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The Algebra of Generalized Full Terms

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Abstract

In this paper the algebra of generalized full terms is defined. The well-known connection between hyperidentities of an algebra and identities satisfied by the clone of this algebra is studied in a restricted setting, that of n_i-ary generalized full hyperidentities and identities of the n_i-ary clone of term operations of an algebra induced by generalized full terms.

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1 Introduction

Let $X := \{x_1, x_2, ...\}$ be a countably infinite set of symbols called variables. We often refer to these variables as letters, to X as an alphabet, and also refer to the set $X_n =: \{x_1, x_2, ..., x_n\}$ as an n-element alphabet. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called an n_i-ary operation symbol, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The function τ, on the values of τ written as $(n_i)_{i\in I}$ is called a type.

An n-ary term of type τ is defined inductively as follows :

(i) The variables $x_1, ..., x_n$ are n-ary terms of type τ.

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(ii) If $t_1, ..., t_{n_i}$ are n-ary terms of type τ then $f_i(t_1, ..., t_{n_i})$ is an n-ary term of type τ.
By $W_\tau(X_n)$ we denote the smallest set which contains $x_1, ..., x_n$ and is closed under finite application of (ii). Then the set $W_\tau(X) := \cup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type τ. This set can be used as the universe of an algebra of type τ. For every $i \in I$, an n_i-ary operation $\bar{f}_i$ on $W_\tau(X)$ is defined by $\bar{f}_i : W_\tau(X)^{n_i} \longrightarrow W_\tau(X)$ with $(t_1, ..., t_{n_i}) \longmapsto f_i(t_1, ..., t_{n_i})$.

The algebra $\mathcal{F}_\tau(X) := (W_\tau(X); (\bar{f}_i)_{i \in I})$ is called the absolutely free algebra of type τ over the set X. The term algebra $\mathcal{F}_\tau(X)$ is generated by set X and has the property called *absolute freeness* means that for every algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I}) \in Alg(\tau)$ and every mapping $f : X \longrightarrow A$ there exists a unique homomorphism $\hat{f} : \mathcal{F}_\tau(X) \longrightarrow \mathcal{A}$ which extends the mapping f such that $\hat{f} \circ \varphi = f$ where $\varphi : X \longrightarrow \mathcal{F}_\tau(X)$ is the embedding of X into $\mathcal{F}_\tau(X)$.

Then for any $m \geq 1$; $m \in \mathbb{N}$, we define $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ inductively by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Let P_{n_i} be the set of all permutations on $\{1, 2, ..., n_i\}$. Generalized full terms of type τ are inductively defined as in the following definition.

Definition 1.1 Let $\tau = (n_i)_{i \in I}$ and let f_i be an n_i-ary operation symbol,

(i) if $s : \{1, ..., n_i\} \rightarrow \{1, ..., n_i\}$ is a permutation, then $f_i(x_{s(1)}, ..., x_{s(n_i)})$ is a generalized full term of type τ;

(ii) if $j_1, j_2, ..., j_{n_i}$ are natural numbers and greater than n_i, then $f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})$ is a generalized full term of type τ where s' is a permutation on $\{j_1, ..., j_{n_i}\}$;

(iii) if $t_1, ..., t_{n_i}$ are generalized full terms of type τ, then $f_i(t_1, ..., t_{n_i})$ is a generalized full term of type τ.

Let $W_\tau^{GF}(X)$ be the set of all n_i-ary generalized full terms of type τ. By Definition 1.1 the set $W_\tau^{GF}(X)$ is closed under the operation $\bar{f}_i$. Therefore $(W_\tau^{GF}(X); (\bar{f}_i)_{i \in I})$ is a subalgebra of $\mathcal{F}_\tau(X)$. Then we define a superposition operation S^{n_i} on $W_\tau^{GF}(X)$ by the following steps:

Definition 1.2 Let $s \in P_{n_i}, t_1, ..., t_{n_i}, p_1, ..., p_{n_i} \in W_\tau^{GF}(X), j_1, ..., j_{n_i}$ are natural numbers and greater than n_i and $s^{'}$ is a permutation on $\{j_1, ..., j_{n_i}\}$, $S^{n_i} : W_\tau^{GF}(X)^{n+1} \to W_\tau^{GF}(X)$ is defined by

(i) $S^{n_i}(f_i(x_{s(1)}, ..., x_{s(n_i)}), t_1, ..., t_{n_i}) := f_i(t_{s(1)}, ..., t_{s(n_i)})$;

(ii) $S^{n_i}(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})), t_1, ..., t_{n_i}) := f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})$;

(iii) $S^{n_i}(f_i(p_1, ..., p_{n_i}), t_1, ..., t_{n_i}) := f_i(S^{n_i}(p_1, t_1, ..., t_{n_i}), ..., S^{n_i}(p_{n_i}, t_1, ..., t_{n_i}))$.

Then the algebra $clone_{GF}\tau := (W_\tau^{GF}; S^{n_i})$ of type $\tau = (n_i+1)$ with $\mathcal{F}_{GS\tau} = \{f_i(x_{s(1)}, ..., x_{s(n_i)}) \mid i \in I, s \in P_{n_i}\} \cup \{f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) \mid j_1, ..., j_{n_i} > n_i, s^{'}$ is a permutation on $\{j_1, ..., j_{n_i}\}\}$ as a generating system is called the clone of generalized full terms of type τ.

If $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ is an algebra of type τ, then for every generalized full term t of type τ induces a term operation $t^{\mathcal{A}}$ on $\mathcal{A}$ via the following steps :

Definition 1.3 Let $s \in P_{n_i}$ and $s^{'}$ be a permutation on $\{j_1, ..., j_{n_i}\}$ where $j_1, ..., j_{n_i}$ are natural numbers and greater than n_i. Then an n_i-ary term operation $t^{\mathcal{A}}$ induced by a generalized full term t is defined as follows:

(i) If $t = f_i(x_{s(1)}, ..., x_{s(n_i)})$ then
$t^{\mathcal{A}} = [f_i(x_{s(1)}, ..., x_{s(n_i)})]^{\mathcal{A}} := f_i^{\mathcal{A}}(x_{s(1)}^{\mathcal{A}}, ..., x_{s(n_i)}^{\mathcal{A}}) := (f_i^{\mathcal{A}})_s$.

(ii) If $t = f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})$ then
$t^{\mathcal{A}} = [f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})]^{\mathcal{A}} := f_i^{\mathcal{A}}(x_{s'(j_1)}^{\mathcal{A}}, ..., x_{s'(j_{n_i})}^{\mathcal{A}}) = (f_i^{\mathcal{A}})_{s'}$ where $x_{s'(j_k)}^{\mathcal{A}} := c_{a_k}^{n_i,A}$ is the n_i-ary constant operation on $\mathcal{A}$ with value a_k, and each element from $\mathcal{A}$ is uniquely induced by an element from $X \setminus X_{n_i}$, i.e. $f_i^{\mathcal{A}}(x_{s'(j_1)}^{\mathcal{A}}, ..., x_{s'(j_{n_i})}^{\mathcal{A}}) := f_i^{\mathcal{A}}(a_{s'(j_1)}, ..., a_{s'(j_{n_i})})$.

(iii) If $t = f_i(t_1, ..., t_{n_i})$ then $t^{\mathcal{A}} = [f_i(t_1, ..., t_{n_i})]^{\mathcal{A}} := f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}})$ where $t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}}$ are n_i-ary term operations which are induced by generalized full terms $t_1, ..., t_{n_i} \in W_\tau^{GF}(X)$.

Here the right hand side of the equation in (iii) means the n_i-ary operation defined by $f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}})(a_1, ..., a_{n_i}) := f^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, ..., a_{n_i}), ..., t_{n_i}^{\mathcal{A}}(a_1, ..., a_{n_i}))$ for every $(a_1, ..., a_{n_i}) \in A^{n_i}$. Let $\mathcal{T}_{GF}(\mathcal{A})$ be the set of all these term operations. On the set $\mathcal{T}_{GF}(\mathcal{A})$ we defined inductively an (n_i+1)-ary superposition operation $S^{n_i,A}$ by the following steps:

Definition 1.4 Let $s \in P_{n_i}$, $t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}}, p_1^{\mathcal{A}}, ..., p_{n_i}^{\mathcal{A}} \in T_{GF}(\mathcal{A})$ and $s^{'}$ be a permutation on $\{j_1, ..., j_{n_i}\}$ where $j_1, ..., j_{n_i}$ are natural numbers and greater than n_i, $S^{n_i,A} : T_{GF}(\mathcal{A})^{n_i+1} \to T_{GF}(\mathcal{A})$ is defined by

(i) $S^{n_i,A}(f_i^{\mathcal{A}}(x_{s(1)},...,x_{s(n_i)}),t_1^{\mathcal{A}},...,t_{n_i}^{\mathcal{A}}) := f_i^{\mathcal{A}}(t_{s(1)}^{\mathcal{A}},...,t_{s(n_i)}^{\mathcal{A}})$;

(ii) $S^{n_i,A}(f_i^{\mathcal{A}}(x_{s'(j_1)},...,x_{s'(j_{n_i})}),t_1^{\mathcal{A}},...,t_{n_i}^{\mathcal{A}}) := f_i^{\mathcal{A}}(x_{s'(j_1)}^{\mathcal{A}},...,x_{s'(j_{n_i})}^{\mathcal{A}})$;

(iii) $S^{n_i,A}(f_i^{\mathcal{A}}(p_1^{\mathcal{A}},...,p_{n_i}^{\mathcal{A}}),t_1^{\mathcal{A}},...,t_{n_i}^{\mathcal{A}}) := f_i^{\mathcal{A}}(S^{n_i,A}(p_1^{\mathcal{A}},t_1^{\mathcal{A}},...,$ $t_{n_i}^{\mathcal{A}}),...,S^{n_i,A}(p_{n_i}^{\mathcal{A}},t_1^{\mathcal{A}},...,t_{n_i}^{\mathcal{A}}))$.

This gives the algebra $\mathcal{T}_{GF}(\mathcal{A}) := (T_{GF}(\mathcal{A}); S^{n_i,A})$ called an n_i-ary generalized full (term) clone of an n_i-ary algebra $\mathcal{A}$. Let τ be a type. An equation of type τ is a pair (s,t) from $W_\tau(X)$; such pair is commonly written as $s \approx t$. An equation $s \approx t$ is an *identity* of an algebra $\mathcal{A}$, denoted by $\mathcal{A} \models s \approx t$ if $s^{\mathcal{A}} = t^{\mathcal{A}}$. The equation means that for every mapping $f : X \longrightarrow A$; we have $\hat{f}(s) = \hat{f}(t)$, where $\hat{f}$ is the uniquely determined extension of f.

2 Main Results

Using the new set of variables $\mathcal{X} = (Y_i)_{i\in I}$ indexed by an indexed set I, and an (n_i+1)-ary operation symbol $\tilde{S^{n_i}}$ we define a new language of type $\tau = (n_i+1)$ and consider equations formulated in this new language.

Proposition 2.1 The algebra $clone_{GF}\tau$ satisfies the so-called superassociative law (C)

$$\tilde{S^{n_i}}(X_0,\tilde{S^{n_i}}(Y_1,X_1,...,X_{n_i}),...,\tilde{S^{n_i}}(Y_{n_i},X_1,...,X_{n_i})) \approx \tilde{S^{n_i}}(\tilde{S^{n_i}}(X_0,Y_1,...,Y_{n_i}),X_1,...,X_{n_i}),$$

where $\tilde{S^{n_i}}$ is (n_i+1)-ary operation symbol and X_i, Y_j are variables.
Proof. We give a proof by induction on the complexity of a generalized full term t which is substituted for X_0.
Substituting for X_0 a generalized full term $t = f_i(x_{s(1)},...,x_{s(n_i)})$ for all $s \in P_{n_i}$, then $S^{n_i}(f_i(x_{s(1)},...,x_{s(n_i)}),S^{n_i}(t_1,s_1,...,s_{n_i}),...,S^{n_i}(t_{n_i},s_1,...,s_{n_i}))$

$$\begin{aligned}
&= f_i(S^{n_i}(x_{s(1)},S^{n_i}(t_1,s_1,...,s_{n_i}),...,S^{n_i}(t_{n_i},s_1,...,s_{n_i})),....,\\
&\quad S^{n_i}(x_{s(n_i)},S^{n_i}(t_1,s_1,...,s_{n_i}),...,S^{n_i}(t_{n_i},s_1,...,s_{n_i})))\\
&= f_i(S^{n_i}(t_{s(1)},s_1,...,s_{n_i}),...,S^{n_i}(t_{s(n_i)},s_1,...,s_{n_i}))\\
&= S^{n_i}(f_i(t_{s(1)},...,t_{s(n_i)}),s_1,...,s_{n_i})\\
&= S^{n_i}(S^{n_i}(f_i(x_{s(1)},...,x_{s(n_i)}),t_1,...,t_{n_i}),s_1,...,s_{n_i}).
\end{aligned}$$

If we substitute for X_0 a generalized full term $t = f_i(x_{s'(j_1)},...,x_{s'(j_{n_i})})$ where $j_1,...,j_{n_i}$ are natural numbers and greater than n_i and s' is a permutation on

$\{j_1, ..., j_{n_i}\}$, then
$S^{n_i}(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}), S^{n_i}(t_1, s_1, ..., s_{n_i}), ..., S^{n_i}(t_{n_i}, s_1, ..., s_{n_i}))$

$$\begin{aligned}
&= f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) \\
&= S^{n_i}(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}), s_1, ..., s_{n_i}) \\
&= S^{n_i}(f_i(S^{n_i}(x_{s'(j_1)}, t_1, ..., t_{n_i}), ..., S^{n_i}(x_{s'(j_{n_i})}, .t_1, ..., t_{n_i})), s_1, ..., s_{n_i}) \\
&= S^{n_i}(S^{n_i}(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}), t_1, ..., t_{n_i}), s_1, ..., s_{n_i}).
\end{aligned}$$

If we substitute for X_0 a generalized full term $t = f_i(r_1, ..., r_{n_i})$ and assume that (C) is satisfied for $r_1, ..., r_{n_i}$, then
$S^n(f_i(r_1, ..., r_{n_i}), S^{n_i}(t_1, s_1, ..., s_{n_i}), ..., S^{n_i}(t_{n_i}, s_1, ..., s_{n_i}))$

$$\begin{aligned}
&= f_i(S^{n_i}(r_1, S^{n_i}(t_1, s_1, ..., s_{n_i}), ..., S^{n_i}(t_{n_i}, s_1, ..., s_{n_i})), ..., \\
&\quad S^{n_i}(r_{n_i}, S^{n_i}(t_1, s_1, ..., s_{n_i}), ..., S^{n_i}(t_{n_i}, s_1, ..., s_{n_i}))) \\
&= S^{n_i}(f_i(S^{n_i}(r_1, t_1, ..., t_{n_i}), ..., S^{n_i}(r_{n_i}, t_1, ..., t_{n_i})), s_1, ..., s_{n_i}) \\
&= S^{n_i}(S^{n_i}(f_i(r_1, ..., r_{n_i})), t_1, ..., t_{n_i}), s_1, ..., s_{n_i}).
\end{aligned}$$

Proposition 2.2 The algebra $\mathcal{T}_{GF}(\mathcal{A})$ satisfies the superassociative law (C).
Proof. We give a proof by induction on the complexity of a term operation $t^{\mathcal{A}}$ on $\mathcal{A}$ which is substituted for X_0. Substituting for X_0 an n_i-ary term operation $t^{\mathcal{A}}$ of the form $t^{\mathcal{A}} = [f_i(x_{s(1)}, ..., x_{s(n_i)})]^{\mathcal{A}}$ for all $s \in P_{n_i}$, then
$S^{n_i,A}(f_i^{\mathcal{A}}(x_{s(1)}, ..., x_{s(n_i)}), S^{n_i,A}(t_1^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(t_{n_i}^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}))$

$$\begin{aligned}
&= f_i^{\mathcal{A}}(S^{n_i,A}(x_{s(1)}, S^{n_i,A}(t_1^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(t_{n_i}^{\mathcal{A}}, s_1, ..., s_{n_i}^{\mathcal{A}})), ..., \\
&\quad S^{n_i,A}(x_{s(n_i)}, S^{n_i,A}(t_1^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(t_{n_i}^{\mathcal{A}}, s_1, ..., s_{n_i}^{\mathcal{A}}))) \\
&= f_i^{\mathcal{A}}(S^{n_i,A}(t_{s(1)}^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(t_{s(n_i)}^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}})) \\
&= S^{n_i,A}(f_i^{\mathcal{A}}(t_{s(1)}^{\mathcal{A}}, ..., t_{s(n_i)}^{\mathcal{A}}), s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}) \\
&= S^{n_i,A}(S^{n_i,A}(f_i^{\mathcal{A}}(x_{s(1)}, ..., x_{s(n_i)}), t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}}), s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}).
\end{aligned}$$

If we substitute for X_0 an n_i-ary term operation $t^{\mathcal{A}}$ of the form
$t^{\mathcal{A}} = [f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})]^{\mathcal{A}}$ where $j_1, ..., j_{n_i}$ are natural numbers and greater than n_i and s' is a permutation on $\{j_1, ..., j_{n_i}\}$, then
$S^{n_i,A}(f_i^{\mathcal{A}}(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}), S^{n_i,A}(t_1^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(t_{n_i}^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}))$

$$\begin{aligned}
&= f_i^{\mathcal{A}}(x_{s'(j_1)}^{\mathcal{A}}, ..., x_{s'(j_{n_i})}^{\mathcal{A}}) \\
&= S^{n_i,A}(f_i^{\mathcal{A}}(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}), s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}) \\
&= S^{n_i,A}(f_i^{\mathcal{A}}(S^{n_i,A}(x_{s'(j_1)}, t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(x_{s'(j_{n_i})}, .t_1^{\mathcal{A}}, ..., t_{n_i})), s_1, ..., s_{n_i}^{\mathcal{A}}) \\
&= S^{n_i,A}(S^{n_i,A}(f_i^{\mathcal{A}}(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}), t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}}), s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}).
\end{aligned}$$

If we substitute for X_0 an n_i-ary term operation $t^{\mathcal{A}}$ of the form

$$t^{\mathcal{A}} = [f_i(r_1, ..., r_{n_i})]^{\mathcal{A}}$$

and assume that (C) is satisfied for $r_1^{\mathcal{A}}, ..., r_{n_i}^{\mathcal{A}}$, then
$S^{n_i,A}(f_i^{\mathcal{A}}(r_1^{\mathcal{A}}, ..., r_{n_i}^{\mathcal{A}}), S^{n_i,A}(t_1^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(t_{n_i}^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}))$

$$\begin{aligned}
&= f_i^{\mathcal{A}}(S^{n_i,A}(r_1^{\mathcal{A}}, S^{n_i,A}(t_1^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(t_{n_i}^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}})), ...,\\
&\quad S^{n_i,A}(r_{n_i}^{\mathcal{A}}, S^{n_i,A}(t_1^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(t_{n_i}^{\mathcal{A}}, s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}})))\\
&= S^{n_i,A}(f_i^{\mathcal{A}}(S^{n_i,A}(r_1^{\mathcal{A}}, t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}}), ..., S^{n_i,A}(r_{n_i}^{\mathcal{A}}, t_1^{\mathcal{A}}, ..., t_{n_i})), s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}})\\
&= S^{n_i,A}(S^{n_i,A}(f_i^{\mathcal{A}}(r_1^{\mathcal{A}}, ..., r_{n_i}^{\mathcal{A}})), t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}}), s_1^{\mathcal{A}}, ..., s_{n_i}^{\mathcal{A}}).
\end{aligned}$$

Let V_τ^{GFC} be the variety of type $\tau = (n_i+1)$ generated by the identity (C). Both algebras belong to this variety. Now let $\mathcal{F}_{V_\tau^{GFC}}(\{Y_l \mid l \in J\})$ be the free algebra with respect to V_τ^{GFC}, freely generated by an alphabet $\{Y_l \mid l \in J\}$ where $J = \{(i,s) \mid i \in I\ , s \in P_{n_i}\}$. The operation of $\mathcal{F}_{V_\tau^{GFC}}(\{Y_l \mid l \in J\})$ is denoted by $\tilde{S}^{n_i}$. Then we have:

Theorem 2.3 The algebra $clone_{GF}\tau$ is isomorphic to $\mathcal{F}_{V_\tau^{GFC}}(\{Y_l \mid l \in J\})$ and therefore free with respect to the variety V_τ^{GFC}, and freely generated by the set $\{f_i(x_{s(1)}, ..., x_{s(n_i)}) \mid i \in I, s \in P_{n_i}\} \cup \{f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) \mid j_1, ..., j_{n_i} > n_i, s'$ is a permutation on $\{j_1, ..., j_{n_i}\}\}$
Proof. We define the mapping $\varphi : W_\tau^{GF}(X) \longrightarrow \mathcal{F}_{V_\tau^{GFC}}(\{Y_l \mid l \in J\})$ inductively as follows:

(i) $\varphi(f_i(x_{s(1)}, ..., x_{s(n_i)}) = y_{(i,s)}$;

(ii) $\varphi(f_i(t_{s(1)}, ..., t_{s(n_i)})) = \tilde{S}^{n_i}(y_{(i,s)}, \varphi(t_1), ..., \varphi(t_{n_i}))$;

(iii) $\varphi(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})) = y_{(i,s')}$ where s' is a permutation on $\{j_1, ..., j_{n_i}\}$.

Since φ maps the generating system of $clone_{GF}\tau$ onto the generating system of $\mathcal{F}_{V_\tau^{GFC}}(\{Y_l \mid l \in J\})$, it is surjective. We prove the homomorphism property $\varphi(S^{n_i}(t_0, t_1, ..., t_{n_i})) = \tilde{S}^{n_i}(\varphi(t_0), \varphi(t_1), ..., \varphi(t_{n_i}))$ by induction on the complexity of the generalized full term t_0. If $t_0 = f_i(x_{s(1)}, ..., x_{s(n_i)})$, then

$$\begin{aligned}
\varphi(S^{n_i}(f_i(x_{s(1)}, ..., x_{s(n_i)}), t_1, ..., t_{n_i})) &= \varphi(f_i(t_{s(1)}, ..., t_{s(n_i)}))\\
&= \tilde{S}^{n_i}(y_{(i,s)}, \varphi(t_1), ..., \varphi(t_{n_i}))\\
&= \tilde{S}^{n_i}(\varphi(f_i(x_{s(1)}, ..., x_{s(n_i)})), \varphi(t_1), ..., \varphi(t_{n_i})).
\end{aligned}$$

If $t_0 = f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})$, then $\varphi(S^{n_i}(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}), t_1, ..., t_{n_i}))$

$$\begin{aligned}
&= \varphi(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}))\\
&= y_{(i,s')}\\
&= \tilde{S}^{n_i}(y_{(i,s')}, \varphi(t_1), ..., \varphi(t_{n_i}))\\
&= \tilde{S}^{n_i}(\varphi(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})), \varphi(t_1), ..., \varphi(t_{n_i})).
\end{aligned}$$

If $t_0 = f_i(r_1, ..., r_{n_i})$ and assume that
$\varphi(S^{n_i}(r_l, t_1, ..., t_{n_i})) = \tilde{S}^{n_i}(\varphi(r_l), \varphi(t_1), ..., \varphi(t_{n_i}))$ for all $1 \leq l \leq n_i$. Then
$$\begin{aligned}
\varphi(S^{n_i}(f_i(r_1, ..., r_{n_i}), t_1, ..., t_{n_i})) \\
&= \varphi(f_i(S^{n_i}(r_1, t_1, ..., t_{n_i}), ..., S^{n_i}(r_{n_i}, t_1, ..., t_{n_i}))) \\
&= \tilde{S}^{n_i}(y_{(i,id)}, \varphi(S^{n_i}(r_1, t_1, ..., t_{n_i})), ..., \\
&\quad \varphi(S^{n_i}(r_{n_i}, t_1, ..., t_{n_i}))) \\
&= \tilde{S}^{n_i}(y_{(i,id)}, \tilde{S}^{n_i}(\varphi(r_1), \varphi(t_1), ..., \varphi(t_{n_i})), ..., \\
&\quad \tilde{S}^{n_i}(\varphi(r_{n_i}), \varphi(t_1), ..., \varphi(t_{n_i}))) \\
&= \tilde{S}^{n_i}(\tilde{S}(y_{(i,id)}, \varphi(r_1), ..., \varphi(r_{n_i})), \varphi(t_1), ..., \varphi(t_{n_i})) \\
&= \tilde{S}^{n_i}(\varphi(f_i(r_1, .., r_{n_i})), \varphi(t_1), ..., \varphi(t_{n_i})).
\end{aligned}$$
Thus φ is a homomorphism. The mapping φ is bijective since $\{y_{(i,s)} \mid i \in I,\ s \in P_{n_i}\}$ is free independent set. Therefore we have
$$\begin{aligned}
y_{(i,s_1)} = y_{(j,s_2)} &\Longrightarrow (i, s_1) = (j, s_2) \\
&\Longrightarrow i = j,\ s_1 = s_2
\end{aligned}$$
and
$$\begin{aligned}
y_{(i,s_1')} = y_{(j,s_2')} &\Longrightarrow (i, s_1') = (j, s_2') \\
&\Longrightarrow i = j,\ s_1' = s_2'.
\end{aligned}$$
So $f_i(x_{s(1)}, ..., x_{s(n_i)}) = f_j(x_{s(1)}, ..., x_{s(n_i)})$ and $f_i(x_{s'(1)}, ..., x_{s'(n_i)}) = f_j(x_{s'(1)}, ..., x_{s'(n_i)})$. Thus φ is a bijection between the generating sets of $clone_{GF}\tau$ and $\mathcal{F}_{V_\tau^{GFC}}(\{Y_l \mid l \in J\})$ and therefore φ is an isomorphism.

Since the free algebra $clone_{GF}\tau$ is generated by the set $\{f_i(x_{s(1)}, ..., x_{s(n_i)}) \mid i \in I, s \in P_{n_i}\} \cup \{f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) \mid j_1, ..., j_{n_i} > n_i, s'$ is a permutation on $\{j_1, ..., j_{n_i}\}\}$. Therefore any mapping η from $\mathcal{F}_{GS_\tau}$ into $W_\tau^{GF}(X)$ can be uniquely extended to an endomorphism $\bar{\eta}$ of $clone_{GF}\tau$. The mappings are called generalized full clone substitutions. The set of all generalized full clone substitutions is denoted by $Subst_{GFC}$.

The set $Subst_{GFC}$ with a binary operation $\odot$ defined by $\eta_1 \odot \eta_2 := \bar{\eta_1} \circ \eta_2$ where $\circ$ is the usual composition of functions and together with $id_{\mathcal{F}_{GS_\tau}}$, the identity mapping on $\mathcal{F}_{GS_\tau}$ we conclude that $(Subst_{GFC}; \odot, id_{\mathcal{F}_{GS_\tau}})$ is a monoid. Next, we give the definition of a generalized full hypersubstitution and a generalized full hyperidentity and introduce some properties about generalized full hyperidentities and generalized full hypersubstitutions.

Definition 2.4 A generalized full hypersubstitution of type τ is mapping σ from the set $\{f_i \mid i \in I\}$ of n_i-ary operation symbols of the type τ to the set $W_\tau^{GF}(X)$ of all n_i-ary generalized full terms of type τ.

Any generalized full hypersubstitution σ induces a mapping $\hat{\sigma}$ defined on the set $W_\tau^{GF}(X)$ of all n_i-ary generalized full terms of type τ, as follows.

Definition 2.5 Let σ be a generalized full hypersubstitution of type τ and $s \in P_{n_i}$ and s' is a permutation on $\{j_1, ..., j_{n_i}\}$. Then σ induces a mapping $\hat{\sigma} : W_\tau^{GF}(X) \longrightarrow W_\tau^{GF}(X)$ by setting

(i) $\hat{\sigma}[f_i(x_{s(1)}, ..., x_{s(n_i)})] := (\sigma(f_i))_s$,

(ii) $\hat{\sigma}[f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})] := (\sigma(f_i))_{s'}$,

(iii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$.

Let $Hyp_{GF}(\tau)$ be the set of all generalized full hypersubstitutions of type τ. We define a binary operation $\circ_{GF}$ on $Hyp_{GF}(\tau)$ by $\sigma_1 \circ_{GF} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of functions. Together with the hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) := f_i(x_1, ..., x_{n_i})$, one has a monoid $(Hyp_{GF}(\tau); \circ_{GF}, \sigma_{id})$.

Let M be any submonoid of $Hyp_{GF}(\tau)$. If $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ is an n_i-ary algebra, then an identity $s \approx t$ in $\mathcal{A}$ is said to be an M- strong full hyperidentity in $\mathcal{A}$ if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in $\mathcal{A}$ for every generalized full hypersubstitution $\sigma \in M$. In the special case that M is all of $Hyp_{GF}(\tau)$, an M- strong full hyperidentity is usually called a strong full hyperidentity. An identity is an M- strong full hyperidentity of a variety V if it is an M-strong full hyperidentity of every algebra in V. A variety in which each of its identities holds as an M- strong full hyperidentity is called an M- full strongly solid variety, or a GF- strongly solid variety in the special case $M = Hyp_{GF}(\tau)$.

Proposition 2.6 The monoids $(Subst_{GF}; \odot, id)$ and $(Hyp_{GF}(\tau); \circ_{GF}, \sigma_{id})$ are isomorphic.

Proof. Firstly, we define a mapping $\psi : Subst_{GF} \longrightarrow Hyp_{GF}(\tau)$ by $\psi(\eta) := \eta\circ\sigma_{id}$. Then $\eta\circ\sigma_{id}$ is a generalized full hypersubstitution, so ψ is a well-defined mapping between $Subst_{GF}$ and $Hyp_{GF}(\tau)$. The mapping ψ is surjective, since any generalized full hypersubstitution σ can be obtained as $\psi(\eta)$ for $\eta = \sigma\circ\sigma^{-1}$. The mapping φ is also injective, since $\psi(\eta_1) = \psi(\eta_2) \Longrightarrow \eta_1 \circ \sigma_{id} = \eta_2 \circ \sigma_{id} \Longrightarrow \eta_1 = \eta_2$ since σ_{id} is a bijection.

Next, to show that ψ is a homomorphism, we first verify the following addition property:

$$(\eta \circ \sigma_{id})\hat{}\,[t] = \bar{\eta}(t) \qquad (*)$$

when $\bar{\eta}$ is the unique extension of η.

For a generalized full term $t = f_i(x_{s(1)}, ..., x_{s(n_i)})$ where $s \in P_{n_i}$, we have

$$\begin{aligned}(\eta \circ \sigma_{id})\hat{}\,[f_i(x_{s(1)}, ..., x_{s(n_i)})] &= (\eta \circ \sigma_{id}(f_i))_s \\ &= \eta(f_i(x_{s(1)}, ..., x_{s(n_i)})) \\ &= \bar{\eta}(f_i(x_{s(1)}, ..., x_{s(n_i)})).\end{aligned}$$

For a generalized full term $t = f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})$ where s' is a permutation on $\{j_1, ..., j_{n_i}\}$, we have

$$\begin{aligned}(\eta \circ \sigma_{id})\hat{}\,[f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})] &= (\eta \circ \sigma_{id}(f_i))_{s'} \\ &= \eta(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) \\ &= \bar{\eta}(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}).\end{aligned}$$

For a generalized full term $t = f_i(t_1, ..., t_{n_i})$ and assume that $(\eta\circ\sigma_{id})\hat{}[t_l] = \bar{\eta}(t_l)$ for all $1 \leq l \leq n_i$ and $t_1, ..., t_{n_i} \in W_\tau^{GF}(X)$, we have

$$\begin{aligned}
&(\eta \circ \sigma_{id})\hat{}\,[f_i(t_1, ..., t_{n_i})]\\
&\quad = S^{n_i}((\eta \circ \sigma_{id})(f_i), (\eta \circ \sigma_{id})\hat{}\,[t_1], ..., (\eta \circ \sigma_{id})\hat{}\,[t_{n_i}])\\
&\quad = S^{n_i}(\eta(f_i(x_1, ..., x_{n_i}), \bar{\eta}(t_1), ..., \bar{\eta}(t_{n_i}))\\
&\quad = S^{n_i}(\bar{\eta}(f_i(x_1, ..., x_{n_i}), \bar{\eta}(t_1), ..., \bar{\eta}(t_{n_i}))\\
&\quad = \bar{\eta}(S^{n_i}(f_i(x_1, ..., x_{n_i}), t_1, ..., t_{n_i}))\\
&\quad = \bar{\eta}(f_i(t_1, ..., t_{n_i}))\ .
\end{aligned}$$

For the homomorphism property of ψ we have

$$\begin{aligned}
\psi(\eta_1) \circ_{GF} \psi(\eta_2) &= (\eta_1 \circ \sigma_{id}) \circ_{GF} (\eta_2 \circ \sigma_{id})\\
&= (\eta_1 \circ \sigma_{id})\hat{}\ \circ (\eta_2 \circ \sigma_{id})\\
&= \bar{\eta}_1 \circ (\eta_2 \circ \sigma_{id}) && \text{by property } (*)\\
&= (\bar{\eta}_1 \circ \eta_2) \circ \sigma_{id} && \text{by associativity}\\
&= (\eta_1 \odot \eta_2) \circ \sigma_{id} && \text{by the definition of } \odot\\
&= \psi(\eta_1 \odot \eta_2).
\end{aligned}$$

Let $\mathcal{A}$ be any n_i-ary algebra. We define a mapping $g : \{f_i(x_{s(1)}, ..., x_{s(n_i)}) \mid i \in I, s \in P_{n_i}\} \cup \{f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) \mid j_1, ..., j_{n_i} > n_i, s'$ is a permutation on $\{j_1, ..., j_{n_i}\}\} \longrightarrow \{f_i^{\mathcal{A}} \mid i \in I\}$ by letting

$$g(f_i(x_{s(1)}, ..., x_{s(n_i)})) = (f_i^{\mathcal{A}})_s,$$

$$g(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) = f_i^{\mathcal{A}}(x^{\mathcal{A}}_{s'(j_1)}, ..., x^{\mathcal{A}}_{s'(j_{n_i})}) = (f_i^{\mathcal{A}})_{s'}$$

where $x^{\mathcal{A}}_{s'(j_k)} := c_{a_k}^{n_i,A}$ is the $n_i - ary$ constant operation on $\mathcal{A}$ with value a_k, and each element from $\mathcal{A}$ is uniquely induced by an element from $X\backslash X_{n_i}$, i.e. $f_i^{\mathcal{A}}(x^{\mathcal{A}}_{s'(j_1)}, ..., x^{\mathcal{A}}_{s'(j_{n_i})}) =: f_i^{\mathcal{A}}(a_{s'(j_1)}, ..., a_{s'(j_{n_i})})$.
Since $clone_{GF}\tau$ is free with respect to the variety V_τ^{GFC} and since $\mathcal{T}_{GF}(\mathcal{A})$ is an element of this variety, this mapping g has unique extension to a surjective homomorphism $\bar{g}$. It is clear that the mapping $\bar{g}$ assigns to each generalized full term $t \in W_\tau^{GF}(X)$ the induced generalized full term operation $t^{\mathcal{A}}$. We denote by $Id^{GF}(\mathcal{A})$ the set of all identities $s \approx t$ in $\mathcal{A}$ with $s, t \in W_\tau^{GF}(X)$. Such identities are called generalized full identities. Then we have

Theorem 2.7 Let $\mathcal{A}$ be an algebra of type τ and let $s \approx t \in Id^{GF}(\mathcal{A})$. Then $s \approx t$ is a generalized full hyperidentities in $\mathcal{A}$ iff $s \approx t$ is an identities in $\mathcal{T}_{GF}(\mathcal{A})$.

Proof. Firstly, we assume that $s \approx t$ is a generalized full hyperidentity of $\mathcal{A}$. This means that for every $\sigma \in Hyp_{GF}(\tau)$ we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{GF}(\mathcal{A})$; i.e. $\hat{\sigma}[s]^{\mathcal{A}} = \hat{\sigma}[t]^{\mathcal{A}}$ and thus $\bar{g}(\hat{\sigma}[s]) = \bar{g}(\hat{\sigma}[t])$. To show that $s \approx t$ holds in $\mathcal{T}_{GF}(\mathcal{A})$, we will show that $\bar{v}(s) = \bar{v}(t)$ for every valuation $v : \{f_i(x_{s(1)}, ..., x_{s(n_i)}) \mid i \in I, s \in P_{n_i}\} \cup \{f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) \mid j_1, ..., j_{n_i} > n_i, s'$ is a permutation on $\{j_1, ..., j_{n_i}\}\} \longrightarrow \mathcal{T}_{GF}(\mathcal{A})$. Since $\bar{g}$ is surjective, there exists a generalized full clone substitution η_v such that $v = \bar{g} \circ \eta_v$, using the axiom of choice. Then $\eta_v \circ \sigma_{id}$ is a generalized full hypersubstitution, which we shall denote by σ_v.

Then we have

$$\bar{v}(s) = (\bar{g} \circ \bar{\eta}_v)(s) = (\bar{g} \circ (\eta_v \circ \sigma_{id})\hat{\ })(s) = \bar{g}(\hat{\sigma}_v[s]).$$

Similarly, we have $\bar{v} = \bar{g}(\hat{\sigma}_v[t])$. By our assumption we have $\bar{g}(\hat{\sigma}_v[s]) = \bar{g}(\hat{\sigma}_v[t])$, we get $\bar{v}(s) = \bar{v}(t)$ as required.
Conversely, let $s \approx t \in Id\mathcal{T}_{GF}(\mathcal{A})$, so that $s, t \in W_\tau^{GF}(X)$ and for every valuation mapping v we have $\bar{v}(s) = \bar{v}(t)$. Let $\sigma \in Hyp_{GF}(\tau)$. By the surjectivity from Proposition 2.6 there is a generalized full clone substitution η_σ such that $\eta_\sigma \circ \sigma_{id} = \sigma$. We take v to be the valuation $\bar{g} \circ \bar{\eta}_\sigma$. Then

$$\hat{\sigma}[s]^{\mathcal{A}} = \bar{g}(\hat{\sigma}[s]) = (\bar{g} \circ (\eta_\sigma \circ \sigma_{id})\hat{\ })(s) = (\bar{g} \circ \bar{\eta}_\sigma)(s) = \bar{v}(s).$$

Similarly, we have $\hat{\sigma}[t]^{\mathcal{A}} = \bar{v}(t)$, and from our assumption that $\bar{v}(s) = \bar{v}(t)$, we get the desired equality.

Let $\mathcal{L}(\tau)$ be the lattice of all varieties of type τ. For a variety V of type τ we can form the variety $SGF_{n_i}^{A}(V)$ of type τ, determined by all n_i-ary generalized full identities of V.

Corollary 2.8 Let $\mathcal{A}$ be an algebra of type τ. Then the variety $SGF_{n_i}^{A}(V(\mathcal{A}))$ is GF-strongly solid iff $\mathcal{T}_{GF}(\mathcal{A})$ is free with respect to itself, freely generated by the set $\{f^{\mathcal{A}} \mid i \in I\}$, meaning that every mapping from $\{f^{\mathcal{A}} \mid i \in I\}$ to $\mathcal{T}_{GF}(\mathcal{A})$ can be extended to an endomorphism of $\mathcal{T}_{GF}(\mathcal{A})$.
Proof. For the converse direction we use Theorem 2.7 and we will show that $SGF_{n_i}^{A}(V(\mathcal{A}))$ is GF-strongly solid iff every identity $s \approx t \in IdSGF_{n_i}^{A}(V(\mathcal{A}))$ is also identity in $\mathcal{T}_{GF}(\mathcal{A})$. Suppose that $\mathcal{T}_{GF}(\mathcal{A})$ is free with respect to itself, freely generated by the set $\{f^{\mathcal{A}} \mid i \in I\}$. Let $s \approx t \in IdSGF_{n_i}^{A}(V(\mathcal{A}))$. Then $\bar{g}(s) = \bar{g}(t)$. To show that $s \approx t$ is an identity in $\mathcal{T}_{GF}(\mathcal{A})$, we will show that $\bar{v}(s) = \bar{v}(t)$ for any valuation mapping $v : \mathcal{F}_{GS_\tau} \longrightarrow \mathcal{T}_{GF}(\mathcal{A})$. For any valuation mapping v, we define a mapping $\alpha_v : \{f^{\mathcal{A}} \mid i \in I\} \longrightarrow \mathcal{T}_{GF}(\mathcal{A})$ by

$$\alpha_v((f_i^{\mathcal{A}})_s) = v(f_i(x_{s(1)}, ...x_{s(n_i)})) \text{ and}$$
$$\alpha_v((f_i^{\mathcal{A}})_{s'}) = v(f_i(x_{s'(j_1)}, ...x_{s'(j_{n_i})})).$$

Since $(f_i^{\mathcal{A}})_s = (f_l^{\mathcal{A}})_s \Longrightarrow i = l$

$$\Longrightarrow f_i(x_{s(1)}, ..., x_{s(n_i)}) = f_l(x_{s(1)}, ..., x_{s(n_i)})$$
$$\Longrightarrow v(f_i(x_{s(1)}, ..., x_{s(n_i)})) = v(f_l(x_{s(1)}, ..., x_{s(n_i)}))$$
$$\Longrightarrow \alpha_v((f_i^{\mathcal{A}})_s) = \alpha_v((f_l^{\mathcal{A}})_s)$$

and $(f_i^{\mathcal{A}})_{s'} = (f_l^{\mathcal{A}})_{s'} \Longrightarrow i = l$

$$\Longrightarrow f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}) = f_l(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})$$
$$\Longrightarrow v(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})) = v(f_l(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}))$$
$$\Longrightarrow \alpha_v((f_i^{\mathcal{A}})_{s'}) = \alpha_v((f_l^{\mathcal{A}})_{s'})$$

the mapping α_v is well defined. Since the set $\mathcal{F}_{GS_\tau}$ generates the algebra $clone_{GF}\tau$, the mapping v can be uniquely extended to $\bar{v}$ on the set $W_\tau^{GF}(X)$. Then we have $\bar{g}(s) = \bar{g}(t) \Longrightarrow \bar{\alpha}_v(\bar{g}(s)) = \bar{\alpha}_v(\bar{g}(t)) \Longrightarrow \bar{v}(s) = \bar{v}(t)$. Thus $s \approx t \in IdT_{GF}(\mathcal{A})$. Assume that $SGF_{n_i}^{A}(V(\mathcal{A}))$ is GF-strongly solid. We will show that any mapping $\alpha : \{f^{\mathcal{A}} \mid i \in I\} \longrightarrow \mathcal{T}_{GF}(\mathcal{A})$ can be extended to an endomorphism of $\mathcal{T}_{GF}(\mathcal{A})$. We consider the mapping $\bar{\alpha} = \overline{\alpha \circ g} : W_\tau^{GF}(X) \longrightarrow \mathcal{T}_{GF}(\mathcal{A})$

with $\bar{\alpha}(t^{\mathcal{A}}) = \overline{\alpha \circ g}(t)$, which is a valuation of terms. Then for any terms $s, t \in W_{\tau}^{GF}(X)$, it follows from $s^{\mathcal{A}} = t^{\mathcal{A}}$ that $\bar{g}(s) = \bar{g}(t)$ and thus $\bar{\alpha}(s^{\mathcal{A}}) = \bar{\alpha}(\bar{g}(s)) = \bar{\alpha}(\bar{g}(t)) = \bar{\alpha}(t^{\mathcal{A}})$, since $\bar{\alpha} \circ \bar{g}$ is a valuation and every identity of $SGF_{n_i}^{\mathcal{A}}(V(\mathcal{A}))$ is a $clone_{GF}\tau$-identity. This shows that $\bar{\alpha}$ is well defined. It is also an endomorphism since $\bar{\alpha}(S^{n_i,\mathcal{A}}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, ..., t_{n_i}^{\mathcal{A}})) = \bar{\alpha}(\bar{g}(S^{n_i}(s, t_1, ..., t_{n_i}))) = (\bar{\alpha} \circ \bar{g})(S^{n_i}(s, t_1, ..., t_{n_i})) = S^{n_i,\mathcal{A}}(\bar{\alpha}\circ\bar{g}(s), \bar{\alpha}\circ\bar{g}(t_1), ..., \bar{\alpha}\circ\bar{g}(t_{n_i})) = S^{n_i,\mathcal{A}}(\bar{\alpha}(s^{\mathcal{A}}), \bar{\alpha}(t_1^{\mathcal{A}}), ..., \bar{\alpha}(t_{n_i}^{\mathcal{A}}))$ using the fact that $\bar{\alpha} \circ \bar{g}$ is the homomorphism extending the valuation $\alpha \circ g$ define on the generating set of the free algebra $clone_{GF}\tau$. Finally, $\bar{\alpha}$ extends α since $\bar{\alpha}((f_i^{\mathcal{A}})_s) = \overline{\alpha \circ g}(f_i(x_{s(1)}, ..., x_{s(n_i)})) = (\alpha\circ g)(f_i(x_{s(1)}, ..., x_{s(n_i)}))$ $= \alpha(g(f_i(x_{s(1)}, ..., x_{s(n_i)}))) = \alpha((f_i^{\mathcal{A}})_s)$
and $\bar{\alpha}((f_i^{\mathcal{A}})_{s'}) = \overline{\alpha \circ g}(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}))(\alpha \circ g)(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})})) =$
$\alpha(g(f_i(x_{s'(j_1)}, ..., x_{s'(j_{n_i})}))) = \alpha((f_i^{\mathcal{A}})_{s'})$ for each $i \in I$.

3 Open Problem

Problem: Let V be a variety of type τ and $Id^{GF}V := W_{\tau}^{GF}(X)^2 \cap Id\ V$ be the set of all identities of V consisting of n_i-ary generalized full terms. Prove that $Id^{GF}V$ is a congruence on $clone_{GF}\tau$.

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