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A Lower Bound for an Erdös-Szekeres-Type Problem with Interior Points

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Abstract

A point of a finite planar point set is called an interior point of the set if it is not on the boundary of the convex hull of the set. For any positive integer n, let g(n) be the smallest integer such that every planar point set P with no three collinear points and with at least g(n) interior points has a subset Q whose the interior of the convex hull of Q contains exactly n points of P. In this paper, we prove that $g(n) \geq 4n$ for all $n \geq 4$.

Keywords: *Interior point, Finite planar set, Convex hull, Deficient point set.*

2000 Mathematics Subject Classification: 52C10

1 Introduction

In this paper, we suppose that no three points are collinear. We focus on finite point sets in the plane. In 1935, Erdös and Szekeres [3] posed the problem: for any integer $n \geq 3$, determine the smallest positive integer $f(n)$ such that any finite point set of at least $f(n)$ points has a subset of n points whose the convex hull contains exactly n vertices; moreover, they proved that the existence of $f(n)$. They showed that $f(n) \geq 2^{n-2} + 1$ for all $n \geq 3$ (see [4]). This lower bound may be closed to $f(n)$; particularly, $f(3) = 3$ and $f(4) = 5$ (see [2]), $f(5) = 9$ (see [6]), and $f(6) = 17$ (see [7]).

In 2001, Avis, Hosono and Urabe [1] posed an *Erdös-Szekeres-type problem*: for any positive integer n, determine the smallest positive integer $g(n)$ such that any finite point set P of at least $g(n)$ points has a subset Q whose the interior of the convex hull of Q contains exactly n points in P; moreover, they proved that $g(1) = 1$, $g(2) = 4$, $g(3) \geq 8$, and $g(n) \geq n + 2$ for all $n \geq 4$. In 2003, Fevens [5] presented a lower bound: $g(n) \geq 3n - 1$ for all $n \geq 3$. In 2008, Wei and Ding [8] improved this lower bound to a new lower bound: $g(n) \geq 3n$ for all $n \geq 3$. In 2009, Wei and Ding [9] proved that $g(3) = 9$. In this paper, we presented a lower bound: $g(n) \geq 4n$ for all $n \geq 4$.

2 Preliminaries

Let P be a finite set of points in the plane such that no three points are collinear. The convex hull of P is denoted by $CH(P)$. The interior of $CH(P)$ is denoted by $intCH(P)$. The set of vertices of $CH(P)$ is denoted by $V(P)$. The number of elements in $V(P)$ is denoted by $v(P)$. A point $p \in P$ is called an *interior point* of P if $p \notin V(P)$. The set of interior points of P is denoted by $I(P)$. The number of elements in $I(P)$ is denoted by $i(P)$.

Let $Q \subseteq P$. The set $I(P) \cap intCH(Q)$ is denoted by $I^*(Q)$. The number of elements in $I^*(Q)$ is denoted by $i^*(Q)$.

For any positive integer n, let $g(n)$ be the smallest integer such that every planar point set P with no three collinear points and with at least $g(n)$ interior points has a subset Q whose the interior of the convex hull of Q contains exactly n points of P. Hence, for each positive integer n,

$$g(n) = \min\{\, s : \; i(P) \geq s \;\Rightarrow\; \exists Q \subseteq P \text{ s.t. } i^*(Q) = n \,\}.$$

Proposition 2.1 [8] *For any integer $n \geq 4$, $g(n) \geq 3n$.*

A planar point set P is called a *deficient point set of type $P(m, s, n)$* and denoted by $P = P(m, s, n)$ if $v(P) = m$, $i(P) = s$, and $i^*(Q) \neq n$ for all $Q \subseteq P$. Note that if $P = P(m, s, n)$ then $g(n) \geq s + 1$. To show that $g(n) \geq 4n$ for all $n \geq 4$, we seek for a deficient point set of type $P(m, 4n - 1, n)$. By the following Lemma, it suffices to prove the existence of a deficient point set of type $P(3, 4n - 1, n)$.

Lemma 2.2 (Extension Lemma) [1] *Every deficient point set $P(m, s, n)$ can be extended to a deficient point set $P(m + 1, s, n)$.*

For any three points x, y, z, let $C(x; y, z)$ be the cone formed by the two rays xy and xz, and let $H(xy; z)$ be the open half plane bounded by the straight line xy such that $z \in H(xy; z)$, and let $H^*(xy; z)$ be the open half plane bounded by the straight line xy such that $z \notin H^*(xy; z)$, and let Δxyz be the triangle with vertices x, y, z.

3 Main Results

In this section, we improve the lower bound in Proposition 2.1.

Theorem 3.1 *For any integer* $n \geq 4$, $g(n) \geq 4n$.

Proof. Let n be an integer such that $n \geq 4$. By the Lemma 2.2, it suffices to construct a deficient point set of type $P = P(3, 4n - 1, n)$.

Take $P = \{ v_1, v_2, v_3, a_1, a_2, \ldots, a_{4n-4}, d_1, d_2, d_3 \}$ and $V(P) = \{ v_1, v_2, v_3 \}$ where v_1, v_2, v_3 put into anticlockwise positions respectively.

Then $I(P) = \{ a_1, a_2, \ldots, a_{4n-4}, d_1, d_2, d_3 \}$.

Now, $v(P) = 3$ and $i(P) = 4n - 1$.

Let $a_0 = v_1$ and $a_{4n-3} = v_3$.

Take $P_0 = P \setminus\{v_2\}$ and $V(P_0) = \{ a_0, a_1, a_2, \ldots, a_{4n-4}, a_{4n-3} \}$ where $a_0, a_1, a_2, \ldots, a_{4n-4}, a_{4n-3}$ put into anticlockwise positions respectively

Then $I(P_0) = \{ d_1, d_2, d_3 \}$. We locate d_1, d_2, d_3 as follows:

- $d_1 \in C(v_2; a_{n-1}, a_n)$,
- $d_2 \in C(v_2; a_{2n-2}, a_{2n-1})$,
- $d_3 \in C(v_2; a_{3n-3}, a_{3n-2})$,
- $d_1 \in H(a_0a_{n+1}; v_2) \cap H(a_1a_{n+2}; v_2) \cap \ldots \cap H(a_{n-2}a_{2n-1}; v_2)$,
- $d_2 \in H(a_{n-1}a_{2n}; v_2) \cap H(a_na_{2n+1}; v_2) \cap \ldots \cap H(a_{2n-3}a_{3n-2}; v_2)$,
- $d_3 \in H(a_{2n-2}a_{3n-1}; v_2) \cap H(a_{2n-1}a_{3n}; v_2) \cap \ldots \cap H(a_{3n-4}a_{4n-3}; v_2)$,
- $d_1 \in H^*(a_{n-1}a_{2n}; v_2)$,
- $d_2 \in H^*(a_{n-2}a_{2n-1}; v_2) \cap H^*(a_{2n-2}a_{3n-1}; v_2)$,
- $d_3 \in H^*(a_{2n-3}a_{3n-2}; v_2)$,
- $d_1 \in H^*(a_0a_n; v_2) \cap H^*(a_1a_{n+1}; v_2) \cap \ldots \cap H^*(a_{n-1}a_{2n-1}; v_2)$,
- $d_2 \in H^*(a_{n-1}a_{2n-1}; v_2) \cap H^*(a_na_{2n}; v_2) \cap \ldots \cap H^*(a_{2n-2}a_{3n-2}; v_2)$,
- $d_3 \in H^*(a_{2n-2}a_{3n-2}; v_2) \cap H^*(a_{2n-1}a_{3n-1}; v_2) \cap \ldots \cap H^*(a_{3n-3}a_{4n-3}; v_2)$.

The construction of P is valid. For examples, we presents the set P as shown in Fig. 1 and 2 for $n = 4$ and 5, respectively.

Next, we will show that $i^*(Q) \neq n$ for all $Q \subseteq P$.

Let $Q \subseteq P$.

Case 1: $v_2 \notin Q$. Then $Q \subseteq P \setminus\{v_2\} = P_0$. Since $i^*(P_0) = 3$ and $v_2 \notin I(P)$, we obtain that $i^*(Q) \leq 3 < n$.

Case 2: $v_2 \in Q$.

Subcase 2.1: $d_1 \in intCH(Q)$. Then $\Delta v_2a_ja_{j+n+1} \subseteq CH(Q)$ for some $j = 0, 1, \ldots, n - 2$. This implies that $i^*(Q) \geq n + 1$.

Subcase 2.2: $d_2 \in intCH(Q)$. Then $\Delta v_2a_{j+n-1}a_{j+2n} \subseteq CH(Q)$ for some $j = 0, 1, \ldots, n - 2$. This implies that $i^*(Q) \geq n + 1$.

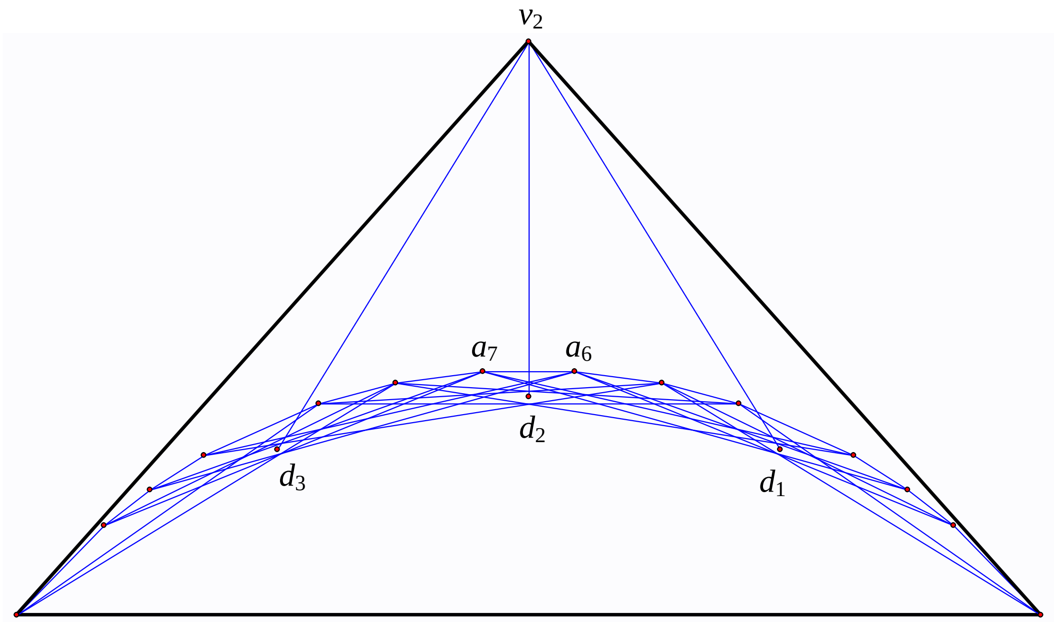

Fig. 1. A deficient point set $P = P(3, 15, 4)$.

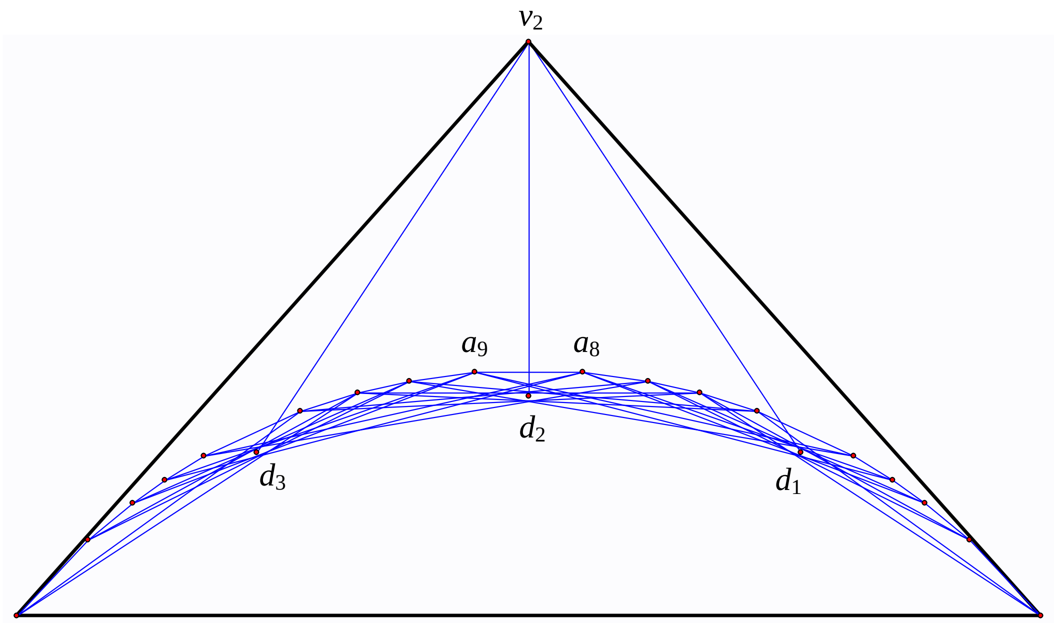

Fig. 2. A deficient point set $P = P(3, 19, 5)$.

Subcase 2.3: $d_3 \in intCH(Q)$. Then $\Delta v_2 a_{j+2n-2} a_{j+3n-1} \subseteq CH(Q)$ for some $j = 0, 1, \ldots, n-2$. This implies that $i^*(Q) \geq n+1$.

Subcase 2.4: $d_1, d_2, d_3 \notin intCH(Q)$. Then $CH(Q)$ contains at most one triangle $\Delta v_2 a_j a_{j+n}$. This implies that $i^*(Q) \leq n - 1$.

Hence $i^*(Q) \neq n$ for all $Q \subseteq P$.

Therefore P is a deficient point set of type $P(3, 4n - 1, n)$. This proof is completed. □

4 Open Problem

In [8], we know that $g(n) \geq 3n$ for all $n \geq 3$. In this paper, we show that $g(n) \geq 4n$ for all $n \geq 4$. In [1], we know that $g(1) = 1 = 1^2$ and $g(2) = 4 = 2^2$. In [9], we know that $g(3) = 9 = 3^2$. From above results, we conjecture a better lower bound for $g(n)$ as follow.

Conjecture 4.1 *For any positive integer n,* $g(n) \geq n^2$.

Moreover, we may conjecture about $g(n)$ as follow.

Conjecture 4.2 *For any positive integer n,* $g(n) = n^2$.

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