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Differential Subordination and Superordination of Analytic Functions Defined by Fractional Derivative Operator

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Abstract

The differential subordinations method is one of the newest methods used in the geometric theory of analytic functions. In this paper, differential subordination and superordination results are obtained for analytic functions in the open unit disk which are associated with the operator $\Delta_{z,p}^{\lambda,\mu,\nu}$ defined in terms of Saigo fractional derivative. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained and we derive certain other related results. Some of the results established would provide extensions of those given in earlier works.

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1 Introduction and Preliminaries

To state our results**,** we need the following preliminaries**.**
Let $\mathrm{H}(U)$ be the class of functions analytic in $U = \{z \in C : |z| < 1\}$ and $\mathrm{H}[a,n]$ be the subclass of $\mathrm{H}(U)$ consisting of functions of the form

$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, with $\mathrm{H}_0 = \mathrm{H}[0,1]$ and $\mathrm{H} = \mathrm{H}[1,1]$. Also, let $A(p)$ denotes the class of all analytic functions of the form $f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$.

The subordination now plays an important role in complex analysis. We recall here the definition of subordination (Aouf and Seoudy [3], Miller and Mocanu ([9], [10]) and Salim [15]) as follows.

Let f and F be members of analytic function. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$.

In particular, if F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let $\Psi : C^3 \times U \to C$ and let h be univalent in U . If p is analytic in U and satisfies the second order differential subordination

$$\Psi\left(p(z), z\ p'(z), z^2 p''(z); z\right) \prec h(z) \qquad (z \in U) \qquad \textbf{(1.1)}$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1).

Similarly, let $\Phi : C^3 \times U \to C$ and let h be univalent in U . If p is analytic in U and satisfies the second order differential superordination

$$h(z) \prec \Phi\left(p(z), z\ p'(z), z^2 p''(z); z\right) \qquad (z \in U) \qquad \textbf{(1.2)}$$

then p is called a solution of the differential superordination.

The univalent function q is called a subordinant if $q \prec p$ for all p satisfying (1.2). A subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinant q of (1.2) is said to be the best subordinant. (see the monograph by Miller and Mocanu [9]).

We use here the Saigo type fractional derivative operator introduced and studied by Saigo ([13], [14]), see, also Raina and Choi [12], Choi [7] and Owa [11].

Let $0 \le \lambda < 1$ and $\mu, \nu \in R$. Then the generalized fractional derivative operator $\Im_{0,z}^{\lambda,\mu,\nu}$ of a function f is defined by

$$\Im_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\zeta)^{-\lambda} \, {}_2F_1\left(\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\zeta}{z} \right) f(\zeta)\ d\zeta \right) \qquad \textbf{(1.3)}$$

In terms of Gamma functions, we have (Srivastava, Saigo and Owa [16])

$$\Im_{0,z}^{\lambda,\mu,\nu} z^p = \frac{\Gamma(p+1)\ \Gamma(p-\mu+\nu+1)}{\Gamma(p-\mu+1)\ \Gamma(p-\lambda+\nu+1)} z^{p-\mu} \quad ,0 \le \lambda < 1,\ p > \max\{0, \mu-\nu\} - 1 \,. \textbf{(1.4)}$$

Under the hypotheses of above definition, the fractional derivative operator $\Im_{0,z}^{\lambda+m,\mu+m,\nu+m}$ of a function $f(z)$ is defined by (Choi [7])

$$\Im_{0,z}^{\lambda+m,\mu+m,\nu+m} f(z) = \frac{d^m}{dz^m} \Im_{0,z}^{\lambda,\mu,\nu} f(z), \quad (\ z \in U, m \in \mathrm{N}_0 := \{0\} \cup \mathrm{N}\ ) . \qquad \textbf{(1.5)}$$

Using the Saigo Derivative operator $\Im_{0,z}^{\lambda,\mu,\nu}$ of order λ of a function f **,** we can define a modification of the fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$ by (Choi [7])

$$\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(p-\mu+1)\ \Gamma(p-\lambda+\nu+1)}{\Gamma(p+1)\ \Gamma(p-\mu+\nu+1)}\ z^{\mu}\ \Im_{0,z}^{\lambda,\mu,\nu} f(z) \qquad \textbf{(1.6)}$$

for $f(z) \in \mathrm{A}(p)$ and $\mu - \nu - p < 1$**.**

It is observed that $\Delta_{z,p}^{\lambda,\mu,\nu}$ also maps $\mathrm{A}(p)$ onto itself as follows **:**

$$\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} \psi_{k,p}(\lambda,\mu,\nu)\ z^{k+p} \,, \qquad \textbf{(1.7)}$$

where $$\psi_{k,p}(\lambda,\mu,\nu) = \frac{(p+1)_k (p+1-\mu+\nu)_k}{(p+1-\mu)_k (p+1-\lambda+\nu)_k} \,,$$

$(z \in U, 0 \le \lambda < 1, \mu - \nu - p < 1)$**.**

It is easily verified (Choi [7]) from (1.7) that

$$(p-\mu)\ \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu\ \Delta_{z,p}^{\lambda,\mu,\nu} f(z) = z\ \left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z)\right)' . \qquad \textbf{(1.8)}$$

Before obtaining our main result, we need to introduce a class of analytic functions defined on the unit disk that has some nice boundary properties.
Denote by $\aleph$ the set of all functions $q(z)$ that are analytic and injective on $\overline{U}/\mathrm{E}(q)$ where

$$\mathrm{E}(q)=\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\},$$

and are such that $q'(\zeta)\neq 0$ for $\zeta\in\partial U/\mathrm{E}(q)$. Further, let the subclass of $\aleph$ for which $q(0)=a$ be denoted by $\aleph(a)$, $\aleph(0)\equiv\aleph_0$ and $\aleph(1)\equiv\aleph_1$.

In the recent publications of Miller and Mocanu [9] and Aouf and Seoudy ([3],[4]) the class of admissible functions $\Psi_n[\Omega,q]$ was defined as follows.

Definition 1.1. Let Ω be a set in $C, q\in\aleph$ and n be a positive integer.
The class of admissible functions $\Psi_n[\Omega,q]$, consists of those functions $\Psi: C^3\times U\to C$ that satisfy the admissibility condition :

$$\Psi(r,s,t;z)\notin\Omega$$

whenever

$$r=q(\zeta), s=k\ \zeta\ q'(\zeta), \Re\left\{\frac{t}{s}+1\right\}\geq k\ \Re\left\{1+\frac{\zeta\ q''(\zeta)}{q'(\zeta)}\right\},$$

where $z\in U, \zeta\in\partial U/\mathrm{E}(q)$ and $k\geq n$. We write $\Psi_1[\Omega,q]$ as $\Psi[\Omega,q]$.

Also, other more class of admissible functions such as $\Psi'_n\ [\Omega,q]$ has been used in the study of superordination. For instant, see the papers by Miller and Mocanu [10] and Aouf and Seoudy [3].

Definition 1.2. Let Ω be a set in $C, q(z)\in \mathrm{H}\ [a,n]$ with $q'(z)\neq 0$. The class of admissible functions $\Psi'_n\ [\Omega,q]$, consists of those functions $\Psi: C^3\times\overline{U}\to C$ that satisfy the admissibility condition :

$$\Psi(r,s,t;\zeta)\in\Omega$$

whenever

$$r=q(z), s=\frac{z\ q'(z)}{m}, \Re\left\{\frac{t}{s}+1\right\}\leq\frac{1}{m}\Re\left\{1+\frac{z\ q''(z)}{q'(z)}\right\},$$

where $z\in U, \zeta\in\partial U$ and $m\geq n\geq 1$. In particular, we write $\Psi'_1\ [\Omega,q]$ as $\Psi'[\Omega,q]$.

In order to prove our subordination results, we make use of the following two lemmas, which were investigated by Miller and Mocanu ([9], [10]) and were used in the paper of Aouf and Seoudy [3].

Lemma 1.3. Let $\Psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function

$g(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ satisfies $\Psi(g(z), z\, g'(z), z^2 g''(z)\ ; z) \in \Omega$,

then $g(z) \prec q(z)$.

Lemma 1.4. Let $\Psi \in \Psi'_n\ [\Omega, q]$ with $q(0) = a$. If $g(z) \in \aleph(a)$ and

$\Psi(g(z), z\, g'(z), z^2 g''(z); z)$ is univalent in U, then

$\Omega \subset \{\ \Psi\ (\ g(z), z\, g'(z), z^2 g''(z)\ ; z): z \in U\ \}$ implies $q(z) \prec g(z)$.

The differential subordinations method is one of the newest methods used in the geometric theory of analytic functions. The basics of this theory were introduced and studied by Miller and Mocanu ([9],[10]) and similar problems were studied by Kim and Srivastava [8], Aouf and Seoudy [3], Aghalary et al. [1], Ali et al. [2], Aouf [6] and Aouf et al. [5]. In this paper, differential subordination and superordination results are obtained for analytic functions in the open unit disk, which are associated with the derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained and we obtain certain other related results. Further, we generalize classical results of the theory of differential subordination and superordination, that is the differential subordination and superordination results of Aouf and Seoudy [3] is extended for functions associated with the derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$. Some of the results established in this paper would provide extensions of those given in earlier works.

2 Subordination Results Involving the Derivative Operator $\Delta_{z,p}^{\lambda,\mu,\nu}$

First, the following class of admissible functions is required in our first result.

Definition 2.1. Let Ω be a set in $C, q(z) \in \aleph_0 \cap \mathrm{H}[0,p]$**.** The class of admissible functions $\Phi_{\Delta}[\Omega, q]$**,** consists of those functions $\Phi : C^3 \times U \to C$ that satisfy the admissibility condition **:**

$$\Phi(u, x, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), x = \frac{k\ \zeta\ q'(\zeta) - \mu\ q(\zeta)}{p - \mu},$$

$$\Re\left\{\frac{(p-\mu)\ (p-\mu-1)\ w - \mu\ (\mu+1)\ u}{(p-\mu)\ x + \mu\ u} + 2\mu + 1\right\} \geq k\ \Re\left\{1 + \frac{\zeta\ q''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U, \zeta \in \partial U / \mathrm{E}(q), \mu \neq p, p \in \mathrm{N}$ and $k \geq p$ **.**

Next**,** by appealing to Lemma 1.3**,** we prove the following Theorem**.**

Theorem 2.2. Let $\Phi \in \Phi_{\Delta}[\Omega, q]$**.** If $f(z) \in \mathrm{A}(p)$ satisfies

$$\left\{\ \Phi\left(\ \Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)\ ; z\right) : z \in U\ \right\} \subset\ \Omega, \quad \mathbf{(2.1)}$$

then $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q(z)$**.** $\quad \left(\ 0\ \leq\ \lambda\ <\ 1, \mu\ \notin \left\{\ p, p-1\ \right\}, z \in U\ , p \in\ \mathrm{N}\ \right)$**.**

Proof. Define the analytic function $g(z)$ in U by

$$g(z) = \Delta_{z,p}^{\lambda,\mu,\nu} f(z)\ \left(\ 0\ \leq\ \lambda\ <\ 1, \mu\ \notin \left\{\ p, p-1\ \right\}, z \in U\ , p \in\ \mathrm{N}\ \right). \quad \mathbf{(2.2)}$$

In view of the relation (1.9)**,** then from (2.2) we get

$$\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) = \frac{z\ g'(z) - \mu\ g(z)}{p - \mu}. \quad \mathbf{(2.3)}$$

Further computations show that

$$\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z) = \frac{z^2 g''(z) - 2\ \mu\ z\ g'(z) + \mu\ (\mu+1)\ g(z)}{(p-\mu)\ (p-\mu-1)}. \quad \mathbf{(2.4)}$$

Define the transformation from C^3 to C by

$$u=r, x=\frac{s-\mu\ r}{p-\mu}, w=\frac{t-2\ \ \mu\ s+\mu\ (\mu+1)\ r}{(p-\mu)\ (p-\mu-1)}\ . \tag{2.5}$$

Let

$$\Psi(r,s,t;z)=\Phi(u,x,w;z)=\Phi\left(r,\frac{s-\mu\ r}{p-\mu},\frac{t-2\ \ \mu\ s+\mu\ (\mu+1)\ r}{(p-\mu)\ (p-\mu-1)};z\right). \tag{2.6}$$

Using equations (2.2),(2.3) and (2.4), then from (2.6), we obtain

$$\Psi\ \left(g(z), z\ g'(z), z^2 g''(z)\ ;z\right)=\Phi\ \left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)\ ;z\right). \tag{2.7}$$

Hence (2.1) becomes $\Psi\ \left(g(z), z\ g'(z), z^2 g''(z); z\right)\in\Omega$.

The proof is completed, if it can be shown that the admissibility condition for $\Phi\in\Phi_{\Delta}[\Omega,q]$ is equivalent to the admissibility condition for Ψ as given in Definition 1.1. Note that

$$\frac{t}{s}+1=\frac{(p-\mu)\ (p-\mu-1)\ w-\mu\ (\mu+1)\ u}{(p-\mu)\ x+\mu\ u}+2\ \ \mu+1\ ,$$

and hence $\Psi\in\Psi_p[\Omega,q]$. By Lemma 1.3, $g(z)\prec q(z)$ or $\Delta_{z,p}^{\lambda,\mu,\nu} f(z)\prec q(z)$. □

If $\Omega\neq C$ is a simply connected domain ,then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_{\Delta}[h(U),q]$ is written as $\Phi_{\Delta}[h,q]$. The following result is immediate consequence of Theorem 2.2.

Theorem 2.3. Let $\Phi\in\Phi_{\Delta}[h,q]$. If $f(z)\in A(p)$ satisfies

$$\Phi\left(\ \Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)\ ;z\right)\prec h(z), \tag{2.8}$$

then $\Delta_{z,p}^{\lambda,\mu,\nu} f(z)\prec q(z)$. $\quad\left(\ 0\ \leq\ \lambda\ <\ 1\ ,\mu\notin\left\{\ p,p-1\ \right\}, z\in U\ , p\in\ \mathrm{N}\ \right)$.

Our next result is an extension of Theorem 2.2 to the case, where the behavior of $q(z)$on ∂U is not known.

Corollary 2.4. Let $\Omega\subset C$ and let $q(z)$ be univalent in U, $q(0)=0$. Let

$\Phi \in \Phi_{\Delta}\left[\Omega, q_{\rho}\right]$ for some $\rho \in (0,1)$, where $q_{\rho}(z) = q(\rho\, z)$. If $f(z) \in \mathrm{A}(p)$ and

$$\Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f\ (z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f\ (z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f\ (z)\ ; z \right) \in \Omega,$$

then $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q(z)$. $\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

Proof. Theorem 2.2 yields $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q_{\rho}(z)$. The result is now deduced from $q_{\rho}(z) \prec q(z)$.
□

Theorem 2.5. Let $h(z)$ and $q(z)$ be univalent in U with $q(0) = 0$ and set $q_{\rho}(z) = q(\rho\, z)$ and $h_{\rho}(z) = h\ (\rho\, z)$.
Let $\Phi : C^3 \times U \to C$ satisfy one of the following conditions :
(1) $\Phi \in \Phi_{\Delta}\left[h, q_{\rho}\right]$, for some $\rho \in (0,1)$, or

(2) there exists $\rho_0 \in (0,1)$ such that $\Phi \in \Phi_{\Delta}\left[h_{\rho}, q_{\rho}\right]$, for all $\rho \in (\rho_0, 1)$.
If $f(z) \in \mathrm{A}(p)$ satisfies (2.8), then $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q(z)$.
$\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

Proof. The proof is similar to the proof of Miller and Mocanu [9] and therefore omitted.
□

The next Theorem yields the best dominant of the differential subordination (2.8).

Theorem 2.6. Let $h(z)$ be univalent in U. Let $\Phi : C^3 \times U \to C$. Suppose that the differential equation

$$\Phi\left(q(z), z\, q'(z), z^2 q''(z)\ ; z\right) = h\ (z) \qquad \textbf{(2.9)}$$

has a solution $q(z)$ with $q(0) = 0$ and satisfy one of the following conditions :

(1) $q(z) \in \aleph_0 \cap \mathrm{H}[0, p]$ and $\Phi \in \Phi_{\Delta}[h, q]$,

(2) $q(z)$ is univalent in U and $\Phi \in \Phi_{\Delta}\left[h, q_{\rho}\right]$, for some $\rho \in (0,1)$, or

(3) $q(z)$ is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\Phi \in \Phi_{\Delta}\left[h_{\rho}, q_{\rho}\right]$, for all $\rho \in (\rho_0, 1)$.

If $f(z) \in \mathrm{A}(p)$ satisfies (2.8), then $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q(z)$,
and $q(z)$ is the best dominant.

$\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

Proof. We deduce that $q(z)$ is a dominant from Theorems 2.3 and 2.5. Since $q(z)$ satisfies (2.9) it is also a solution of (2.8) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. □

In the particular case $q(z) = \mathrm{M}\, z$, $\mathrm{M} > 0$, and in view of the Definition 2.1, the class of admissible functions $\Phi_\Delta[\Omega, q]$, denoted by $\Phi_\Delta[\Omega, \mathrm{M}]$, is described below.

Definition 2.7. Let Ω be a set in C and $\mathrm{M} > 0$. The class of admissible functions $\Phi_\Delta[\Omega, \mathrm{M}]$, consists of those functions $\Phi : C^3 \times U \to C$ such that

$$\Phi\left(\mathrm{M}\, e^{i\theta}, \frac{k-\mu}{p-\mu} \mathrm{M}\, e^{i\theta}, \frac{L + \left(-2\mu\, k + \mu\, (\mu+1)\right) \mathrm{M}\, e^{i\theta}}{(p-\mu)\,(p-\mu-1)}; z \right) \notin \Omega \qquad \textbf{(2.10)}$$

whenever
$z \in U, \theta \in R, \Re\{L\, e^{-i\theta}\} \ge (k-1)\, k\, \mathrm{M}$ for all real $\theta, \mu \notin \{ p, p-1 \}, p \in \mathrm{N}$ and $k \ge p$.

From above definition and Theorem 2.2, we have the following corollary

Corollary 2.8. Let $\Phi \in \Phi_\Delta[\Omega, \mathrm{M}]$. If $f(z) \in \mathrm{A}(p)$ satisfies

$$\Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)\; ; z \right) \in \Omega,$$

then $\left| \Delta_{z,p}^{\lambda,\mu,\nu} f(z) \right| < \mathrm{M}$. $\quad \left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

In the special case $\Omega = q\,(U) = \{ w : |w| < \mathrm{M} \}$, the class $\Phi_\Delta[\Omega, \mathrm{M}]$ is simply denoted by $\Phi_\Delta[\mathrm{M}]$, then Corollary 2.8 takes the following form.

Corollary 2.9. Let $\Phi \in \Phi_\Delta[\mathrm{M}]$. If $f(z) \in \mathrm{A}(p)$ satisfies

$$\left| \Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z) \;; z\right) \right| < \mathrm{M},$$

then $\left| \Delta_{z,p}^{\lambda,\mu,\nu} f(z) \right| < \mathrm{M}$. $\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

If $\Phi\ (u,x,w;z) = x = \dfrac{k-\mu}{p-\mu}\mathrm{M}\ e^{i\theta}$, from Corollary 2.9, we obtain the next corollary.

Corollary 2.10. If $k \ge p$ and $f(z) \in \mathrm{A}(p)$ satisfies

$\left| \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) \right| < \mathrm{M}$, then $\left| \Delta_{z,p}^{\lambda,\mu,\nu} f(z) \right| < \mathrm{M}$.
$\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

Now, we introduce a new class of admissible functions $\Phi_{\Delta,1}[\Omega, q]$.

Definition 2.11. Let Ω be a set in $C, q(z) \in \aleph_0 \cap \mathrm{H}_0$. The class of admissible functions $\Phi_{\Delta,1}[\Omega, q]$, consists of those functions $\Phi : C^3 \times U \to C$

that satisfy the admissibility condition **:**

$$\Phi(u,x,w;z) \notin \Omega$$

whenever

$$u = q(\zeta), x = \frac{k\ \zeta\ q'(\zeta) + (p-\mu-1)\ q(\zeta)}{p-\mu},$$

$$\Re\left\{ \frac{(p-\mu)\ (p-\mu-1)\ w - (p-\mu-1)\ (p-\mu-2)\ u}{(p-\mu)\ x - (p-\mu-1)\ u} - 2(p-\mu) + 3 \right\} \ge k\ \Re\left\{ 1 + \frac{\zeta\ q''(\zeta)}{q'(\zeta)} \right\}$$

,

where $z \in U, \zeta \in \partial U / \mathrm{E}(q), \mu \ne p, p \in \mathrm{N}$ and $k \ge 1$.

By making use of Lemma 1.3, we prove the following subordination result.

Theorem 2.12. Let $\Phi \in \Phi_{\Delta,1}[\Omega, q]$. If $f(z) \in \mathrm{A}(p)$ satisfies

$$\left\{ \Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z \right) : z \in U \right\} \subset \Omega \quad \mathbf{(2.11)}$$

then $\dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \prec q(z)$. $\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathbb{N} \right)$.

Proof. Define the analytic function $g(z)$ in U by

$$g(z) = \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \qquad \left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathbb{N} \right). \quad \mathbf{(2.12)}$$

In view of the relation (1.5), then (2.12) yields

$$\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}} = \frac{z\, g'(z) + (p-\mu-1)\, g(z)}{p-\mu}. \quad \mathbf{(2.13)}$$

Further computations show that

$$\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}} = \frac{z^2 g''(z) + 2\,(p-\mu-1)\, z\, g'(z) + (p-\mu-1)(p-\mu-2)\, g(z)}{(p-\mu)(p-\mu-1)} \quad \mathbf{(2.14)}$$

Define the transformation from C^3 to C by

$$u = r, x = \frac{s + (p-\mu-1)\, r}{p-\mu}, w = \frac{t + 2\,(p-\mu-1)\, s + (p-\mu-1)(p-\mu-2)\, r}{(p-\mu)(p-\mu-1)}. \quad \mathbf{(2.15)}$$

Let $\Psi\,(r,s,t;z) = \Phi\,(u,x,w;z)$

$$= \Phi\left(r, \frac{s + (p-\mu-1)\, r}{p-\mu}, \frac{t + 2\,(p-\mu-1)\, s + (p-\mu-1)(p-\mu-2)\, r}{(p-\mu)(p-\mu-1)}; z \right). \quad \mathbf{(2.16)}$$

Using equations (2.12),(2.13) and (2.14), then from (2.16), we obtain

$$\Psi\left(g(z), z\, g'(z), z^2 g''(z)\,; z\right) = \Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z \right). \quad \mathbf{(2.17)}$$

Hence (2.11) becomes $\Psi\left(g(z), z\, g'(z), z^2 g''(z); z\right) \in \Omega$.

The proof is completed if it can be shown that the admissibility condition for $\Phi \in \Phi_{\Delta,1}[\Omega, q]$ is equivalent to the admissibility condition for Ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{(p-\mu)\,(p-\mu-1)\,w - (p-\mu-1)\,(p-\mu-2)\,u}{(p-\mu)\,x - (p-\mu-1)\,u} - 2(p-\mu) + 3,$$

and hence $\Psi \in \Psi[\Omega, q]$. By Lemma 1.3, $g(z) \prec q(z)$ or $\dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \prec q(z)$.

□

If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_{\Delta,1}[h(U), q]$ is written as $\Phi_{\Delta,1}[h, q]$.

Proceeding similarly as in Theorem 2.3, the following result is an immediate consequence of Theorem 2.12.

Theorem 2.13. Let $\Phi \in \Phi_{\Delta,1}[h, q]$. If $f(z) \in \mathrm{A}(p)$ satisfies

$$\Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z\right) \prec h(z), \qquad \textbf{(2.18)}$$

then $\dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \prec q(z)$. $\left(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, z \in U, p \in \mathrm{N}\right)$.

In the particular case $q(z) = \mathrm{M}\, z, \mathrm{M} > 0$, the class of admissible functions $\Phi_{\Delta,1}[\Omega, q]$, denoted by $\Phi_{\Delta,1}[\Omega, \mathrm{M}]$, is described below.

Definition 2.14. Let Ω be a set in C and $\mathrm{M} > 0$. The class of admissible functions $\Phi_{\Delta,1}[\Omega, \mathrm{M}]$, consists of those functions $\Phi : C^3 \times U \to C$ such that

$$\Phi\left(\mathrm{M}\, e^{i\theta}, \frac{k+p-\mu-1}{p-\mu}\mathrm{M}\, e^{i\theta}, \frac{L + (p-\mu-1)\,(2k+p-\mu-2)\,\mathrm{M}\, e^{i\theta}}{(p-\mu)\,(p-\mu-1)}; z\right) \notin \Omega \; \textbf{(2.19)}$$

whenever

.

$z \in U, \theta \in R, \Re\{L\ e^{-i\theta}\} \geq (k-1)\ k\ \mathrm{M}$ for all real $\theta, \mu \notin \{\ p, p-1\ \}, p \in \mathrm{N}$ and $k \geq 1$.

From above Definition and Theorem 2.12, we have the following corollary

Corollary 2.15. Let $\Phi \in \Phi_{\Delta,1}[\Omega, \mathrm{M}]$. If $f(z) \in \mathrm{A}(p)$ satisfies

$$\Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z \right) \in \Omega ,$$

then $\left| \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \right| < \mathrm{M}$. $\quad \left(0 \leq \lambda < 1, \mu \notin \{\ p, p-1\ \}, z \in U, p \in \mathrm{N} \right)$.

In the special case $\Omega = q\ (U) = \{\ w : |w| < \mathrm{M}\ \}$, the class $\Phi_{\Delta,1}[\Omega, \mathrm{M}]$ is simply denoted by $\Phi_{\Delta,1}[\mathrm{M}]$, then Corollary 2.15 takes the following form.

Corollary 2.16. Let $\Phi \in \Phi_{\Delta,1}[\mathrm{M}]$. If $f(z) \in \mathrm{A}(p)$ satisfies

$$\left| \ \Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z \right) \right| < \mathrm{M} ,$$

then $\left| \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \right| < \mathrm{M}$. $\quad \left(0 \leq \lambda < 1, \mu \notin \{\ p, p-1\ \}, z \in U, p \in \mathrm{N} \right)$.

By taking $\Phi\ (u, x, w; z) = x = \frac{k+p-\mu-1}{p-\mu} \mathrm{M}\ e^{i\theta}$ in Corollary 2.16, we obtain the following corollary.

Corollary 2.17. If $k \geq 1$ and $f(z) \in \mathrm{A}(p)$ satisfies $\left| \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}} \right| < \mathrm{M}$,

then $\left| \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \right| < \mathrm{M}$. $\quad \left(0 \leq \lambda < 1, \mu \notin \{\ p, p-1\ \}, z \in U, p \in \mathrm{N} \right)$.

Next, we introduce down a new class of admissible functions $\Phi_{\Delta,2}[\Omega, q]$.

Definition 2.18. Let Ω be a set in $C, q(z) \in \aleph_1 \cap \mathrm{H}$. The class of admissible functions $\Phi_{\Delta,2}[\Omega, q]$, consists of those functions $\Phi : C^3 \times U \to C$ that satisfy the admissibility condition :

$$\Phi(u, x, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), x = \frac{1}{p-\mu-1}\left[-1+(p-\mu)\ q\ (\zeta)+\frac{k\ \zeta\ q'(\zeta)}{q\ (\zeta)}\right],$$

$$\Re\left\{\frac{(p-\mu-1)\ x\ [\ (p-\mu-2)\ w-(p-\mu-1)\ x+1]}{(p-\mu-1)\ x-(p-\mu)\ u+1}-2(p-\mu)\ u+(p-\mu-1)\ x+1\right\}$$

$$\geq k\ \Re\left\{1+\frac{\zeta\ q''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U, \zeta \in \partial U / \mathrm{E}(q), \mu \neq p-1, p \in \mathrm{N}$ and $k \geq 1$.

By making use of Lemma 1.3, we prove the following subordination result.

Theorem 2.19. Let $\Phi \in \Phi_{\Delta,2}[\Omega, q]$ and $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \neq 0$. If $f(z) \in \mathrm{A}(p)$ satisfies

$$\left\{\ \Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}; z\right) : z \in U\ \right\} \subset \Omega \qquad ,$$

(2.20)

then $\dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \prec q(z)$. $\quad \left(\ 0\ \leq\ \lambda\ <\ 1\ , \mu\ \notin \left\{\ p-1, p-2\ \right\}, z \in U\ , p \in\ \mathrm{N}\ \right)$.

Proof. Define the analytic function $g(z)$ in U by

$$g(z) = \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \qquad \left(\ 0\ \leq\ \lambda\ <\ 1\ , \mu\ \notin \left\{\ p-1, p-2\ \right\}, z \in U\ , p \in\ \mathrm{N}\ \right)\ .$$

(2.21)

Using (2.21), we get

$$\frac{z\ g'(z)}{g(z)}=\frac{z\left(\ \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)\ \right)'}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}-\frac{z\left(\ \Delta_{z,p}^{\lambda,\mu,\nu}f(z)\ \right)'}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}. \tag{2.22}$$

By making use of the relation (1.8) in (2.22), we get

$$\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}=\frac{1}{p-\mu-1}\left[\frac{z\ g'(z)}{g(z)}+(p-\mu)\ g(z)-1\right]. \tag{2.23}$$

Further computations show that

$$\frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}=\frac{1}{(p-\mu-2)}\left[-2+\frac{z\ g'(z)}{g(z)}+(p-\mu)\ g(z)\right.$$

$$\left.+\frac{z^2\left(\frac{g''(z)}{g(z)}\right)+(p-\mu)\ z\ g'(z)+\frac{z\ g'(z)}{g(z)}-\left(\frac{z\ g'(z)}{g(z)}\right)^2}{-1+\frac{z\ g'(z)}{g(z)}+(p-\mu)\ g(z)}\right]. \tag{2.24}$$

Define the transformation from C^3 to C by

$$u=r, x=\frac{1}{p-\mu-1}\left[-1+\frac{s}{r}+(p-\mu)\ r\right],$$

$$w=\frac{1}{(p-\mu-2)}\left[-2+\frac{s}{r}+(p-\mu)\ r+\frac{\left(\frac{t}{r}\right)+(p-\mu)\ s+\frac{s}{r}-\left(\frac{s}{r}\right)^2}{-1+\frac{s}{r}+(p-\mu)\ r}\right]. \tag{2.25}$$

Let $\Psi(r,s,t;z)=\Phi(u,x,w;z)$

$$=\Phi\left(r,\frac{1}{p-\mu-1}\left[-1+\frac{s}{r}+(p-\mu)\ r\right],\right.$$

$$\left.\frac{1}{(p-\mu-2)}\left[-2+\frac{s}{r}+(p-\mu)\ r+\frac{\left(\frac{t}{r}\right)+(p-\mu)\ s+\frac{s}{r}-\left(\frac{s}{r}\right)^2}{-1+\frac{s}{r}+(p-\mu)\ r};z\right]\right).\quad \textbf{(2.26)}$$

Using equations (2.21),(2.23) and (2.24), then (2.26) implies

$$\Psi\left(g(z),z\ g'(z),z^2g''(z)\ ;z\right)$$

$$=\Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)},\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)},\frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)};z\right).\quad \textbf{(2.27)}$$

Hence (2.20) becomes $\Psi\ \left(g(z),z\ g'(z),z^2g''(z);z\right)\in\Omega$.

The proof is completed, if it can be shown that the admissibility condition for $\Phi\in\Phi_{\Delta,2}[\Omega,q]$ is equivalent to the admissibility condition for Ψ as given in definition 1.1. Note that

$$\frac{t}{s}+1=\frac{(p-\mu-1)\ x\ \left[\ (p-\mu-2)\ w-(p-\mu-1)\ x+1\right]}{(p-\mu-1)\ x-(p-\mu)\ u+1}-2(p-\mu)\ u+(p-\mu-1)\ x+1,$$

and hence $\Psi\in\Psi[\Omega,q]$. By Lemma 1.3, $g(z)\prec q(z)$ or $\dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}\prec q(z)$.

□

If $\Omega\neq C$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_{\Delta,2}[h(U),q]$ is written as $\Phi_{\Delta,2}[h,q]$. Proceeding similarly as in Theorem 2.13, the following result is immediate consequence of Theorem 2.19.

Theorem 2.20. Let $\Phi\in\Phi_{\Delta,2}[h,q]$ and $\Delta_{z,p}^{\lambda,\mu,\nu}f(z)\neq 0$. If $f(z)\in A(p)$ satisfies

$$\Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)},\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)},\frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)};z\right)\prec h(z),\quad \textbf{(2.28)}$$

then $$\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \prec q(z) \, .$$
$\left(0 \le \lambda < 1 \, , \mu \notin \{ p-1, p-2 \} \, , z \in U \, , p \in \mathrm{N} \right).$

In the particular case $q(z) = \mathrm{M}\, z \, , \mathrm{M} > 0$ **,** the class of admissible functions $\Phi_{\Delta,2}[\Omega, q]$**,** denoted by $\Phi_{\Delta,2}[\Omega, \mathrm{M}]$**,** is described below**.**

Definition 2.21. Let Ω be a set in C and $\mathrm{M} > 0$ **.** The class of admissible functions $\Phi_{\Delta,2}[\Omega, \mathrm{M}]$**,** consists of those functions $\Phi : C^3 \times U \to C$ such that

$$\Phi\Bigg(\mathrm{M}\, e^{i\theta}, \frac{1}{p-\mu-1}\left[k-1+(p-\mu)\,\mathrm{M}\, e^{i\theta}\right], \frac{1}{(p-\mu-2)}\Bigg\{k-2+(p-\mu)\,\mathrm{M}\, e^{i\theta}$$
$$+\frac{(p-\mu)\,k\,\mathrm{M}^2 e^{i\theta}+k\,\mathrm{M}-k^2\mathrm{M}+L\,e^{-i\theta}}{(k-1)\,\mathrm{M}+(p-\mu)\,\mathrm{M}^2 e^{i\theta}}\Bigg\}; z\Bigg) \notin \Omega \, , \qquad \textbf{(2.29)}$$

whenever $z \in U, \theta \in R, \Re\{L\, e^{-i\theta}\} \ge (k-1)\, k\, \mathrm{M}$ for all real $\theta, \mu \notin \{p-1, p-2\}, p \in \mathrm{N}$ and $k \ge 1$**.**

From above definition and Theorem 2**.**19**,** we have the following corollary

Corollary 2.22. Let $\Phi \in \Phi_{\Delta,2}[\Omega, \mathrm{M}]$ and $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \ne 0$**.** If $f(z) \in \mathrm{A}(p)$ satisfies

$$\Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}; z \right) \in \Omega \, ,$$

then $\left| \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right| < \mathrm{M}$ **.** $\quad \left(0 \le \lambda < 1 \, , \mu \notin \{ p-1, p-2 \}, z \in U \, , p \in \mathrm{N} \right).$

In the special case $\Omega = q\,(U) = \{ w : |w| < \mathrm{M} \}$**,** the class $\Phi_{\Delta,2}[\Omega, \mathrm{M}]$ is simply denoted by $\Phi_{\Delta,2}[\mathrm{M}]$**,** then Corollary 2**.**22 takes the following form**.**

Corollary 2.23. Let $\Phi \in \Phi_{\Delta,2}[\mathrm{M}]$ and $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \ne 0$**.** If $f(z) \in \mathrm{A}(p)$ satisfies

$$\left| \Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}; z \right) \right| < \mathrm{M},$$

then $\left| \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right| < \mathrm{M}$. $\left(0 \le \lambda < 1, \mu \notin \{ p-1, p-2 \}, z \in U, p \in \mathrm{N} \right)$.

3 Superordination of the Derivative Operator $\Delta_{z,p}^{\lambda,\mu,\nu}$

The dual problem of differential subordination, that is, differential superordination of the derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$ is investigated in this section. For this purpose, the class of admissible functions is given as the following definition.

Definition 3.1. Let Ω be a set in $C, q(z) \in \mathrm{H}[0, p]$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_{\Delta}[\Omega, q]$, consists of those functions $\Phi : C^3 \times \overline{U} \to C$ that satisfy the admissibility condition :

$$\Phi(u, x, w; \zeta) \in \Omega$$

whenever

$$u = q(z), x = \frac{z\ q'(z) - m\ \mu\ q(z)}{m\ (p-\mu)},$$

$$\Re\left\{ \frac{(p-\mu)\ (p-\mu-1)\ w - \mu\ (\mu+1)\ u}{(p-\mu)\ x + \mu\ u} + 2\mu + 1 \right\} \le \frac{1}{m}\ \Re\left\{ 1 + \frac{z\ q''(z)}{q'(z)} \right\},$$

where $z \in U, \zeta \in \partial U, \mu \neq p, p \in \mathrm{N}$ and $m \ge p$.

Theorem 3.2 follows by using the same technique to prove Theorem 2.2 and by application of Lemma 1.4.

Theorem 3.2. Let $\Phi \in \Phi'_{\Delta}\ [\Omega, q]$. If $f(z) \in \mathrm{A}(p)$, $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \in \aleph_0$ and

$\Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)\ ; z \right)$ is univalent in U,

then

$$\Omega \subset \left\{ \Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z) \ ; z\right) : z \in U \right\}, \quad \textbf{(3.1)}$$

implies $q(z) \prec \Delta_{z,p}^{\lambda,\mu,\nu} f(z)$. $\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

Proof. From (2.7) and (3.1), we have

$$\Omega \subset \left\{ \Psi\left(g(z), z\, g'(z), z^2 g''(z) \ ; z\right) : z \in U \right\}.$$

From (2.5), we see that the admissibility condition for $\Phi \in \Phi'_{\Delta}\,[\Omega, q]$ is equivalent to the admissibility condition for Ψ as given in Definition 1.2. Hence $\Psi \in \Psi'_p[\Omega, q]$, and by Lemma 1.4, $q(z) \prec g(z)$ or $q(z) \prec \Delta_{z,p}^{\lambda,\mu,\nu} f(z)$. □

If $\Omega \ne C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi'_{\Delta}[h(U), q]$ is written as $\Phi'_{\Delta}[h, q]$. The following result is immediate consequence of Theorem 3.2.

Theorem 3.3. Let $\Phi \in \Phi'_{\Delta}[h, q]$. If $f(z) \in \mathrm{A}(p)$, $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \in \aleph_0$ and

$\Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z) \ ; z\right)$ is univalent in U, then

$$h(z) \prec \Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z) \ ; z\right), \quad \textbf{(3.2)}$$

implies $q(z) \prec \Delta_{z,p}^{\lambda,\mu,\nu} f(z)$. $\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

Theorems 3.2 and 3.3 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following Theorem proves the existence of the best subordinant of (3.2) for certain Φ.

Theorem 3.4. Let $h(z)$ be univalent in U and $\Phi : C^3 \times \overline{U} \to C$. Suppose that the differential equation

$$\Phi\left(q(z), z\, q'(z), z^2 q''(z) \ ; z\right) = h\,(z)$$

has a solution $q(z) \in \mathrm{H}[0, p]$. If $\Phi \in \Phi'_{\Delta}\,[h, q]$, $f(z) \in \mathrm{A}(p)$, $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \in \aleph_0$ and

$\Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z) \ ; z\right)$ is univalent in U, then

$$h(z) \prec \Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)\ ; z\right),$$

implies $q(z) \prec \Delta_{z,p}^{\lambda,\mu,\nu} f(z)$ and $q(z)$ is the best subordinant.

$\left(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, z \in U, p \in \mathrm{N}\right)$.

Proof. The proof is similar to the proof of Theorem 2.6 and therefore omitted. □

Combining Theorems 2.3 and 3.3, we obtain the following sandwich-type corollary.

Corollary 3.5. Let $h_1(z)$ and $h_2(z)$ are univalent functions in U and

$\Phi \in \Phi_\Delta[h_2, q_2] \cap \Phi'_\Delta\ [h_1, q_1]$. If $f(z) \in \mathrm{A}(p)$, $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \in \mathrm{H}\ [0, p] \cap \aleph_0$ and

$\Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)\ ; z\right)$ is univalent in U, then

$$h_1(z) \prec \Phi\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z), \Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z), \Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)\ ; z\right) \prec h_2(z),$$

implies $q_1(z) \prec \Delta_{z,p}^{\lambda,\mu,\nu} f(z) \prec q_2(z)$. $\left(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, z \in U, p \in \mathrm{N}\right)$.

Now, we introduce down a new class of admissible functions $\Phi'_{\Delta,1}[\Omega, q]$.

Definition 3.6. Let Ω be a set in C and $q(z) \in \mathrm{H}_0$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_{\Delta,1}[\Omega, q]$, consists of those functions $\Phi : C^3 \times \overline{U} \to C$ that satisfy the admissibility condition :

$$\Phi(u, x, w; \zeta) \in \Omega$$

whenever

$$u = q(z), x = \frac{z\ q'(z) + m\ (p-\mu-1)\ q(z)}{m\ (p-\mu)},$$

$$\Re\left\{\frac{(p-\mu)\ (p-\mu-1)\ w - (p-\mu-1)\ (p-\mu-2)\ u}{(p-\mu)\ x - (p-\mu-1)\ u} - 2(p-\mu) + 3\right\} \leq \frac{1}{m}\Re\left\{1 + \frac{z\ q''(z)}{q'(z)}\right\}$$

,where $z \in U, \zeta \in \partial U, \mu \neq p, p \in \mathrm{N}$ and $m \geq 1$.

Now, we give the dual result of Theorem 2.12 for differential superordination.

Theorem 3.7. Let $\Phi \in \Phi'_{\Delta,1}[\Omega, q]$. If $f(z) \in \mathrm{A}(p)$,$\dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \in \aleph_0$ and

$\Phi\left(\dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \dfrac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z\right)$ is univalent in U, then

$$\Omega \subset \left\{ \Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z\right) : z \in U \right\}, \quad \textbf{(3.3)}$$

implies $q(z) \prec \dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}$. $\quad \left(0 \leq \lambda < 1, \mu \notin \{p, p-1\}, z \in U, p \in \mathrm{N}\right)$.

Proof. From (2.17) and (3.3), we have

$$\Omega \subset \left\{ \Psi\left(g(z), z\, g'(z), z^2 g''(z); z\right) : z \in U \right\}.$$

From (2.15), we see that the admissibility condition for $\Phi \in \Phi'_{\Delta,1}[\Omega, q]$ is equivalent to the admissibility condition for Ψ as given in Definition 1.2.
Hence $\Psi \in \Psi'[\Omega, q]$, and by Lemma 1.4, $q(z) \prec g(z)$ or $q(z) \prec \dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}$.
□

If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi'_{\Delta,1}[h(U), q]$ is written as $\Phi'_{\Delta,1}[h, q]$. The following result is immediate consequence of Theorem 3.7.

Theorem 3.8. Let $\Phi \in \Phi'_{\Delta,1}[h, q]$. If $f(z) \in \mathrm{A}(p)$,$\dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \in \aleph_0$ and

$\Phi\left(\dfrac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \dfrac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z\right)$ is univalent in U, then

$$h(z) \prec \Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z \right), \quad \textbf{(3.4)}$$

implies $q(z) \prec \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}$. $\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

Combining Theorems 2.13 and 3.8, we obtain the following sandwich-type corollary.

Corollary 3.9. Let $h_1(z)$ and $h_2(z)$ are univalent functions in U and $\Phi \in \Phi_{\Delta,1}[h_2, q_2] \cap \Phi'_{\Delta,1}[h_1, q_1]$. If $f(z) \in \mathrm{A}(p)$, $\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \in \mathrm{H}_0 \cap \aleph_0$ and

$$\Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z \right)$$ is univalent in U, then

$$h_1(z) \prec \Phi\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{z^{p-1}}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{z^{p-1}}; z \right) \prec h_2(z),$$

implies $q_1(z) \prec \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^{p-1}} \prec q_2(z)$. $\left(0 \le \lambda < 1, \mu \notin \{ p, p-1 \}, z \in U, p \in \mathrm{N} \right)$.

Finally, we introduce down a new class of admissible functions $\Phi'_{\Delta,2}[\Omega, q]$.

Definition 3.10. Let Ω be a set in $C, q(z) \neq 0, q'(z) \neq 0$ and $q(z) \in \mathrm{H}$.
The class of admissible functions $\Phi'_{\Delta,2}[\Omega, q]$, consists of those functions $\Phi : C^3 \times \overline{U} \to C$ that satisfy the admissibility condition **:**

$\Phi(u, x, w; \zeta) \in \Omega$
whenever

$$u = q(z), x = \frac{1}{p-\mu-1}\left[-1 + (p-\mu)\ q(z) + \frac{z\ q'(z)}{m\ q(z)} \right],$$

$$\Re\left\{\frac{(p-\mu-1)\ x\left[\ (p-\mu-2)\ w-(p-\mu-1)\ x+1\ \right]}{(p-\mu-1)\ x-(p-\mu)\ u+1}-2\ (p-\mu)\ u+(p-\mu-1)\ x+1\right\}$$

$$\leq\frac{1}{m}\Re\left\{1+\frac{z\ q''(z)}{q'(z)}\right\},$$

where $z\in U,\zeta\in\partial U,\mu\neq p-1,p\in\mathrm{N}$ and $m\geq 1$.

Now, we give the dual result of Theorem 2.19 for the differential superordination.

Theorem 3.11. Let $\Phi\in\Phi'_{\Delta,2}[\Omega,q]$ and $\Delta_{z,p}^{\lambda,\mu,\nu}f(z)\neq 0$. If $f(z)\in\mathrm{A}(p)$,

$\dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}\in\aleph_1$ and $\Phi\left(\dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)},\dfrac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)},\dfrac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)};z\right)$ is

univalent in U, then

$$\Omega\subset\left\{\ \Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)},\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)},\frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3}f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2}f(z)};z\right):z\in U\ \right\}\quad\textbf{(3.5)}$$

implies $q(z)\prec\dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}$. $\left(\ 0\ \leq\ \lambda\ <\ 1\ ,\mu\ \notin\left\{\ p-1,p-2\ \right\},z\in U\ ,p\in\ \mathrm{N}\ \right)$.

Proof. From (2.27) and (3.5), we have

$$\Omega\subset\left\{\ \Psi\big(g(z),z\ g'(z),z^2g''(z)\ ;z\big):z\in U\ \right\}.$$

From (2.25), we see that the admissibility condition for $\Phi\in\Phi'_{\Delta,2}[\Omega,q]$ is equivalent to the admissibility condition for Ψ as given in Definition 1.2. Hence $\Psi\in\Psi'[\Omega,q]$, and by Lemma 1.4, $q(z)\prec g(z)$ or $q(z)\prec\dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}$.

□

If $\Omega\neq C$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi'_{\Delta,2}[h(U),q]$ is written as $\Phi'_{\Delta,2}[h,q]$. The following result is immediate consequence of Theorem 3.11.

Theorem 3.12. Let $\Phi \in \Phi'_{\Delta,2}\ [h,q]$ and $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \neq 0$.

If $f(z) \in \mathrm{A}(p)$, $\dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \in \aleph_1$ and

$$\Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}; z \right)$$

is univalent in U, then

$$h(z) \prec \Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}; z \right), \quad \textbf{(3.6)}$$

implies $$q(z) \prec \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}.$$

$\left(0 \leq \lambda < 1, \mu \notin \{ p-1, p-2 \}, z \in U, p \in \mathrm{N} \right)$.

Combining Theorems 2.20 and 3.12, we obtain the following sandwich-type corollary.

Corollary 3.13. Let $h_1(z)$ and $h_2(z)$ are univalent functions in U,

$\Phi \in \Phi_{\Delta,2}[h_2, q_2] \cap \Phi'_{\Delta,2}\ [h_1, q_1]$ and $\Delta_{z,p}^{\lambda,\mu,\nu} f(z) \neq 0$.

If $f(z) \in \mathrm{A}(p)$, $\dfrac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \in \mathrm{H} \cap \aleph_1$ and

$$\Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}; z \right)$$

is univalent in U, then

$$h_1(z) \prec \Phi\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}, \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}, \frac{\Delta_{z,p}^{\lambda+3,\mu+3,\nu+3} f(z)}{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}; z \right) \prec h_2(z)$$

,

implies

$$q_1(z) \prec \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \prec q_2(z). \quad \left(0 \le \lambda < 1 , \mu \notin \{ p-1, p-2 \}, z \in U , p \in \mathrm{N} \right).$$

4 Open Problem

One can define another class by using another fractional calculus operator or a multiplier operator the samae way as in this paper and hence new results can be obtained.

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