Int. J. Open Problems Compt. Math., Vol. 4, No. 4, December 2011
ISSN 1998-6262;
www.i-csrs.org

Equivalent fuzzy strong n-inner product space

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Abstract

The purpose of this paper is to introduce the notion of equivalent fuzzy strong n-inner product space and to provide some results on it.

AMS Subject Classification : 46S40, 03E72.
Keywords: n-inner product, fuzzy n-inner product, fuzzy strong n-inner product, α-n-inner product.

1 Introduction

Significant contribution in the theory of 2-inner product space and n-inner product space have been made by eminent researchers in $[3, 4, 5, 6, 7]$. Recently S.Vijayabalaji and N.Thillaigovindan [9] have introduced the notion of fuzzy n-inner product space. As a natural generalization of fuzzy n-inner product space, the notion of fuzzy strong n-inner product space has been introduced in [10].

S.Vijayabalaji and N.Thillaigovindan [9] raised a problem of constructing α-n-inner product space and answer to this problem is provided in [10] by constructing α- strong n-inner product space.

Analogue of α-n-inner product space [9] the notion of α-strong n-inner product space is introduced in [10].

In this paper, for a given two fuzzy strong n-inner product spaces we define their equivalent conditions and provide some interesting reults on it.

2 Preliminaries

In this section we recall some concepts which will be needed in the sequel.

Definition 2.1 [2]**.** Let n be a natural number greater than 1 and X be a real linear space of dimension greater than or equal to n and let $(\bullet, \bullet|\bullet, ..., \bullet)$ be a real valued function on $\underbrace{X \times ... \times X}_{n+1} = X^{n+1}$ satisfying the following conditions :

(1) $(i)(x, x|x_2, ..., x_n) \geq 0$,
$(ii)(x, x|x_2, ..., x_n) = 0$ if any only if $x, x_2, ..., x_n$ are linearly dependent,
(2) $(x, y|x_2, ..., x_n) = (y, x|x_2, ..., x_n)$,
(3) $(x, y|x_2, ..., x_n)$ is invariant under any permutation of $x_2, ..., x_n$,
(4) $(x, x|x_2, ..., x_n) = (x_2, x_2|x, x_3, ..., x_n)$,
(5) $(ax, x|x_2, ..., x_n) = a(x, x|x_2, ..., x_n)$ for every $a \in R$(real),
(6) $(x + x', y|x_2, ..., x_n) = (x, y|x_2, ..., x_n) + (x', y|x_2, ..., x_n)$.
Then $(\bullet, \bullet|\bullet, ..., \bullet)$ is called an n-inner product on X and $(X, (\bullet, \bullet|\bullet, ..., \bullet))$ is called an n-inner product space.

Definition 2.2.[9] Let X be a linear space over a field F. A fuzzy subset $J : X^{n+1} \times R\,(R-$set of real numbers) is called a fuzzy n-inner product on X if and only if:
(1) For all $t \in R$ with $t \leq 0$, $J(x, x|x_2, ..., x_n, t) = 0$;
(2) For all $t \in R$ with $t > 0$, $J(x, x|x_2, ..., x_n, t) = 1$ if and only if $x, x_2, ..., x_n$ are linearly dependent;
(3) For all $t > 0$, $J(x, y|x_2, ..., x_n, t) = J(y, x|x_2, ..., x_n, t)$;
(4) $J(x, y|x_2, ..., x_n, t)$ is invariant under any permutation of $x_2, ..., x_n$;
(5) For all $t > 0$, $J(x, x|x_2, ..., x_n, t) = J(x_2, x_2|x, x_3, ..., x_n, t)$;
(6) For all $t > 0$, $J(ax, bx|x_2, ..., x_n, t) = J(x, x|x_2, ..., x_n, \frac{t}{|ab|})$, $a, b \in R$(real);
(7) For all $s, t \in R$,
$J(x + x', y|x_2, ..., x_n, t + s) \geq \min\{J(x, y|x_2, ..., x_n, t), J(x', y|x_2, ..., x_n, s)\}$;
(8) For all $s, t \in R$ with $s > 0$, $t > 0$,
$J(x, y|x_2, ..., x_n, \sqrt{ts}) \geq \min\{J(x, x|x_2, ..., x_n, t), J(y, y|x_2, ..., x_n, s)\}$;
(9) $J(x, y|x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and
$\lim_{t\to\infty} J(x, y|x_2, ..., x_n, t) = 1$.

Then (X, J) is called a fuzzy n-inner product space or in short f-n-IPS.

Definition 2.3 [10] Let X be a linear space over a field F. A fuzzy subset $J : X^{n+1} \times R\,(R-$set of real numbers) is called a fuzzy strong n-inner product on X if and only if:
(1) For all $t \in R$ with $t \leq 0$, $J(x,x|x_2,...,x_n,t) = 0$;
(2) For all $t \in R$ with $t > 0$, $J(x,x|x_2,...,x_n,t) = 1$ if and only if $x, x_2, ..., x_n$

are linearly dependent;
(3) For all $t > 0$, $J(x,y|x_2,...,x_n,t) = J(y,x|x_2,...,x_n,t)$;
(4) $J(x,y|x_2,...,x_n,t)$ is invariant under any permutation of $x_2,...,x_n$;
(5) For all $t > 0$, $J(x,x|x_2,...,x_n,t) = J(x_2,x_2|x,x_3,...,x_n,t)$;
(6) For all $t > 0$, $J(ax,bx|x_2,...,x_n,t) = J(x,x|x_2,...,x_n,\frac{t}{|ab|})$, $a, b \in R$(real);
(7) For all $s, t \in R$,

$J(x + x', y|x_2,...,x_n,t+s) = \min\{J(x,y|x_2,...,x_n,t), J(x',y|x_2,...,x_n,s)\}$;
(8) For all $s, t \in R$ with $s > 0$, $t > 0$,

$J(x,y|x_2,...,x_n,\sqrt{ts}) = \min\{J(x,x|x_2,...,x_n,t), J(y,y|x_2,...,x_n,s)\}$;
(9) $J(x,y|x_2,...,x_n,t)$ is a non-decreasing function of $t \in R$ and

$\lim_{t\to\infty} \mathrm{J}(x,y|x_2,...,x_n,t)$=1.
Then (X, J) is called a fuzzy strong n-inner product space or in short f-ST-n-IPS.

Example 2.4.[10] Let $(X, (\bullet, \bullet|\bullet, ..., \bullet))$ be an n-inner product space. Define

$$J(x,y|x_2,...,x_n,t) = \begin{cases} \dfrac{t}{t + |(x,y|x_2,...,x_n)|}, & \text{when } t > 0, t \in R, \\ & (x,y|x_2,...,x_n) \in X^{n+1} \\ 0, & \text{when } t \leq 0. \end{cases}$$

Then (X, J) is a f-ST-n-IPS.

Theorem 2.5 [10]. Let (X,J) be a f-ST-n-IPS. Assume the condition that (10) $J(x,x|x_2,...,x_n,t) > 0$ implies $x, x_2, ..., x_n$ are linearly dependent.
Define $(x,x|x_2,...,x_n)_\alpha$=inf$\{t :$ J$(x,x|x_2,...,x_n,t) \geq \alpha\}$,$\alpha \in (0,1)$.
Then $\{(\bullet,\bullet|\bullet,...,\bullet)_\alpha : \alpha \in (0,1)\}$,is an ascending family of strong n-inner products on X. We call these n-inner products as strong α-n-inner product on X corresponding to the fuzzy strong n- inner product on X.

Remark 2.6. [10] We assume that (11) For $x_1, x_2, ..., x_n$ linearly independent, $J(x,y|x_2,...,x_n,t)$ is continuous functions and strictly increasing on the subset $\{t : 0 < J(x,y|x_2,...,x_n,t) < 1\}$ of R.

3 Equivalent fuzzy strong n-inner product spaces

We now enter into our new notion of equivalent f-ST-n-IPS as follows.

Definition 3.1. Let $A = (X, J_1)$ and $B = (X, J_2)$ be two f-ST-n-IPS. Then A and B are said to be equivalent if there exists positive constants a and b such that
$J_2(x, y|x_2, ..., ax_n, t) \leq J_1(x, y|x_2, ..., x_n, t) \leq J_2(x, y|x_2, ..., bx_n, t)$
$\forall t \in R$. We denote it by $J_1 \sim J_2$.

Example 3.2. Let $(X, (\bullet, \bullet|\bullet, ..., \bullet))$ be an n-inner product space. Define

$$J_1(x, y|x_2, ..., x_n, t) = \begin{cases} \dfrac{k_1 t}{k_1 t + |(x, y|x_2, ..., x_n)|}, & \text{when } t > 0, t \in R, \\ & (x, y|x_2, ..., x_n) \in X^{n+1} \\ 0, & \text{when } t \leq 0. \end{cases}$$

Then $A = (X, J_1)$ is a f-ST-n-IPS.
Also define

$$J_2(x, y|x_2, ..., x_n, t) = \begin{cases} \dfrac{k_2 t}{k_2 t + |(x, y|x_2, ..., x_n)|}, & \text{when } t > 0, t \in R, \\ & (x, y|x_2, ..., x_n) \in X^{n+1} \\ 0, & \text{when } t \leq 0. \end{cases}$$

Then $B = (X, J_2)$ is a f-ST-n-IPS.
Choose $k_1 < k_2$ and $a > b$, where $k_1, k_2, a, b > 0$.
Then A and B are equivalent f-ST-n-IPS.

Theorem 3.3. The relation $\sim$ defined above is an equivalence relation.

Proof. (i) The relation is reflexive, since
$J_2(x, y|x_2, ..., 1.x_n, t) \leq J_1(x, y|x_2, ..., x_n, t) \leq J_2(x, y|x_2, ..., 1.x_n, t)\ \forall t \in R.$
(ii) To prove symmetry, let
$J_2(x, y|x_2, ..., ax_n, t) \leq J_1(x, y|x_2, ..., x_n, t) \leq J_2(x, y|x_2, ..., bx_n, t)\ \forall t \in R.$
We have to prove that there exists positive numbers c and d such that
$J_1(x, y|x_2, ..., cx_n, t) \leq J_2(x, y|x_2, ..., x_n, t) \leq J_1(x, y|x_2, ..., dx_n, t)\ \forall t \in R.$
We have
$J_2(x, y|x_2, ..., ax_n, t) \leq J_1(x, y|x_2, ..., x_n, t)$
$\Rightarrow J_2(x, y|x_2, ..., x_n, \frac{t}{a}) \leq J_1(x, y|x_2, ..., x_n, t).$
Putting $s = \frac{t}{a}$ we get,
$J_2(x, y|x_2, ..., x_n, s) \leq J_1(x, y|x_2, ..., x_n, as)$

$$= J_1(x, y|x_2, ..., x_n, \tfrac{s}{\frac{1}{a}})$$
$$= J_1(x, y|x_2, ..., \tfrac{x_n}{a}, s)$$
$$J_2(x, y|x_2, ..., x_n, s) \le J_1(x, y|x_2, ..., \tfrac{x_n}{a}, s) \quad (3.1)$$

On the other hand

$$J_1(x, y|x_2, ..., x_n, t) \le J_2(x, y|x_2, ..., bx_n, t)$$
$$= J_2(x, y|x_2, ..., x_n, \tfrac{t}{b})$$

Putting $\frac{bt}{a}$ for t we get,

$$J_1(x, y|x_2, ..., x_n, \tfrac{bt}{a}) \le J_2(x, y|x_2, ..., x_n, \tfrac{t}{a})$$
$$\Rightarrow J_1(x, y|x_2, ..., x_n, bs) \le J_2(x, y|x_2, ..., x_n, s)$$
$$\Rightarrow J_1(x, y|x_2, ..., \tfrac{x_n}{b}, s) \le J_2(x, y|x_2, ..., x_n, s) \quad (3.2)$$

Now by (3.1) and (3.2) we get,

$$J_1(x, y|x_2, ..., \tfrac{x_n}{b}, s) \le J_2(x, y|x_2, ..., x_n, s) \le J_1(x, y|x_2, ..., \tfrac{x_n}{a}, s)$$
$$\Rightarrow J_1(x, y|x_2, ..., cx_n, s) \le J_2(x, y|x_2, ..., x_n, s) \le J_1(x, y|x_2, ..., dx_n, s) \quad (3.3)$$

where $c = \frac{1}{b}$ and $d = \frac{1}{a}$

From (3.3) it follows that $\sim$ is symmetric.

(iii) To prove transitivity, let

$$J_0(x, y|x_2, ..., ax_n, t) \le J(x, y|x_2, ..., x_n, t) \le J_0(x, y|x_2, ..., bx_n, t)$$
$$J_1(x, y|x_2, ..., cx_n, t) \le J_0(x, y|x_2, ..., x_n, t) \le J_1(x, y|x_2, ..., dx_n, t).$$

Then we show that there exist two positive numbers e and f such that

$$J_1(x, y|x_2, ..., ex_n, t) \le J(x, y|x_2, ..., x_n, t) \le J_1(x, y|x_2, ..., fx_n, t).$$

Now $J_1(x, y|x_2, ..., cx_n, t) \le J_0(x, y|x_2, ..., x_n, t)$

$$\Rightarrow J_1(x, y|x_2, ..., x_n, \tfrac{t}{c}) \le J_0(x, y|x_2, ..., x_n, t)$$
$$\Rightarrow J_1(x, y|x_2, ..., ax_n, \tfrac{t}{c}) \le J_0(x, y|x_2, ..., ax_n, t)$$
$$\Rightarrow J_1(x, y|x_2, ..., acx_n, t) \le J_0(x, y|x_2, ..., ax_n, t)$$

So, $J_1(x, y|x_2, ..., acx_n, t) \le J_0(x, y|x_2, ..., ax_n, t)$

$$\le J(x, y|x_2, ..., x_n, t)$$
$$\le J_0(x, y|x_2, ..., bx_n, t) \quad (3.4)$$

Also $J_0(x, y|x_2, ..., x_n, t) \le J_1(x, y|x_2, ..., dx_n, t)$

$$\Rightarrow J_0(x, y|x_2, ..., bx_n, t) \le J_1(x, y|x_2, ..., bdx_n, t) \quad (3.5)$$

From (3.4) and (3.5)

$$J_1(x, y|x_2, ..., acx_n, t) \le J(x, y|x_2, ..., x_n, t) \le J_1(x, y|x_2, ..., bdx_n, t).$$

Choose $ac = e$ and $bd = f$

$$\Rightarrow J_1(x, y|x_2, ..., ex_n, t) \le J(x, y|x_2, ..., x_n, t) \le J_1(x, y|x_2, ..., fx_n, t) \quad (3.6)$$

From (3.6) we see that $\sim$ is transitive. □

Theorem 3.4. Let A and B be two f-ST-n-IPS satisfying (10) and (11). Then A and B are equivalent if and only if their corresponding strong α-n-inner products are equivalent for all $\alpha \in (0, 1)$.

Proof. Let A and B be two equivalent f-ST-n-IPS. Then there exists positive constants a, b and c, d such that

$$J_2(x, y|x_2, ..., ax_n, t) \le J_1(x, y|x_2, ..., x_n, t) \le J_2(x, y|x_2, ..., bx_n, t) \ \forall t \in R.$$

Let $(\bullet, \bullet|\bullet, ..., \bullet)^1_\alpha$ and $(\bullet, \bullet| \bullet ..., \bullet)^2_\alpha$ where $\alpha \in (0, 1)$ be the corresponding

strong α-n-inner products of A and B respectively.
First we show that
$J_2(x,y|x_2,...,ax_n,t) \leq J_1(x,y|x_2,...,x_n,t)$ for all $t \in R$
$\Leftrightarrow (x,y|x_2,...,x_n)^1_\alpha \leq (x,y|x_2,...,ax_n)^2_\alpha$ for all $\alpha \in (0,1)$.
Suppose $J_2(x,y|x_2,...,ax_n,t) \leq J_1(x,y|x_2,...,x_n,t)$ holds for all $t \in R$.
Now $(x,y|x_2,...,ax_n)^2_\alpha < t$
$\Rightarrow \inf\{s : J_2(x,y|x_2,...,ax_n,s) \geq \alpha,\} < t$
$\Rightarrow \exists s_0 < t$ such that $J_2(x,y|x_2,...,ax_n,s_0) \geq \alpha$
$\Rightarrow J_1(x,y|x_2,...,x_n,s_0) \geq \alpha, \alpha \in (0,1)$
$\Rightarrow (x,y|x_2,...,x_n)^1_\alpha \leq s_0 < t$
$\Rightarrow (x,y|x_2,...,x_n)^1_\alpha \leq (x,y|x_2,...,ax_n)^2_\alpha$ (3.7)
Next we suppose that $(x,y|x_2,...,x_n)^1_\alpha \leq (x,y|x_2,...,ax_n)^2_\alpha$ for all $\alpha \in (0,1)$.
Now $\nu < J_2(x,y|x_2,...,ax_n,t)$
$\Rightarrow \nu < \sup\{\alpha \in (0,1) : (x,y|x_2,...,ax_n)^2_\alpha \leq t\}$
$\Rightarrow \exists \alpha_0 \in (0,1)$ such that $\nu < \alpha_0$ and $(x,y|x_2,...,ax_n)^2_{\alpha_0} \leq t$
$\Rightarrow (x,y|x_2,...,x_n)^1_{\alpha_0} \leq t$
$\Rightarrow \nu < J_1(x,y|x_2,...,x_n,t)$
$\Rightarrow J_2(x,y|x_2,...,ax_n,t) \leq J_1(x,y|x_2,...,x_n,t)$ (3.8)
From (3.7) and (3.8) it follows that
$J_2(x,y|x_2,...,ax_n,t) \leq J_2(x,y|x_2,...,bx_n,t)$
$\Leftrightarrow (x,y|x_2,...,x_n)^1_\alpha \leq (x,y|x_2,...,ax_n)^2_\alpha$ for all $\alpha \in (0,1)$ □

4 Open Problems

1. Is it possible to interrelate fuzzy n-inner product space and fuzzy n-normed linear space.
2. Efforts can be made to apply this theory to infinite dimensional space.

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