overline{\Omega}$.By **Pr-itsker**'s theorem ([3]) and differentiating k-times the both sides of (20) we get as-sertion (22). With help of more general conditions and by **Pritsker**'s formula we get $F_n(z_0)\to\alpha$ .As n→∞.Thus property (23) is proved.

The following three theorems are the main result of this work.

3.2 Relative asymptotic formula of polar BPODC.

Theorems 3.3.*We have uniformly for compacts subsets of* $\overline{\Omega}$

$$\lim_{n\to\infty}\frac{L_n(z)}{P_n(z)}=(z-\xi)F(z) \tag{24}$$

Where uniformly for compacts subsets of $\overline{\Omega}$.

$$F(z)=\lim_{n\to\infty}\frac{P_n'(z)}{nP_n(z)} \tag{25}$$

Hence uniformly for compacts subsets of $\overline{\Omega}$.

$$\lim_{n\to\infty}\frac{P_{n+1}(z)}{P_n(z)}=v^{-1}\varphi(z) \tag{26}$$

Proof. To prove (24) let us remark from this equality $\left((z-\xi)P_n(z)\right)'=(n+1)L_n(z)$

We get uniformly for compacts subsets of $\overline{\Omega}$.

$$\lim_{n\to\infty}\frac{L_n(z)}{P_n(z)}=(z-\xi)\lim_{n\to\infty}\frac{P_n'(z)}{nP_n(z)}$$

Now use (24) we obtain $\lim_{n\to\infty}\frac{P_{n+1}(z)}{P_n(z)}=\lim_{n\to\infty}\frac{L_{n+1}(z)}{L_n(z)}$. Accordingly to (21) we readily conclude that (26).

Theorem.3.4 *For each* $a\in\Omega$ *we have uniformly for compacts subsets of dom-ain* $\overline{\Omega}-\{a\}$.

$$\lim_{n\to\infty}\left(\frac{P_{n+1}(z)}{P_n(z)}\right)\left(\frac{P_{n+1}(a)}{P_n(a)}\right)^{-1}=\exp\left(\int_\xi^z F(t)dt\right). \tag{27}$$

Where F *is defined by (25)*. This gives for every complex number a.

$$\lim_{n\to\infty}\frac{P_{n+1}(z)}{P_n(z)}=v^{-1}\varphi(a)\exp\left(\int_\xi^z F(t)dt\right) \tag{28}$$

Thus uniformly for compacts subsets of Ω

$$\lim_{n\to\infty}\frac{P_n'(z)}{nP_n(z)}=\frac{\varphi'(z)}{\varphi(z)} \tag{29}$$

Proof. We can put

$$\frac{\left((z-\xi)P_n(z)\right)'}{(z-\xi)P_n(z)}=(n+1)\frac{L_n(z)}{(z-\xi)P_n(z)}$$

Taking exponential yields : $\dfrac{(z-\xi)P_n(z)}{(a-\xi)P_n(a)}=\exp(n+1)\displaystyle\int_a^z\frac{L_n(t)dt}{(t-\xi)P_n(t)}$.

Thus we get

$$P_n(z)=P_n(a)\frac{a-\xi}{z-\xi}\exp(n+1)\int_a^z\frac{L_n(t)dt}{(t-\xi)P_n(t)}$$

Which leads to prove (27) if we taking into account that

$$\exp\left((n+2)\int_a^z\frac{L_{n+1}(t)dt}{(t-\xi)P_{n+1}(t)}-(n+1)\int_a^z\frac{L_n(t)dt}{(t-\xi)P_n(t)}\right)\to\exp\int_a^z\frac{L_n(t)dt}{(t-\xi)P_n(t)}.$$

In order to complete proving the rest. By (27) with help of (26) we deduce (28).
We can state for each $z\neq a$

$$\exp\left(\int_\xi^z F(t)dt\right)=\frac{\varphi(z)}{\varphi(a)}$$

Differentiating both sides of this equation yields to (29).

3.3 Polar modified BPODC by affine transforms in variable

Next we denote by $R_n(z)$ the sequence of the monic polynomial of degree n polar of $l_n(z)$.i-e

$$(z-\xi)R_n(z)=(n+1)\int_\xi^z l_n(t)dt \qquad n=0,1,2.... \tag{30}$$

Theorem 3.5. *For each* ν

$$(z-\xi)R_n(z)-(\nu-\xi)R_n(\nu)=a^{-n-1}\left[(az+b-\xi)P_n(az+b)-(a\nu+b-\xi)P_n(a\nu+b)\right]$$

Hence

$$-\xi R_n(0)-(\nu-\xi)R_n(\nu)=a^{-n-1}\left[(b-\xi)P_n(b)-(a\nu+b-\xi)P_n(a\nu+b)\right]$$

Moreover

$$(z-\xi)R_n'(z)+R_n(z)=a^{-n}\left[(az+b-\xi)P_n'(az+b)+P_n(az+b)\right]$$

In addition

$$-\xi^2R_n'(0)-(\nu-\xi)R_n(\nu)=a^{-n-1}\left[(b-\xi)P_n(b)-(a\nu+b-\xi)P_n(a\nu+b)\right]+$$
$$a^{-n}\xi\left[(b-\xi)P_n'(b)+P_n(b)\right]$$

Therefore for $z\neq\xi$

$$a^{-n}(az+b-\xi)P_n(az+b)=$$
$$a((z-\xi)R_n(z)-(\nu-\xi)R_n(\nu))+a^{-n}(a\nu+b-\xi)P_n(a\nu+b) \quad (31)$$

Proof. When l_n is replaced by $a^{-n}L_n(at+b)$ in equation (30) we get

$$(z-\xi)R_n(z)-(\nu-\xi)R_n(\nu)=a^{-n-1}(n+1)\int_{a\nu+b}^{az+b}L_n(t)dt$$

Hence the first above assertion is proved. Replacing $z=0$ in it we readily conclu--de that the second assertion. Differentiation of $(z-\xi)R_n(z)$ yields to the third as--sertion. Taking $z=0$ in it and use the first assertion we get the fourth one. Than--ks to the first assertion we readily conclude that (31).

Proposition 3.6. *We have uniformly for compacts subsets of* $\Omega-\{\xi\}$

$$G(z)=\left(1+\frac{\xi^2-\xi+b}{\xi(z-\xi)}\sigma(z)\right)aF(az+b) \quad (32)$$

Where $G(z)=\lim_{n\to\infty}\frac{R_n'(z)}{nR_n(z)}$ *and* $\sigma(z)=\lim_{n\to\infty}\frac{R_n\left(\frac{z-b}{a}\right)}{R_n(z)}$ *uniformly for compacts su--bsets of* $\Omega.-\{\xi\}$.

If $\sigma(z)=0$ *.Then uniformly for compacts subsets of* $\Omega-\{\xi\}$

$$G(z)=a\frac{\varphi'(az+b)}{\varphi(az+b)} \quad (33)$$

Proof. Notice that

$$(z-\xi)G(z)=\lim_{n\to\infty}\frac{l_n(z)}{R_n(z)},\ \textit{and}\ (z-\xi)F(z)=\lim_{n\to\infty}\frac{L_n(z)}{P_n(z)}$$

Hence

$$F(az+b)=\lim_{n\to\infty}\frac{a^{-n}L_n(az+b)}{a^{-n}(az+b-\xi)P_n(az+b)}=\lim_{n\to\infty}\frac{l_n(z)}{a^{-n}(az+b-\xi)P_n(az+b)}$$

Therefore

$$F(az+b)=\lim_{n\to\infty}\frac{l_n(z)}{a(z-\xi)R_n(z)+a^{-n}(a(\xi-1)+b)P_n(a\xi+b)}$$

Let remark now

$$(a\xi+b-\xi)P_n(\xi)=a^n(n+1)\int_{\xi}^{a\xi+b}a^{-n}L_n(t)dt=a^n(n+1)\int_{\xi}^{a\xi+b}l_n\left(\frac{t-b}{a}\right)dt$$
$$=a^{n+1}(n+1)\int_{\frac{\xi-b}{a}}^{\xi}l_n(t)dt=-a^{n+1}\left(\frac{\xi-b}{a}-\xi\right)R_n\left(\frac{\xi-b}{a}\right)$$

We arrive at the representation

$$F(az+b)=\lim_{n\to\infty}\frac{l_n(z)}{a(z-\xi)R_n(z)-a\left(\frac{\xi-b}{a}-\xi\right)R_n\left(\frac{\xi-b}{a}\right)}.$$

Hence

$$F(az+b)=\lim_{n\to\infty}\frac{\dfrac{l_n(z)}{R_n(z)}}{a(z-\xi)-a\left(\dfrac{\xi-b}{a}-\xi\right)\dfrac{R_n\left(\frac{\xi-b}{a}\right)}{R_n(z)}}.$$

We readily conclude that (32).Thus (33) is confirmed.

Corollary 3.7. ***(Asymptotic formulas)***

$$\lim_{n\to\infty}\frac{P_n(az+b)P_n(z)}{a^nL_n(z)R_n(z)}=\frac{a}{(az+b-\xi)}\frac{\varphi(z)}{\varphi'(z)}. \tag{34}$$

Uniformly for compacts subsets of $\overline{\Omega}$.

Hence uniformly for compacts subsets of $\overline{\Omega}$.

$$\lim_{n\to\infty}\frac{a^nL_n(z)}{P_n(az+b)}=(az+b-\xi)\frac{\varphi'(az+b)}{\varphi(az+b)}. \tag{35}$$

Analogously to this equation we derive

$$\lim_{n\to\infty}\frac{a^nR_n(z)}{P_n(az+b)}=\frac{(az+b-\xi)}{a(z-\xi)}. \tag{36}$$

Uniformly for compacts subsets of $\overline{\Omega}$.

Proof. In order to prove (34).Applying the first assertion of theorem 3.5 to (9). With help of (24) and taking into account that

$$\lim_{n\to\infty}\frac{L_n(az+b)}{P_n(az+b)}=(az+b-\xi)\frac{\varphi'(az+b)}{\varphi(az+b)}.$$

It follows that

$$\frac{G(z)}{F(z)}=\lim_{n\to\infty}\frac{l_n(z)}{R_n(z)}\frac{P_n(z)}{L_n(z)}=\lim_{n\to\infty}\frac{a^{-n}L_n(az+b)}{P_n(az+b)R_n(z)}\frac{P_n(az+b)P_n(z)}{L_n(z)}.$$

Thus

$$\frac{G(z)}{(az+b-\xi)F(z)}\frac{\varphi(az+b)}{\varphi'(az+b)}=\lim_{n\to\infty}\frac{a^{-n}}{R_n(z)}\frac{P_n(az+b)P_n(z)}{L_n(z)}.$$

Applying (33) we get

$$\frac{a}{(az+b-\xi)}\frac{\varphi(z)}{\varphi'(z)}=\lim_{n\to\infty}\frac{a^{-n}}{R_n(z)}\frac{P_n(az+b)P_n(z)}{L_n(z)}.$$

Of course $\lim_{n\to\infty}\frac{P_n(z)}{R_n(z)}=\frac{G(z)}{F(z)}.=a\frac{F(az+b)}{F(z)}=a\frac{\varphi(z)}{\varphi(az+b)}\frac{\varphi'(az+b)}{\varphi'(z)}.$

Taking into account that

$$\frac{(az+b-\xi)}{a}\frac{\varphi'(z)}{\varphi(z)}=\lim_{n\to\infty}\frac{a^n L_n(z)}{P_n(az+b)}\frac{R_n(z)}{P_n(z)}.$$

Becomes to (35).Thus (36) is established.

4 Open Problem

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose bo- -undary $E=\partial G$ is a **Jordan** curve. Let σ be a positive finite area measure defined on the **Borel** set $B(\mathbb{C})\cap G$ with compact support G. The L_2 extremal polynomial- -s $(L_{n,m})_{n=0,1,2....}$ associated to the measure σ and the support G are defined as the s- -olutions of the extremal problems in the space $L^m(E,d\sigma)$. Let $\mu_{n,m}(\sigma)$ (n =0,1,2 3,4…....) denotes the extremal constants associated with σ and G. $\mu_{n,m}(\sigma)$ satisfy

$$\mu_{n,m}(\sigma)=\min_{q_1,q_2,\dots q_n}\left\{\|Q_n\|_{L^m(E,d\sigma)},Q_n=z^n+q_1z^{n-1}+q_2z^{n-2}+..q_n\right\}$$

Where

$$\|Q_n\|_{L^m(E,d)}=\left\{\int_G |L_n(z)|^m d\sigma_z\right\}^{\frac{1}{m}}.$$

For a fixed complex number ξ we are going to define $(P_{n,m}(z))_{n=0}^{\infty}$ as a monic po- -lynomials such that

$$(z-\xi)P_{n,m}(z)=(n+1)\int_{\xi}^{z}L_{n,m}(t)dt.$$

How it is possible to characterize the asymptotic formulas of $(P_{n,m}(z))_{n=0}^{\infty}$ in terms of the asymptotic properties of the corresponding extremal polynomials.

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