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Generalizations of Prime Ideals in Near-rings

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Abstract :
The purpose of this paper is to define the notions of almost prime ideals in near rings and explore various properties of almost prime ideals and their generalizations in near-rings

Keywords : *Near-ring, prime ideal, m-system, weakly prime ideal.*

1 Introduction

Throughout this paper, N denotes a zero-symmetric near-ring. For any $x \in N$, $< x >$ denote the ideal of N generated by x. For any subsets A, B of N, we denote $(A : B) = \{n \in N \ : \ nB \subseteq A\}$. For basic terminology in near-ring we refere to Pilz [7].

Almost prime ideals arise from the study of factoriation in Notherian domains. They were introduced by S.M. Bhatwadekar and P. K. Sharma in [5]. Weakly prime ideals arise from the study of factorization in commutative rings with zero-divisors. They were studied by A. G. Agargun, et al. in [1] and later studied by D.D. Anderson and E. Smith [3].

A proper ideal P of a commutative ring is almost prime if $ab \in P \backslash P^2$ implies $a \in P$ or $b \in P$. A proper ideal P of a ring R is weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. In [6], P. Dheena and B. Elavarasan extended the notion of weakly primes to near-rings (not necessarily commutative). Following [6], a proper ideal P of N to be weakly prime if $\{0\} \neq AB \subseteq P,$ A and B are ideals of N, implies $A \subseteq P$ or $B \subseteq P$.

In this paper, we define the notion of almost prime ideal in near-ring (not necessarily commutative). A proper ideal P of N is said to be almost prime if for any ideals A and B of N such that $AB \subseteq P$ and $AB \nsubseteq P^2$, we have $A \subseteq P$ or $B \subseteq P$. As every prime ideal is a weakly prime, and a weakly prime ideal is an almost prime, weakly prime ideals and almost prime ideals are both generalizations of prime ideals. However, $\{0\}$ is always a weakly prime ideal and hence almost prime. We next give a non-trivial example of an almost prime ideal which is neither prime nor weakly prime.

Example 1.1 *Let* $N = \{0, a, b, c, d, e, f, 1\}$*. Define addition and multiplication in* N *as follows:*

+	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	0	c	b	e	d	1	f
b	b	c	0	a	f	1	d	e
c	c	b	a	0	1	f	e	d
d	d	e	f	1	0	a	b	c
e	e	d	1	f	a	0	c	b
f	f	1	d	e	b	c	0	a
1	1	f	e	d	c	b	a	0

.	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	a	0	a
b	0	0	b	b	0	0	b	b
c	0	0	b	b	0	a	b	c
d	0	a	0	a	d	d	d	d
e	0	a	0	a	d	e	d	e
f	0	a	b	c	d	d	f	f
1	0	a	b	c	d	e	f	1

Then $(N, +, .)$ *is a near-ring (see[4], Library Near-ring (8/3,696)). Here* $\{0, b\}$ *is an almost prime ideal of* N*, but not a weakly prime, since* $\{0\} \neq \{0, a, b, c\}^2 \subseteq \{0, b\}$*.* □

It is easy to verify that for any near-ring N, P is an almost prime ideal of N if and only if $P/ < P^2 >$ is a weakly prime ideal of $N/ < P^2 >$. Also, if P is almost prime, then P/I is a prime ideal of N/I for any ideal $I \subseteq P$.

2 Main Results

Theorem 2.1 *Let* N *be a near-ring with identity and* P *an almost prime ideal of* N*. If* P *is not prime, then* $P^2 = P$*.*

Proof: suppose that $P \nsubseteq P^2$. We show that P is prime. Let A and B be ideals of N such that $AB \subseteq P$. If $AB \nsubseteq P^2$, then $A \subseteq P$ or $B \subseteq P$. So assume that $AB \subseteq P^2$. Since $P \nsubseteq P^2$, there exist $p \in P$ such that $< p > \nsubseteq P^2$, so $(A+ < p >)(B + N) \nsubseteq P^2$. Suppose $(A+ < p >)(B + N) \nsubseteq P$. Then there exist $a \in A; b \in B$ and $p_0 \in < p >$ and $q_0 \in N$ such that $(a + p_0)(b + q_0) \notin P$ which implies $a(b + q_0) \notin P$, but $a(b + q_0) = a(b + q_0) - ab + ab \in P$ since $AB \subseteq P$, a contradiction. So $(A+ < p >)(B + N) \nsubseteq P$ which implies $A \subseteq P$. □

Remark 2.2 *The above theorem shows the relationship between prime and almost prime ideals in near-rings. From the above theorem, we have the following corollary.*

Corollary 2.3 *Let N be a near-ring with identity and P an ideal of N. If $P^2 \neq P$, then P is prime if and only if P is almost prime.*

Lemma 2.4 *Let P be a nonzero proper ideal of N and if P is almost prime and $(P^2 : P) \subseteq P$, then P is prime.*

Proof: Suppose that P is not a prime ideal of N. Then there exist $x \notin P$ and $y \notin P$ such that $< x >< y > \subseteq P$. If $< x >< y > \nsubseteq P^2$, we are done. So $< x >< y > \subseteq P^2$. Consider $< x > (< y > +P) \subseteq P$. If $< x > (< y > +P) \nsubseteq P^2$, then we have $x \in P$ or $y \in P$, a contradiction. Otherwise $< x > (< y > +P) \subseteq P^2$. Then $< x > P \subseteq P^2$ implies $x \in (P^2 : P) \subseteq P$. $\square$

The next two theorems gives the equivalent conditions for an ideal to be almost prime.

Theorem 2.5 *Let N be a near-ring and P an ideal of N. Then the following are equivalent:*

i) For any $a, b, c \in N$ with $a(< b > + < c >) \subseteq P$, and $a(< b > + < c >) \nsubseteq P^2$, we have $a \in P$ or b, c in P.

ii) For $x \in N \backslash P$, we have $(P :< x > + < y >) = P \cup (P^2 :< x > + < y >$ for any $y \in N$.

iii) For $x \in N \backslash P$, we have $(P :< x > + < y >) = P$ or $(P :< x > + < y >) = (P^2 :< x > + < y >)$ for any $y \in N$.

iv) P is almost prime.

Proof: $(i) \Rightarrow (ii)$ Let $t \in (P :< x > + < y >$ for any $x \in N \backslash P$ and $y \in N$. Then $t(< x > + < y >) \subseteq P$. If $t(< x > + < y >) \subseteq P^2$, then $t \in (P^2 :< x > + < y >)$. Otherwise $t(< x > + < y >) \nsubseteq P^2$. Then $t \in P$ by hypothesis. $(ii) \Rightarrow (iii)$ follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them. $(iii) \Rightarrow (iv)$ Let A and B be ideals of N such that $AB \subseteq P$ and suppose $A \nsubseteq P$ and $B \nsubseteq P$. Then there exist $a \in A$ and $b \in B$ with $a, b \notin P$. Now we claim that $AB \nsubseteq P^2$.

Let $b_1 \in B$. Then $A(< b > + < b_1 >) \subseteq P$ which implies $A \subseteq (P :< b > + < b_1 >)$. Then by assumption, $A(< b > + < b_1 >) \subseteq P^2$ which gives $Ab_1 \subseteq P^2$. Thus $AB \subseteq P^2$ and hence P is an almost prime ideal of N. $(iv) \Rightarrow (i)$ is clear. $\square$

Theorem 2.6 *Let N be a near-ring and P an ideal of N. Then the following are equivalent:*

i) P is almost prime.

ii) For any ideals $I,\ J$ of N with $P \subset I$ and $P \subset J$, we have either $IJ \subseteq P^2$ or $IJ \nsubseteq P$.

iii) For any ideals $I,\ J$ of N with $I \nsubseteq P$ and $J \nsubseteq P$, we have $IJ \subseteq P^2$ or $IJ \nsubseteq P$.

Proof: $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are clear. $(ii) \Rightarrow (iii)$ Let I and J be ideals of N with $I \nsubseteq P$ and $J \nsubseteq P$. Then there exist $i_1 \in I$ and $j_1 \in J$ such that $i_1, j_1 \notin P$.

Suppose that $< i >< j > \nsubseteq P^2$ for some $i \in I$ and some $j \in J$. Then $(< i > + < i_1 > + P)(< j > + < j_1 > + P) \nsubseteq P^2$ and $P \subset < i > + < i_1 > + P; P \subset < j > + < j_1 > + P$. By hypothesis, $(< i > + < i_1 > + P)(< j > + < j_1 > + P) \nsubseteq P$ which implies $< i > (< j > + < j_1 > + P) + < i_1 > (< j > + < j_1 > + P) \nsubseteq P$. So there exist $i' \in < i >; i'_1 \in < i_1 >; j', j'' \in < j >; j'_1, j''_1 \in < j_1 >$ and $p_1, p_2 \in P$ such that $i'(j' + j'_1 + p_1) + i'_1(j'' + j''_1 + p_2) \notin P$. Therefore $i'(j'+j'_1+p_1) - i'(j'+j'_1) + i'(j'+j'_1) + i'_1(j''+j''_1+p_2) - i'_1(j''+j''_1) + i'_1(j''+j''_1) \notin P$. But since $i'(j' + j'_1 + p_1) - i'(j' + j'_1) \in P$ and $i'_1(j'' + j''_1 + p_2) - i'_1(j'' + j''_1) \in P$, we have P does not contain either $i'(j' + j'_1)$ or $i'_1(j'' + j''_1)$ which shows that $IJ \nsubseteq P$. □

Theorem 2.7 *Let N_1 and N_2 be any two near-rings with identity and P a proper ideal of N_1. Then P is almost prime if and only if $(P \times N_2)$ is an almost prime ideal of $N_1 \times N_2$.*

Proof: Let P be an almost prime ideal of N_1 and let $(A_1 \times B_1)$ and $(A_2 \times B_2)$ be ideals of $N_1 \times N_2$ such that $(A_1 \times B_1)(A_2 \times B_2) \subseteq (P \times N_2)$ and $(A_1 \times B_1)(A_2 \times B_2) \nsubseteq (P \times N_2)^2$. Then $(A_1A_2 \times B_1B_2) \subseteq (P \times N_2)$ and $(A_1A_2 \times B_1B_2) \nsubseteq (P^2 \times N_2)$, so $A_1A_2 \subseteq P$ and $A_1A_2 \nsubseteq P^2$ which implies $A_1 \subseteq P$ or $A_2 \subseteq P$. Conversely, suppose that $(P \times N_2)$ is an almost prime ideal of $N_1 \times N_2$ and let I and J be ideals of N_1 such that $IJ \subseteq P$ and $IJ \nsubseteq P^2$. Then $(I \times N_2)(J \times N_2) \subseteq (P \times N_2)$ and $(I \times N_2)(J \times N_2) \nsubseteq (P \times N_2)^2$. By assumption, we have $(I \times N_2) \subseteq (P \times N_2)$ or $(J \times N_2) \subseteq (P \times N_2)$. So $I \subseteq P$ or $J \subseteq P$. □

Theorem 2.8 *Let N_1 and N_2 be any two near-rings with identity. Then an ideal of $N_1 \times N_2$ is almost prime if and only if it has one of the following three forms.*

i) $(I \times N_2)$, where I is an almost prime ideal of N_1.

ii) $(N_1 \times J)$, where J is an almost prime ideal of N_2.

iii) $(I \times J)$, where I is an idempotent ideal of N_1 and J is an idempotent ideal of N_2.

Proof: Let P be an almost prime ideal of $N_1 \times N_2$. If P is of the form $(I \times N_2)$ or of the form $(N_1 \times J)$, where I and J are proper ideals of N_1 and N_2, respectively, then we can quote Theorem 2.7. Let $P = (I \times J)$ with I and J are the proper ideals of N_1 and N_2, respectively. We now claim that I is idempotent. Suppose $a \in I - I^2$. Then $(< a > \times \{0\}) \subseteq P$ and $(< a > \times \{0\}) \nsubseteq P^2$. This implies that either $(< a > \times N_2) \subseteq P$ or $(N_1 \times \{0\}) \subseteq P$. If $(< a > \times N_2) \subseteq P$, then $1 \in J$ and if $(N_1 \times \{0\}) \subseteq P$, then $1 \in I$. This contradicts I and J being proper ideals. So $I - I^2$ is empty and hence I is idempotent. Similarly J is idempotent. So $P = (I \times J)$ is idempotent and hence P is almost prime. □

Corollary 2.9 *Let N_1 and N_2 be any two near-rings with identity and P a ideal of $N_1 \times N_2$. If P is not prime, then P is almost prime if and only if $P^2 = P$.*

Proof: It is clear from Theorem 2.1, 2.7 and 2.8. □

Lemma 2.10 *Let N_1 and N_2 be two near-rings with identity. If every proper ideal of N_1 and N_2 is a product of almost prime ideals, then every proper ideal of $N_1 \times N_2$ is a product of almost prime ideals.*

Proof: Let I and J be a proper ideals of N_1 and N_2, respectively, where $I = A_1...A_n$ and $J = B_1...B_m$ with each A_i and B_j almost prime. If the proper ideal of $N_1 \times N_2$ is of the form $I \times N_2$, then we can write $I \times N_2 = A_1...A_n \times N_2 = (A_1 \times N_2)...(A_n \times N_2)$ which is a product of almost prime ideals by Theorem 2.7. Now if the proper ideals is of the form $N_1 \times J$, then in a similar way we get that it is a product of almost prime ideals. If the proper ideals is of the form $(I \times J)$, then we can also write it as $A_1...A_n \times B_1...B_m = (A_1...A_n \times N_2)(N_1 \times B_1...B_m) = (A_1 \times N_2)...(A_n \times N_2)(N_1 \times B_1)...(N_1 \times B_m)$. Thus we get a product of almost prime ideals. □

Lemma 2.11 *Let P be an almost prime ideal of N and if $\overline{A}$ is an ideal of N/I with $\overline{A}\ \overline{B} = \{\overline{0}\}$ for some non zero ideal $\overline{B}$ of N/P. Then either $A \subseteq P$ or $PB \subseteq P^2$.*

Proof: Assume that $A \nsubseteq P$ and let $x \in P$. Then $(< x > +A) \nsubseteq P$ and $(< x > +A)B \subseteq P$ which implies $(< x > +A)B \subseteq P^2$ as P is almost prime. Thus $< x > B \subseteq P^2$ and hence $PB \subseteq P^2$. □

Theorem 2.12 *Let N be a near-ring with unique maximal ideal M and suppose P is an ideal of N with $M^2 \subseteq P \subseteq M$. Then P is almost prime if and only if $M^2 = P^2$.*

Proof: Let P be an almost prime ideal of N and for any $x, y \in M$, we have $< x >< y >\subseteq M^2 \subseteq P$. Now we claim that $< x >< y >\subseteq P^2$. If not, then as P is almost prime, we have $x \in P$ or $y \in P$. Let $x \in P$. Then $y \notin P$, since otherwise $< x >< y >\subseteq P^2$. Since $< y >^2\subseteq M^2 \subseteq P$ and by Lemma 2.11, we have $< x >< y >\in P < y >\subseteq P^2$. This shows that $M^2 \subseteq P^2$. Remaining parts are trivial. □

A subset M of N is called almost m-system if $M \cap A \neq \phi$ and $M \cap B \neq \phi$ for any ideals A, B of N, then either $AB \cap M \neq \phi$ or $AB \neq (N \backslash M)^2$. It is easy to verify that an ideal $P \subset N$ is almost prime if and only if $N \backslash P$ is m-system. A well known result that, if M is non-void m-system of N and I is an ideal of N with $I \cap M = \phi$, then there exists a prime ideal $P \neq N$ containing I with $P \cap M = \phi$. The similar result does not hold for almost m-system M. For example, let $N = Z_{16},\ I = \{\overline{0}, \overline{8}\}$ and $M = \{\overline{1}, \overline{4}\}$. Then M is an almost m-system and I is an ideal of N with respect to $I \cap M = \phi$, but there exists no almost prime ideal P containing I with $P \cap M = \phi$.

A proper ideal P of N is said to be almost semi prime if for any ideal A of N such that $A^2 \subseteq P$ and $A^2 \nsubseteq P^2$, we have $A \subseteq P$. Clearly every almost prime ideal is almost semi prime, but the converse need not be true in general as in Z_{16}, $\{\overline{0}, \overline{8}\}$ is almost semiprime ideal, but not almost prime. A subset S of N is called almost n-system if $S \cap A \neq \phi$ for any ideal A of N, then either $A^2 \cap S \neq \phi$ or $A^2 \subseteq (N \backslash S)^2$. It is clear that an ideal P of N is almost semi prime if and only if $N \backslash P$ is almost n-system.

3 Open Problem

It is well known that, if S is a n- system of N and let $s \in S$. Then there is some m-system M of N with $s \in M \subseteq S$. Does the similar result holds for almost n-system?

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