Int. J. Open Problems Compt. Math., Vol. 4, No. 4, December 2011
ISSN 1998-6262; Copyright ©ICSRS Publication, 2011
www.i-csrs.org

Faedo-Galerkin's method for a non linear boundary value problem

Rahmoune Abita, Benabderrahmane Benyattou

Laboratory of Computer Sciences and Mathematics
Faculty of Sciences, Laghouat University,
Road of Ghardaia, BP 37G, Laghouat (03000), Algeria
e-mail: abitarahmoune@yahoo.fr
e-mail: bbenyattou@yahoo.com

Abstract

In this paper, a nonlinear boundary value problem is considered. The use of Faedo-Galerkin techniques and a compactness result, when passing to the limit, permits to prove the existence of the variational solution of the considered problem. The most important results given in this paper, consists to demonstrating the uniqueness of the solution without hypothesis that has been considered by many authors for a similar problem governed by the Laplace operator.

Keywords: *Existence and uniqueness of solution, Faedo-Galerkin, Holder's inequality, Non linear hyperbolic equation, Variational problem.*

1 Introduction

In this work, we consider a nonlinear hyperbolic boundary value problem governed by partial differential equations which describe the evolution of linear elastic materials with Dirichlet-Neumann boundary conditions. Assume certain hypotheses on the data functions. Then, by using Faedo-Galerkin techniques and compactness method, we will prove the existence of the solution. Our main goal is, without taking into account the condition imposed by Lions (cf. J.L. Lions [?]), to prove the uniqueness of the solution. This presents an important result in this work which deserves to be announced. The techniques used are that of Lions (cf. J.L. Lions [?]) for a particular problem by changing the elasticity equation by the Laplace operator and with the Neumann boundary conditions.

2 Problem Formulations

Let Ω be an open and bounded domain in $\mathbb{R}^n$, the boundary Γ of Ω is assumed to be regular and is divided as follows: $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1, Γ_2 are two disjoint parts. We assume that $meas\ (\Gamma_1) > 0$. We pose $\Sigma_i = \Gamma_i \times (0, T)\,,\ i = 1, 2$, where T is a finite real. We indicate by u a vector $u = (u_1, u_2 ..., u_n)$, where $\forall i,\ u_i : Q = \Omega \times]0, T[\to \mathbb{R}$. $u' = \frac{\partial u}{\partial t}, u'' = \frac{\partial^2 u}{\partial t^2}$ denote the time derivative. Let η be the unit outwarde normal vector on Γ.

The classical formulation of the problem is as follows.

Find a displacement $u : Q \to \mathbb{R}^n$, a stress $\sigma : Q \to S_n$, such that

$$\frac{\partial^2 u}{\partial t^2} - div\sigma\,(u) + |u|^\rho\, u = f,\ in\ Q, \tag{1}$$

$$\sigma\,(u) = F\,(\varepsilon(u))\ \ in\ Q, \tag{2}$$

$$\begin{cases} a)\ u = 0\ on\ \Sigma_1, \\ b)\ \sigma(u)\eta = 0\ on\ \Sigma_2, \end{cases} \tag{3}$$

$$\begin{cases} a)\ u(x, 0) = u_0(x), \\ b)\ u'(x, 0) = u_1(x), \end{cases} \quad in\ \Omega. \tag{4}$$

Where ρ is an integer > -1, $\mathcal{S}_n$ will denote the space of second-order symmetric tensors on $\mathbb{R}^n$. u , f and $\sigma\,(u)$ represent the displacement field, the density of volume forces and the tensor of constraints, respectively. *div* denotes the divergence operator of the tensor valued functions and $\sigma = (\sigma_{ij}),\ i, j = 1, ..., n$, stands for the stress tensor field. The latter is obtained from the displacement field by the constitutive law of linear elasticity defined by (2). F is a linear elastic constitutive law, and $\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^T u\right)$ represents the linearized strain tensor. The equation (1), without the nonlinear term $|u|^\rho\, u$, describes the evolution of linear elastic materials, while relations (3) and (4) are the boundary conditions on $\Sigma_i,\ i = 1, 2$ and the initial conditions, respectively. We define the space:

$$\mathcal{H} = L^2(\Omega)_s^{n\times n} = \left\{\sigma = (\sigma_{ij}) \in \mathcal{S}_n :\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$

which is a *Hilbert* space endowed with the inner product

$$\langle \sigma, \tau \rangle = \int_\Omega \sigma_{ij}\tau_{ij}\,dx,$$

and the associated norm is denoted $\|.\|_{\mathcal{H}}$. When no ambiguousness is to fear, we will put :

$$\|v\|_{L^2(\Omega)} = |v|,$$

and we will use the notation $\|v\|_{L^2(\Omega)}$ in possible ambiguousness case.

We assume that the function $F : \Omega \times S_n \to S_n$ satisfies the following hypotheses:

$$\left\{ \begin{array}{l} (a)\ \exists\ m > 0;\ (F(x,\varepsilon),\varepsilon) \geq m\,||\varepsilon||^2,\ \forall \varepsilon \in S_n,\ a.e.\ x \in \Omega, \\ (b)\ \ (F(x,\varepsilon),\tau) = (F(x,\tau),\varepsilon),\ \forall\ \varepsilon,\tau \in S_n,\ a.e.\ x \in \Omega, \\ (c)\ For\ any\ \ \varepsilon \in S_n, x \to F(x,\varepsilon)\ is\ measurable\ on\ \Omega. \end{array} \right. \tag{5}$$

And we assume that the following given data verify

$$f \in L^2(Q), \tag{6}$$

$$u_0 \in V \cap L^p(\Omega),\ p = \rho + 2, \tag{7}$$

$$u_1 \in L^2(\Omega). \tag{8}$$

where

$$V = \left\{ v \in H^1(\Omega), v = 0\ on\ \Sigma_1 \right\}.$$

Remark 2.1 *The hypothesis (5) permits us to consider the operator, noted again by F, define by*

$$F : \mathcal{H} \longrightarrow \mathcal{H}, \quad F(\varepsilon(.)) = F(.,\varepsilon(.)),\ \ a.e. on\ \ \Omega$$

Remark 2.2 *As the linear operator F satisfy*

$$(F(\varepsilon),\varepsilon) \geq 0,\ \forall \varepsilon \in \mathcal{H}.$$

Then F is continuous on $\mathcal{H}$.

Under hypotheses (6),(7) and (8), then by multiplying the equation (1) by $v \in H^1(\Omega)$ and integrating on Ω, using the density of $\mathcal{D}(\Omega)$ in $V \cap L^p(\Omega)$, $p = \rho + 2$ and the *Green's* formula, it is easy to verify that the problem (1)-(4) is equivalent to the following variational problem.

$$\left\{ \begin{array}{c} Find\ u \in V \cap L^p(\Omega)\ such\ that \\ (u'', v) + a(u,v) + (|u|^\rho u, v) = (f,v),\ \forall v \in V \cap L^p(\Omega),\ p = \rho + 2, \\ u(x,0) = u_0(x),\ u'(x,0) = u_1(x),\ x \in \Omega, \end{array} \right.$$

where $a(u,v) = \int_\Omega \sigma(u)\varepsilon(v)dx$.

3 Existence and uniqueness

Our main existence and uniqueness result concerning problem (1)-(4), which we establish in this section, is the following.

3.1 Existence

Theorem 3.1 *Assume that (5)-(8) hold. Then there exists at least one solution to problem (1)-(4) and it satisfies*

$$u \in L^{\infty}(0, T; V \cap L^{p}(\Omega)), \quad p = \rho + 2, \tag{9}$$

$$u' \in L^{\infty}(0, T; L^{2}(\Omega)). \tag{10}$$

Lemma 3.2 *Assume that (5)-(8) hold. Then the initial conditions (4) have a sense.*

Proof of Lemma 3.2

Since hypotheses (5)-(8) are satisfied. Then the results of Theorem 3.1 are satisfied. Since $V \cap L^{p}(\Omega) \subset L^{2}(\Omega)$, from (9), (10) it follows

$$u, \; u' \in L^{\infty}\left(0, T; L^{2}(\Omega)\right).$$

Thus, referring to [?] it results that $u : [0, T] \to L^{2}(\Omega)$, is continuous, possibly after a modification on a subset of $[0, T]$ with zero measure, then is well defined at point 0.

It remains to verify that the second condition in (4) has a sense. From the equation (1), we have

$$\frac{\partial^{2} u}{\partial t^{2}} = f + div\sigma\left(u\right) - \left|u\right|^{\rho} u. \tag{11}$$

Using the fact that $\varepsilon(u) \in L^{\infty}\left(0, T; L^{2}(\Omega)\right)$ and F is continuous in $L^{2}(\Omega)$, then

$$F\left(\varepsilon(u)\right) \in L^{\infty}\left(0, T; L^{2}(\Omega)\right).$$

Therefore,

$$divF\left(\varepsilon(u)\right) \in L^{\infty}\left(0, T; V'\right),$$

where V' designates the dual space of V. It is easy to verify that if $u \in L^{p}(\Omega)$, then $\left|u\right|^{\rho} u \in L^{p'}(\Omega)$ such as $\frac{1}{p} + \frac{1}{p'} = 1$. Then, using (11) we have

$$\frac{\partial^{2} u}{\partial t^{2}} \in L^{2}\left(0, T; L^{2}(\Omega)\right) + L^{\infty}\left(0, T; V' + L^{p'}(\Omega)\right).$$

Hence, in particular case we have:

$$\frac{\partial^2 u}{\partial t^2} \in L^2\left(0,T;V' + L^{p'}(\Omega)\right).$$

Then, referring to *J.L.Lions* [**?**], by (10) it results that $u' : [0,T] \longrightarrow V' + L^{p'}(\Omega)$, is continuous, possibly after a modification on a subset of $[0,T]$ with zero measure, then u' is well defined at point 0.

Proof of Theorem 3.1

A sequence (w_n) of functions having the following properties is introduced:

$*$ $\forall j = 1, ..., m : w_j \in V \cap L^p(\Omega)$;

$*$ The family $\{w_1, w_2, ..., w_m\}$ is linearly independent;

$*$ The space $V_m = [w_1, w_2, ..., w_m]$ generated by the family, $\{w_1, w_2, ..., w_m\}$, is dense in $V \cap L^p(\Omega)$.

Let $u_m = u_m(t)$ be an approached solution such that

$$u_m(t) = \sum_{i=1}^{m} K_{jm}(t) w_i. \tag{12}$$

The K_{jm} being to determined by the following expression:

$$(u''_m(t), w_j) + a(u_m, w_j) + (|u_m|^\rho u_m, w_j) = (f, w_j),\ 1 \le j \le m, \tag{13}$$

which is a nonlinear system of ordinary deferential equations and will be completed by the following initial conditions

$$\left\{ \begin{array}{c} u_m(0) = u_{0m}, \\ u_{0m} = \sum\limits_{i=1}^{m} \alpha_{im} w_i \overset{m \longrightarrow \infty}{\longrightarrow} u_0,\ in\ V \cap L^p(\Omega), \end{array} \right. \tag{14}$$

$$\left\{ \begin{array}{c} u'(0) = u_{1m}, \\ u_{1m} = \sum\limits_{i=1}^{m} \beta_{im} w_i \overset{m \longrightarrow \infty}{\longrightarrow} u,\ in\ \ L^2(\Omega). \end{array} \right. \tag{15}$$

As the family $\{w_1, w_2, ..., w_m\}$ is linearly independent, the system (13), (14) and (15) admits at least one solution u_m in the interval $[0, t_m]$ having the following regularity

$$u_m(t) \in L^2(0, t_m; V_m),\ u'_m(t) \in L^2(0, t_m; V_m).$$

A priori estimates which follow will show that t_m is independent of m.

a) A priori estimates

If we pose

$$\|u\|_1 = (a(u,u))^{\frac{1}{2}} = \left(\int_\Omega F(\varepsilon(u))\, \varepsilon(u) dx\right)^{\frac{1}{2}}.$$

Using the hypothesis (5), it is easily to show that $\|u\|_1$ is a norm on V equivalent to the norm $\|u\|$ of $H^1(\Omega)$.

Multiplying the equation (13) by $K'_{jm}(t)$ and performing the summation over $j = 1, ..., m$, yields

$$(u''_m(t), u'_m(t)) + a(u_m(t), u'_m(t)) + (|u_m|^\rho u_m(t), u'_m(t)) = (f, u'_m(t)). \quad (16)$$

But as $u_m \in L^2(0, t_m; V_m)$, $u'_m \in L^2(0, t_m; V_m)$. Therefore,

$$\varepsilon(u_m),\ \varepsilon(u'_m) \in L^2\left(0, T; L^2(\Omega)\right),$$

using Remark 2.2, we have

$$F\varepsilon(u_m)),\ F(\varepsilon(u'_m) \in L^2\left(0, T; L^2(\Omega)\right).$$

Also, we have

$$\begin{aligned}\frac{d}{dt}a(u_m(t), u_m(t)) &= (F(\varepsilon(u_m(t))), \varepsilon(u'_m(t))) + (F(\varepsilon(u'_m(t))), \varepsilon(u_m(t))) \\ &= a(u_m(t), u'_m(t)) + a(u'_m(t), u_m(t)).\end{aligned}$$

Then, using hypothesis (5b), it exists a constant $C_1 > 0$ such that

$$\begin{aligned}a(u_m(t), u'_m(t)) = \tfrac{1}{2}\tfrac{d}{dt}a(u_m(t), u_m(t)) &= \tfrac{1}{2}\tfrac{d}{dt}\|u_m(t)\|_1^2 \\ &\geq \tfrac{1}{2}C_1\tfrac{d}{dt}\|u_m(t)\|^2.\end{aligned} \quad (17)$$

Therefore, from (16) it follows

$$(u''_m(t), u'_m(t)) + \frac{1}{2}C_1\frac{d}{dt}\|u_m(t)\|^2 + (|u_m|^\rho u_m(t), u'_m(t)) = (f(t), u'_m(t)). \quad (18)$$

On the other hand, we have

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}|u'_m(t)|^2 &= (u''_m(t), u'_m(t)), \\ \tfrac{1}{p}\frac{d}{dt}\|u_m(x,t))\|^p_{L^p(\Omega)} &= (|u_m|^\rho u_m(t), u'_m(t)), \quad p = \rho + 2.\end{aligned}$$

Then from (18), it results

$$\frac{1}{2}\frac{d}{dt}\left[|u'_m(t)|^2 + C_1\|u_m(t)\|^2\right] + \frac{1}{p}\frac{d}{dt}\|u_m(x,t))\|^p_{L^p(\Omega)} \leq (f, u'_m). \quad (19)$$

Then, by passing to the absolute value in (19), it comes

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\left[|u'_m(t)|^2 + C_1\|u_m(t)\|^2\right] &+ \tfrac{1}{p}\frac{d}{dt}\|u_m(x,t))\|^p_{L^p(\Omega)} \\ &\leq |(f(s)|\,|u'_m(s)|.\end{aligned} \quad (20)$$

Hence, by integrating over $(0,t)$, using *Cauchy-Schwarz's* inequality, from (20) it follows

$$\begin{aligned} &\tfrac{1}{2}\left(|u'_m(t)|^2 + C_1 \|u_m(t)\|^2\right) + \tfrac{1}{p}\|u_m(t)\|^p_{L^p(\Omega)} \\ &\leq \tfrac{1}{2}|u_{1m}|^2 + \tfrac{1}{2}C_1\|u_{0m}\|^2 + \tfrac{1}{p}\|u_m(0)\|^p_{L^p(\Omega)} + \int_0^t |(f(s)|\,|u'_m(s)|\,ds. \end{aligned} \tag{21}$$

Then, using *Young's* inequality we have:

$$|(f(s)|\,|u'_m(s)| \leq \frac{1}{2}|(f(s)|^2 + \frac{1}{2}|u'_m(s)|^2.$$

Then from (21), it results

$$\begin{aligned} &\tfrac{1}{2}\left(|u'_m(t)|^2 + C_1\|u_m(t)\|^2\right) + \tfrac{1}{p}\|u_m(t)\|^p_{L^p(\Omega)} \leq \tfrac{1}{2}|u_{1m}|^2 \\ &+\tfrac{1}{2}C_1\|u_{0m}\|^2 + \tfrac{1}{p}\|u_{0m}\|^p_{L^p(\Omega)} + \tfrac{1}{2}\int_0^t |(f(s)|^2\,ds + \tfrac{1}{2}\int_0^t |u'_m(s)|^2\,ds. \end{aligned} \tag{22}$$

Thus, using (14), (15) and (6) it results

$$\frac{1}{2}|u_{1m}|^2 + \frac{1}{2}C_1\|u_{0m}\|^2 + \frac{1}{p}\|u_{0m}\|^p_{L^p(\Omega)} + \frac{1}{2}\int_0^t |(f(s)|^2\,ds \leq C_2,\ \forall m \in \mathbb{N}^*.$$

Then, from (22) it deduces

$$\frac{1}{2}\left(|u'_m(t)|^2 + C_1\|u_m(t)\|^2\right) + \frac{1}{p}\|u_m(t)\|^p_{L^p(\Omega)} \leq C_2 + \frac{1}{2}\int_0^t |u'_m(s)|^2\,ds. \tag{23}$$

Therefore,

$$|u'_m(t)|^2 \leq 2C_2 + \int_0^t |u'_m(s)|^2\,ds,\ \forall t \in [0,T].$$

Using *Gronwall's* inequality, it is concluded that

$$|u'_m(t)| \leq C, \tag{24}$$

where C is a constant independent of m.

Consequently, from (23), we deduct

$$\|u_m(t)\|_{L^p(\Omega)} + \|u_m(t)\| \leq C. \tag{25}$$

So, the independence of t_m with respect to m. Passing to the limit where $m \to \infty$, from (24), (25), we conclude

$$\begin{cases} u_m & \textit{remains in a bounded set of } \ L^\infty(0,T;V \cap L^p(\Omega)), \\ u'_m & \textit{remains in a bounded set of } \ L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{26}$$

b) Passage to the limit

From (26), we deduce that it can extract a sub sequence (u_μ) of (u_m) such as

$$u_\mu \overset{weak\ star}{\longrightarrow} u \ \ in\ Ł^\infty\left(0,T;V\cap L^p(\Omega)\right), \tag{27}$$

$$u'_\mu \overset{weak\ star}{\longrightarrow} u' \ in\ L^\infty\left(0,T;L^2(\Omega)\right). \tag{28}$$

Since $V\cap L^p(\Omega)\subset L^2(\Omega)$, from (26) it concludes that the sequences (u_m) and (u'_m) are bounded in $L^2\left(0,T;L^2(\Omega)\right)$, Then, in particular, (u_m) is bounded in $H^1(Q)$. Referring to *J.L. Lions* [?], it is known that the injection of $H^1(Q)$ in $L^2(Q)$ is compact.

This permit us to assume that the extracted sub sequence (u_μ) verify, in addition to relations (27) and (28),

$$u_\mu \longrightarrow u\ in\ L^2(Q). \tag{29}$$

As $|u_m|^\rho u_m$ is in a bounded set of $L^\infty\left(0,T;L^{p'}(\Omega)\right)$, $\frac{1}{p}+\frac{1}{p'}=1$, then we can verify that

$$|u_\mu|^\rho u_\mu \overset{weak\ star}{\rightarrow} |u|^\rho u \ \ in\ \ L^\infty\left(0,T;L^{p'}(\Omega)\right). \tag{30}$$

Let j be fixed and $\mu > j$. Then, using (13) we have

$$\left(u''_\mu(t),w_j\right)+a\left(u_\mu,w_j\right)+\left(|u_\mu|^\rho u_\mu,w_j\right)=\left(f,w_j\right),\quad j=1,...,m. \tag{31}$$

Then, from (27) and (28), it results

$$\begin{aligned} a(u_\mu,w_j) &\overset{weak\ star}{\longrightarrow} a(u,w_j)\ \ in\ \ L^\infty(0,T),\\ (u'_\mu,w_j) &\overset{weak\ star}{\longrightarrow} (u',w_j)\ \ in\ \ L^\infty(0,T). \end{aligned}$$

Therefore,

$$\left(u''_\mu(t),w_j\right)\longrightarrow (u''(t),w_j)\ \ in\ \mathcal{D}'(0,T). \tag{32}$$

Thus, using (30) we deduce that

$$\left(|u_\mu|^\rho u_\mu,w_j\right)\overset{weak\ star}{\longrightarrow}\left(|u|^\rho u,w_j\right)\ in\ L^\infty(0,T).$$

Then (31) takes the form

$$\frac{\partial^2}{\partial t^2}(u,w_j)+a(u,w_j)+\left(|u|^\rho u,w_j\right)=\left(f,w_j\right).$$

Finally, be using the density of V_m in $V\cap L^p(\Omega)$ we obtain

$$\left(u'',v\right)+a\left(u,v\right)+\left(|u|^\rho u,v\right)=\left(f,v\right),\ \forall v\in V\cap L^p(\Omega).$$

Then u satisfies (1).

Remains to verify the initial conditions. Using (27), we have

$$u_\mu(0) \to u(0) \text{ in } L^2(\Omega) \text{ weak.}$$

Then, using (14) we deduce in particular that

$$u_\mu(0) = u_{0\mu} \to u_0 \text{ in } V \cap L^p(\Omega).$$

Hence, it results the first condition in (4). On the other hand, by using (32) we have

$$\left(u''_\mu(t), w_j\right) \overset{weak\ star}{\longrightarrow} \left(u''(t), w_j\right) \ in\ L^\infty(0,T).$$

Therefore,

$$\left(u'_\mu(0), w_j\right) \longrightarrow (u'(0), w_j).$$

Since $\left(u'_\mu(0), w_j\right) \longrightarrow (u_1, w_j)$, we have $(u'(0), w_j) = (u_1, w_j), \forall j$. Then the second condition in (4) is satisfied.

3.2 Uniqueness

Many authors, for some particular problems have showed the uniqueness of the solution basing on the condition, see [?], $\rho \leq \dfrac{2}{n-2}$, for particular problems. However, we will give in this subsection a new result concerning the uniqueness of the solution of the problem considered, where we will demonstrate the uniqueness of the solution without this assumption.

Theorem 3.3 *Under the hypotheses of the Theorem 3.1, then the solution u obtained is unique*

Proof of Theorem 3.3 Let u, v be two solutions of problem (1)-(4), to the sense of the Theorem 3.1.

Setting $w = u - v$, from then the linearity of F we conclude that

$$w''(t) - divF(\varepsilon(w)) + (|u|^\rho u - |v|^\rho v) = 0,\ in\ Q, \tag{33}$$

$$w(0) = w'(0) = 0,\ on\ \Omega, \tag{34}$$

$$w = 0\ on\ \ \Sigma_1,\ \sigma(w)\eta = 0\ on\ \Sigma_2, \tag{35}$$

$$w(t) \in L^\infty(0,T; V \cap L^p(\Omega)),\ p = \rho + 2, \tag{36}$$

$$w'(t) \in L^\infty(0,T; L^p(\Omega)),\ p = \rho + 2. \tag{37}$$

Multiplying the equation (33) by w' and integrating on Ω. Then, by using *Green*'s formula together with the conditions (34), (35), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|w'(t)|^2\right) + a\left(w(t), w'(t)\right) = \int_\Omega \left(|v|^\rho v - |u|^\rho u\right) w' dx. \tag{38}$$

Then, using hypothesis (5b), we have

$$\begin{aligned} &a\left(w(t), w'(t)\right) = \frac{d}{dt}a(w(t), w(t)) - \textstyle\int_\Omega \frac{d}{dt}\left(F(\varepsilon(w))\right)\varepsilon(w)dx \\ &\geq C_1\frac{d}{dt}\|w\|^2 - \textstyle\int_\Omega \left(F(\varepsilon(w'))\right)\varepsilon(w)dx \geq C_1\frac{d}{dt}\|w\|^2 - a\left(w(t), w'(t)\right), \end{aligned}$$

Then, from (38) it follows

$$\frac{1}{2}\frac{d}{dt}\left(|w'(t)|^2 + C_1\|w\|^2\right) \leq \int_\Omega (|v|^\rho v - |u|^\rho u) w' dx. \tag{39}$$

Without loss of generality, we assume that $|u| \leq |v|$.
Then, using *Holder's* inequality we can write

$$\begin{cases} \left|\int_\Omega (|v|^\rho v - |u|^\rho u) w' dx\right| \leq (\rho+1)\int_\Omega |v|^\rho |w| |w'| dx \\ \leq (\rho+1) \||v|^\rho\|_{L^n(\Omega)} \||w|\|_{L^q(\Omega)} |w'|, \quad \frac{1}{n}+\frac{1}{q}+\frac{1}{2}=1. \end{cases}$$

Then, referring to [4], we have

$$\|v\|_{L^{kq}(\Omega)} = \left\||v|^k\right\|_{L^q(\Omega)}^{\frac{1}{k}} \quad \forall k, q \in \mathbb{N}^*. \tag{40}$$

For all $\rho > -1$, we put

$$k = E\left(\frac{\rho(n-2)}{2}\right) + 1,\ k \in \mathbb{N}^*,\ \forall n \geq 2,$$

where $E(x)$, denotes the integer part of x. Then, we have

$$\rho \leq \frac{2k}{n-2}, \quad k \in \mathbb{N}^*,\ n \neq 2 \ \ (\rho \textit{ any finished so } n = 2).$$

Thus, if $\frac{1}{n}+\frac{1}{q}+\frac{1}{2}=1$, then $\rho n \leq qk$ and referring to [?] we have

$$H^1(\Omega) \subset L^q(\Omega). \tag{41}$$

Then, using (40) and (41) we have

$$\||v|^\rho\|_{L^n(\Omega)} = \|v\|_{L^{\rho n}(\Omega)}^\rho \leq \|v\|_{L^{kq}(\Omega)}^\rho = \left\||v|^k\right\|_{L^q(\Omega)}^{\frac{\rho}{k}} \leq \left\||v|^k\right\|^{\frac{\rho}{k}} \leq C\|v\|^\rho,$$

which implies that

$$\left|\int_{\Omega}\left(|v|^{\rho}v-|u|^{\rho}u\right)w'dx\right|\leq C\left\||v|^{\rho}\right\|_{L^{n}(\Omega)}||w||\,|w'|\leq C\left\|v\right\|^{\rho}||w||\,|w'|\,.$$

Since $v\in L^{\infty}\left(0,T;V\cap L^{p}(\Omega)\right)$, then $\|v\|^{\rho}\leq C$, consequently

$$\left|\int_{\Omega}(|v|^{\rho}v-|u|^{\rho}u)w'dx\right|\leq C_{2}\,||w||\,|w'|\,.$$

Therefore, using the *Young*'s inequality, then (39) becomes

$$\frac{1}{2}\frac{d}{dt}\left(|w'(t)|^{2}+C_{1}\left\|w(t)\right\|^{2}\right)\leq C_{2}\,||w(t)||\,|w'(t)|\leq\frac{1}{2}C_{2}\left(||w(t)||^{2}+|w'(t)|^{2}\right).$$

Integrating equation above together with the initial conditions (34), we obtain

$$|w'(t)|^{2}+\|w(t)\|^{2}\leq C_{3}\int_{0}^{t}\left(|w'(s)|^{2}+\|w(s)\|^{2}\right)ds.$$

Finally, use *Gronwall's* inequality to find $w=0$.

4 Conclusion

If we put

$$\sigma\left(u\right)=F\left(\varepsilon(u)\right)=2\varepsilon(u)-Trace(\varepsilon(u))I,$$

where I denotes the identity operator and denotes the trace operator. Then, the problem (1)-(4), without the condition $\sigma(u)\eta=0$ on Σ_2, is reduced to the particular problem studied by *J.L. Lions* [**?**]. Since F is linear and satisfies the hypotheses (5). Then, Theorems 3.1 and 3.3 are verified of particular problem considered in [**?**].

5 Open Problem

In this section, we should present an open problem, which consists in obtaining the same results as in Theorems 3.1 and 3.3 by reducing the assumptions on the operator F , where F is a non linear. On the other hand, obtaining the same results found in this work for contact with or without friction problems is of major concern.

ACKNOWLEDGEMENTS.

The author is highly grateful to the anonymous referee for his/her valuable comments and suggestions for the improvement of the paper.

References

[1] B. Benabderrahmane, B. Nouiri and Y. Boukhatem, " Index of the elasticity operator with contact without friction boundary", *Studia Univ. Babes-Bolyai Mathematica*, Vol. LIIII, No. 3, (2008), pp.3-11.

[2] Y. Boukhatem, B. Benabderrahmane, A. Rahmoune, " Résolvante du système de Lamé dans un polygone", *Anal. Univ. Oradea, fasc. Matematica*, Tom XV, (2008), pp.219-238.

[3] R. Dautry et J.L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques, tome 3*, Masson, Paris, (1985).

[4] Y. Choquet-Bruhat, C. Dewitt-Morette, M. Dillard-Bleick, Analysis, *Manifolds and Physics*, North-Holland Publishing Company, Amsterdam, New York, Oxford, (1977).