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Boundedness for Multilinear Commutator of Bochner-Riesz Operator with Weighted Lipschitz Functions

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Abstract

In this paper, we obtain the boundedness for the multilinear commutators related to the Bochner-Riesz operator with weighted Lipschitz functions.

Keywords: *Multilinear commutator; Bochner-Riesz operator;Weighted Lipschitz function.*
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1 Introduction and Preliminaries

Let b be a locally integrable function on R^n and T be the Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

Janson [3][8] proved that $[b, T]$ is bounded on L^p for $1 < p < \infty$ if and only if $b \in BMO$. Chanillo(see [2]) proved that the commutator $[b, I_\alpha]$ generated by $b \in BMO$ and the fractional integral operator I_α is bounded from $L^p(R^n)$ to $L^q(R^n)$, where $1 < p < q < \infty$ and $1/p - 1/q = \alpha/n$. Then Paluszyński(see [12]) showed that $b \in Lip_\beta$(the homogeneous Lipschitz space) if and only if the commutator $[b, T]$ is bounded from L^p to L^q, where $1 < p < q < \infty$, $0 < \beta < 1$ and $1/q = 1/p - \beta/n$. Also Paluszyński (see [12]) obtain that $b \in Lip_\beta$ if and only if the commutator $[b, I_\alpha]$ is bounded from L^p to L^r, where $1 < p < r < \infty$, $0 < \beta < 1$ and $1/r = 1/p - (\beta + \alpha)/n$ with $1/p > (\beta + \alpha)/n$.

On the other hand, In [1][6], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained. The purpose of this paper is to establish boundedness for the multilinear commutators related to the Bochner-Riesz operator with $b \in Lip_{\beta,\nu}(R^n)$(the weighted Lipschitz space).

2 Notations and Results

A non-negative function ν defined on R^n is called a weight if it is locally integral function. A weight ν is said to belong to the Muckenhoupt class $A_p(R^n)$ for $1 < p < \infty$, if there exists a constant C such that

$$\frac{1}{|B|}\int_B \nu(x)dx\left(\frac{1}{|B|}\int_B (\nu(x))^{-\frac{1}{p-1}}dx\right)^{p-1} \le C$$

for every ball $B \subset R^n$. The class $A_1(R^n)$ is defined replacing the above inequality by

$$\frac{1}{|B|}\int_B \nu(x)dx \le C\nu(x), \qquad \text{a.e. } x \in R^n,$$

for every ball $B \subset R^n$(see [5]).

A locally integral non-negative function ν is said to $A(p,q)(1 < p, q < \infty)$(see [11]) if there exists C such that

$$\left(\frac{1}{|B|}\int_B \nu(x)^q dx\right)^{1/q}\left(\frac{1}{|B|}\int_B (\nu(x))^{-p'}dx\right)^{1/p'} \le C$$

for every ball $B \subset R^n$ and $1/p' + 1/p = 1$.

Then let us introduce some notations(see [5][10][14][15]). In this paper, B will denote a ball of R^n, and for a ball B let $f_B = |B|^{-1}\int_B f(x)dx$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y) - f_B|dy.$$

It is well-known that (see [14])

$$f^{\#}(x) \approx \sup_{B\ni x}\inf_{c\in C}\frac{1}{|B|}\int_B |f(y) - C|dy.$$

For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|dy.$$

We write that $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$. Let M_γ be the fractional maximal operator, that is

$$M_\gamma(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{\gamma-1}} \int_B |f(y)|dy, \quad 0 < \gamma < 1.$$

And following [6], we will say that a locally integral function f belongs to the weighted Lipschitz space $Lip^p_{\beta,\nu}$ for $1 \le p \le \infty$, $0 < \beta < 1$ and $\nu \in A_\infty(R^n)$, that is

$$\sup_B \frac{1}{\nu(B)^{\beta/n}} \left[\frac{1}{\nu(B)} \int_B |f(x) - f_B|^p \nu(x)^{1-p} dx \right]^{1/p} \le C < \infty,$$

where the supremum is taken over all balls $B \subset R^n$.

Modulo constants, the Banach space of such functions is defined by $Lip^p_{\beta,\nu}$. The smallest bound C satisfying conditions above is then taken to be the norm of f in these spaces, and is denoted by $||f||_{Lip^p_{\beta,\nu}}$. Put $Lip_{\beta,\nu} = Lip^1_{\beta,\nu}$. Obviously, for the case $\nu = 1$, then the $Lip_{\beta,\nu}(R^n)$ is the classical $Lip_\beta(R^n)$ space.

Let $\nu \in A_1(R^n)$, García-Cuerva in [4] proved that the spaces $||f||_{Lip^p_{\beta,\nu}}$ coincide, and the norm of $|| \cdot ||_{Lip^p_{\beta,\nu}}$ are equivalent with respect to different values of provided that $1 \le p \le \infty$.

For $b_j \in Lip_{\beta,\nu}(R^n)(j = 1, \cdots, m)$, set

$$||\vec{b}||_{Lip_{\beta,\nu}} = \prod_{j=1}^m ||b_j||_{Lip_{\beta,\nu}}.$$

Given a positive integer m and $1 \le j \le m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C^m_j$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C^m_j$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Lip_{\beta,\nu}} = ||b_{\sigma(1)}||_{Lip_{\beta,\nu}} \cdots ||b_{\sigma(j)}||_{Lip_{\beta,\nu}}$.

In this paper, we will study some multilinear commutators as follows.

Definition. Suppose $b_j's$ are the fixed locally integral functions on R^n and $m \in N, (j = 1, \cdots, m)$. The maximal operator $B^{\vec{b}}_{\delta,*}$ associated with the multilinear commutator generated by the Bochner-Riesz operator is defined by

$$B^{\vec{b}}_{\delta,*}(f)(x) = \sup_{t>0} |B^{\vec{b}}_{\delta,t}(f)(x)|,$$

where

$$B^{\vec{b}}_{\delta,t}(f)(x) = \int_{R^n} B^\delta_t(x-y) f(y) \prod_{j=1}^m (b_j(x) - b_j(y)) dy,$$

$B_t^\delta(x) = t^{-n}B^\delta(x/t)$ and $(B_t^\delta(f))\hat{}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$. We also define

$$B_*^\delta(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)| = \sup_{t>0} \left| \int_{R^n} B_t^\delta(x-y)f(y)dy \right|,$$

which is the Bochner-Riesz operator([7][9][10][15]).

Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then,$B_{\delta,t}^{\vec{b}}(f)(x)$ may be viewed as a mapping from R^n to H, and it is clear that

$$B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||$$

and

$$B_{\delta,*}^{\vec{b}}(f)(x) = ||B_{\delta,t}^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $B_{\delta,*}^{\vec{b}}$ is just the commutator of order m. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors.

Now we state our theorems as following.

Theorem 2.1 Let $\nu \in A_1(R^n)$ and $b_j \in Lip_{\beta,\nu}(R^n)$ for $j = 1, \cdots, m$, $1/q = 1/p - m\beta/n$ for $0 < \beta < 1$, $0 < \varepsilon < 1 < s < n/\beta$. Then there exists a constant $C > 0$ such that

$$M_\varepsilon^\#(B_{\delta,*}^{\vec{b}}(f))(\tilde{x}) \le C\nu(\tilde{x})^m ||\vec{b}||_{Lip_{\beta,\nu}} \left(\sum_{j=1}^m \sum_{\sigma \in C_j^m} M_{m\beta,\nu,s}(B_{\delta,*}^{\vec{b}_{\sigma^c}}(f))(\tilde{x}) + M_{m\beta,\nu,s}(f)(\tilde{x}) \right)$$

for any smooth function f and a.e. $\tilde{x} \in R^n$, and where

$$M_{m\beta,\nu,s}(f)(x) = \sup_{B \ni x} \left(\frac{1}{\nu(B)^{1-sm\beta/n}} \int_B |f(y)|^s \nu(y)dy \right)^{1/s}$$

Theorem 2.2 Let $\nu \in A_1(R^n)$, $1/q = 1/p - m\beta/n$ for $0 < \beta < 1$ and $1 < p < q < \infty$. If $b_j \in Lip_{\beta,\nu}(R^n)$ for $j = 1, \cdots, m$, then the commutator $B_{\delta,*}^{\vec{b}}$ is bounded from $L^p(\nu)$ to $L^q(\nu^{1-q})$.

3. Some lemmas

Lemma 3.1(see [5]) Let $0 < p, \varepsilon < \infty$ and $\nu \in \bigcup_{1 \le \tau < \infty} A_\tau(R^n)$. There exists a positive C such that

$$\int_{R^n} M_\varepsilon f(x)^p \nu(x)dx \le C \int_{R^n} M_\varepsilon^\# f(x)^p \nu(x)dx$$

for any smooth function f for which the left-hand side is finite.

Lemma 3.2([5, p.485]) Let $0 < p < q < \infty$ and for any function $f \ge 0$. We define that, for $1/r = 1/p - 1/q$

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}||f||_{WL^q}.$$

4. Proof of Theorem 2.1 and 2.2

Proof of Theorem 2.1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B |B^{\vec{b}}_{\delta,*}(f)(x) - C_0|^\varepsilon dx)\right)^{1/\varepsilon}\\ \leq\ & C\nu(\tilde{x})^m||\vec{b}||_{Lip_{\beta,\nu}}\left(\sum_{j=1}^m\sum_{\sigma\in C_j^m} M_{m\beta,\nu,s}(B^{\vec{b}_{\sigma^c}}_{\delta,*}f)(\tilde{x}) + M_{m\beta,\nu,s}(f)(\tilde{x})\right).\end{aligned}$$

Fix a ball $B = B(x_0, r)$ and $\tilde{x} \in B$. We first consider the **Case m=1** . Write, for $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$,

$$B^{b_1}_{\delta,t}(f)(x) = (b_1(x)-(b_1)_B)B^\delta_t(f)(x) - B^\delta_t((b_1-(b_1)_B)f_1)(x) - B^\delta_t((b_1-(b_1)_B)f_2)(x).$$

Let $C_0 = B^\delta_*(((b_1)_B - b_1)f_2)(x_0)$, then

$$\begin{aligned}&|B^{b_1}_{\delta,*}(f)(x) - B^\delta_*(((b_1)_B - b_1)f_2)(x_0)|\\ =\ & \Big|||B^{b_1}_{\delta,t}(f)(x)|| - ||B^\delta_t(((b_1)_B - b_1)f_2)(x_0)||\Big|\\ \leq\ & ||B^{b_1}_{\delta,t}(f)(x) - B^\delta_t(((b_1)_B - b_1)f_2)(x_0)||\\ \leq\ & ||(b_1(x) - (b_1)_B)B^\delta_t(f)(x)|| + ||B^\delta_t((b_1 - (b_1)_B)f_1)(x)||\\ & +||B^\delta_t((b_1 - (b_1)_B)f_2)(x) - B^\delta_t((b_1 - (b_1)_B)f_2)(x_0)||\\ =\ & I(x) + II(x) + III(x).\end{aligned}$$

For $I(x)$, by Hölder's inequality with exponent $1/s+1/s' = 1$ and $1 < s < n/\beta$, we get

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B |I(x)|^\varepsilon dx\right)^{1/\varepsilon} \leq \frac{1}{|B|}\int_B |I(x)|dx\\ \leq\ & \left(\frac{1}{|2B|}\int_{2B}|b_1(x) - (b_1)_{2B}|^{s'}\nu(x)^{1-s'}dx\right)^{1/s'}\left(\frac{1}{|B|}\int_B |B^\delta_*(f)(x)|^s\nu(x)dx\right)^{1/s}\\ \leq\ & C\frac{1}{|2B|^{1/s'}}\nu(2B)^{\beta/n}\frac{1}{\nu(2B)^{\beta/n}}\nu(2B)^{1/s'}\left(\frac{1}{\nu(2B)}\int_{2B}|b_1(x) - (b_1)_{2B}|^{s'}\nu(x)^{1-s'}dx\right)^{1/s'}\\ & \times\frac{1}{|B|^{1/s}}\nu(B)^{1/s-\beta/n}\left(\frac{1}{\nu(B)^{1-s\beta/n}}\int_B |B^\delta_*(f)(x)|^s\nu(x)dx\right)^{1/s}\\ \leq\ & C\frac{\nu(B)}{|B|}||b_1||_{Lip_{\beta,\nu}}M_{\beta,\nu,s}(B^\delta_*f)(\tilde{x})\\ \leq\ & C\nu(\tilde{x})||b_1||_{Lip_{\beta,\nu}}M_{\beta,\nu,s}(B^\delta_*f)(\tilde{x}).\end{aligned}$$

For $II(x)$, by Lemma 3.2 and Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |II(x)|^\varepsilon dx\right)^{1/\varepsilon}\\
\leq\ & C\frac{1}{|B|}\int_{R^n} |(b_1(x)-(b_1)_{2B})f(x)\chi_{2B}(x)|dx\\
\leq\ & C\frac{1}{|B|}\int_{2B} |b_1(x)-(b_1)_{2B}||f(x)|dx\\
\leq\ & C\nu(\tilde{x})||b_1||_{Lip_{\beta,\nu}} M_{\beta,\nu,s}(f)(\tilde{x}),
\end{aligned}$$

For $III(x)$, we have, for $x \in B$,

$$\begin{aligned}
C(x) &= ||B_t^\delta((b_1-(b_1)_{2B})f_2)(x) - B_t^\delta((b_1-(b_1)_{2B})f_2)(x_0)||\\
&= \sup_{t>0}\left|\int_{(2B)^c}(b_1(y)-(b_1)_{2B})f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy\right|.
\end{aligned}$$

We consider the following two cases:
Case 1. $0 < t \leq d$. In this case, notice that([9])

$$|B_1^\delta(x)| \leq C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned}
&\left|\int_{(2B)^c}(b_1(y)-(b_1)_{2B})f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy\right|\\
\leq\ & Ct^{-n}\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} |b_1(y)-(b_1)_{2B}||f(y)|(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy\\
\leq\ & C(t/d)^{\delta-(n-1)/2}\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_1(y)-(b_1)_{2B}||f(y)|dy\right)\\
\leq\ & C\sum_{k=1}^\infty \frac{2^{k((n-1)/2-\delta)}}{|2^{k+1}B|}\int_{2^{k+1}B}|b_1(y)-(b_1)_{2^{k+1}B}||f(y)|dy\\
& +C\sum_{k=1}^\infty \frac{2^{k((n-1)/2-\delta)}}{|2^{k+1}B|}|(b_1)_{2B}-(b_1)_{2^{k+1}B}|\int_{2^{k+1}B}|f(y)|dy\\
\leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\nu(\tilde{x})||b_1||_{Lip_{\beta,\nu}} M_{\beta,\nu,s}(f)(\tilde{x})\\
& +C\sum_{k=1}^\infty k2^{k((n-1)/2-\delta)}\nu(\tilde{x})||b_1||_{Lip_{\beta,\nu}} M_{\beta,\nu,s}(f)(\tilde{x})\\
\leq\ & C\nu(\tilde{x})||b_1||_{Lip_{\beta,\nu}} M_{\beta,\nu,s}(f)(\tilde{x}).
\end{aligned}$$

Case 2. $t > d$. In this case, we choose δ_0 such that $(n-1)/2 < \delta_0 < \min(\delta, (n+1)/2)$, notice that (see [9])

$$|(\partial/\partial x)B_1^\delta(x)| \le C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned}
&\left|\int_{(2B)^c}(b_1(y)-(b_1)_{2B})f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy\right| \\
\le\ & Ct^{-n}\int_{(2B)^c}|b_1(y)-(b_1)_{2B}||f(y)||B^\delta((x-y)/t)-B^\delta((x_0-y)/t)|dy \\
\le\ & Ct^{-n-1}\int_{(2B)^c}|b_1(y)-(b_1)_{2B}||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta+(n+1)/2)}dy \\
\le\ & Ct^{-n-1}\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|b_1(y)-(b_1)_{2B}||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\
\le\ & C(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta_0)}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_1(y)-(b_1)_{2B}||f(y)|dy\right) \\
\le\ & C\nu(\tilde{x})||b_1||_{Lip_{\beta,\nu}}M_{\beta,\nu,s}(f)(\tilde{x}),
\end{aligned}$$

where we use the fact that $|(b_1)_{2B}-(b_1)_{2^{k+1}B}| \le C\nu(\tilde{x})\nu(2^{k+1}B)^\beta||b_1||_{Lip_{\beta,\nu}}$, thus

$$\left(\frac{1}{|B|}\int_B|III(x)|^\varepsilon dx\right)^{1/\varepsilon} \le C\nu(\tilde{x})||b_1||_{Lip_{\beta,\nu}}M_{\beta,\nu,s}(f)(\tilde{x}).$$

Now, we consider the **Case** $m \ge 2$. we have, for $b = (b_1, \cdots, b_m)$,

$$\begin{aligned}
&B_{\delta,t}^{\vec{b}}(f)(x) = \int_{R^n}\prod_{j=1}^m(b_j(x)-b_j(y))B_t^\delta(x-y)f(y)dy \\
=\ & \int_{R^n}\prod_{j=1}^m[(b_j(x)-(b_j)_{2B})-(b_j(y)-(b_j)_{2B})]B_t^\delta(x-y)f(y)dy \\
=\ & \sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2B})_\sigma\int_{R^n}((b(y)-(b)_{2B})_{\sigma^c}B_t^\delta(x-y)f(y)dy \\
=\ & \prod_{j=1}^m(b_j(x)-(b_j)_{2B})B_t^\delta(f)(x)+(-1)^mB_t^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2B})f)(x) \\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2B})_\sigma\int_{R^n}(b(y)-(b)_{2B})_{\sigma^c}B_t^\delta(x-y)f(y)dy
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{j=1}^{m}(b_j(x)-(b_j)_{2B})B_t^{\delta}(f)(x)\\
&\quad +(-1)^m B_t^{\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f)(x)\\
&\quad +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} c_{m,j}(b(x)-(b)_{2B})_{\sigma}B_{\delta,t}^{\vec{b}_{\sigma^c}}(f)(x),
\end{aligned}
$$

thus, recall that $C_0 = B_*^{\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x_0)$,

$$
\begin{aligned}
& |B_{\delta,*}^{\vec{b}}(f)(x) - B_*^{\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x_0)|\\
= & \left| ||B_{\delta,t}^{\vec{b}}(f)(x)|| - ||B_t^{\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x_0)|| \right|\\
\le & ||B_{\delta,t}^{\vec{b}}(f)(x) - B_t^{\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x_0)||\\
\le & ||\prod_{j=1}^{m}(b_j(x)-(b_j)_{2B})B_t^{\delta}(f)(x)||\\
+ & \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_{2B})_{\sigma}B_{\delta,t}^{\vec{b}_{\sigma^c}}(f)(x)||\\
+ & ||B_t^{\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_1)(x)||\\
+ & ||B_t^{\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x) - B_t^{\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2B})f_2)(x_0)||\\
= & I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

For $I_1(x)$, by Hölder's inequality with exponent $1/s+1/s_1+\cdots+1/s_m=1$ and $1<s<n/\beta$, set $p>m$, we get

$$
\begin{aligned}
& \left(\frac{1}{|B|}\int_B |I_1(x)|^{\varepsilon}dx\right)^{1/\varepsilon} \le \frac{1}{|B|}\int_B |I_1(x)|dx\\
\le & \prod_{j=1}^{m}\left(\frac{1}{|2B|}\int_{2B}|b_j(x)-(b_j)_{2B}|^{s_j}\nu(x)^{1-s_j/m}dx\right)^{1/s_j}\left(\frac{1}{|B|}\int_B |B_*^{\delta}(f)(x)|^{s}\nu(x)dx\right)^{1/s}\\
\le & C\frac{1}{|2B|^{1/s_1}}\nu(2B)^{1-1/m}\nu(2B)^{\beta/n}\frac{1}{\nu(2B)^{\beta/n}}\nu(2B)^{1/s_1}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\frac{1}{\nu(2B)}\int_{2B}|b_1(x)-(b_1)_{2B}|^{s_1}\nu(x)^{1-s_1}dx\right)^{1/s_1}\cdots\\
&\times\frac{1}{|2B|^{1/s_m}}\nu(2B)^{1-1/m}\nu(2B)^{\beta/n}\frac{1}{\nu(2B)^{\beta/n}}\nu(2B)^{1/s_m}\\
&\times\left(\frac{1}{\nu(2B)}\int_{2B}|b_1(x)-(b_1)_{2B}|^{s_m}\nu(x)^{1-s_m}dx\right)^{1/s_m}\\
&\times\frac{1}{|B|^{1/s}}\nu(B)^{1/s-m\beta/n}\left(\frac{1}{\nu(B)^{1-sm\beta/n}}\int_B|B_*^\delta(f)(x)|^s\nu(x)dx\right)^{1/s}\\
\leq\ & C\frac{\nu(B)^m}{|B|}||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(B_*^\delta f)(\tilde{x})\\
\leq\ & C\frac{\nu(B)}{|B|^{1/m}}\cdots\frac{\nu(B)}{|B|^{1/m}}||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(B_*^\delta f)(\tilde{x})\\
\leq\ & C\left(\frac{1}{|2B|}\int_{2B}\nu(x)^m dx\right)^{1/m}\cdots\left(\frac{1}{|2B|}\int_{2B}\nu(x)^m dx\right)^{1/m}||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(B_*^\delta f)(\tilde{x})\\
\leq\ & C|2B|^{-1/m}\left[\left(\int_{2B}\nu(x)^{m\cdot\frac{p}{m}}\right)^{\frac{m}{p}}|2B|^{1-\frac{m}{p}}\right]^{1/m}\cdots\\
&\times|2B|^{-1/m}\left[\left(\int_{2B}\nu(x)^{m\cdot\frac{p}{m}}\right)^{\frac{m}{p}}|2B|^{1-\frac{m}{p}}\right]^{1/m}||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(B_*^\delta f)(\tilde{x})\\
\leq\ & C\left(\frac{1}{|2B|}\int_{2B}\nu(x)^p dx\right)^{1/p}\cdots\left(\frac{1}{|2B|}\int_{2B}\nu(x)^p dx\right)^{1/p}||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(B_*^\delta f)(\tilde{x})\\
\leq\ & C\nu(\tilde{x})^m||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(B_*^\delta f)(\tilde{x}),
\end{aligned}
$$

where we use the fact that ν satisfies the reverse of Hölder's inequality:

$$
\left(\frac{1}{|B|}\int_B\nu(x)^q dx\right)^{1/q}\leq C\frac{C}{|B|}\int_B\nu(x)dx
$$

for all balls B and some $1<q<\infty$(see [7]).
For $I_2(x)$, by Hölder's inequality with exponent $1/s+1/s'=1$ and $1<s<n/\beta$, we get

$$
\begin{aligned}
&\left(\frac{1}{|B|}\int_B|I_2(x)|^\varepsilon dx\right)^{1/\varepsilon}\leq\frac{1}{|B|}\int_B|I_2(x)|dx\\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2B|}\int_{2B}|\vec{b}(x)-(\vec{b})_{2B}|^{s'}\nu(x)^{1-s'}dx\right)^{1/s'}\\
&\times\left(\frac{1}{|B|}\int_B|B_{\delta,*}^{\vec{b}_{\sigma^c}}(f)(x)|^s\nu(x)dx\right)^{1/s}
\end{aligned}
$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\nu(\tilde{x})||\vec{b}||_{Lip_{\beta,\nu}}M_{\beta,\nu,s}(B_{\delta,*}^{\vec{b}_{\sigma^c}}f)(\tilde{x}).$$

For $I_3(x)$, similar to $II(x)$, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B|I_3(x)|^\varepsilon dx\right)^{1/\varepsilon}\\
\leq\ & \frac{1}{|B|}\int_{R^n}|((b_1(x)-(b_1)_{2B})\cdots(b_m(x)-(b_m)_{2B})f(x)\chi_{2B}(x)|dx\\
\leq\ & C|2B|^{-1}\int_{2B}|(b_1(x)-(b_1)_{2B})\cdots(b_m-(b_m)_{2B})||f(x)|dx\\
\leq\ & C\frac{\nu(2B)^m}{|2B|}||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(f)(\tilde{x})\\
\leq\ & C\nu(\tilde{x})^m||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(f)(\tilde{x}).
\end{aligned}$$

For $I_4(x)$, similar to the proof of $III(x)$ in the **Case** $m=1$. We have :

$$\begin{aligned}
I_4(x) &= ||B_t^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2B})f_2)(x)-B_t^\delta(\prod_{j=1}^m(b_j(y)-(b_j)_{2B})f_2)(x_0)||\\
&= \sup_{t>0}|\int_{(2B)^c}\prod_{j=1}^m(b_j(y)-(b_j)_{2B})f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy|.
\end{aligned}$$

We consider the following two cases:
Case 1. $0<t\leq d$. In this case, notice that

$$|B_1^\delta(x)|\leq C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned}
&|\int_{(2B)^c}\prod_{j=1}^m(b_j(y)-(b_j)_{2B})f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy|\\
\leq\ & Ct^{-n}\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}|\prod_{j=1}^m(b_j(y)-(b_j)_{2B})||f(y)|(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy\\
\leq\ & C(t/d)^{\delta-(n-1)/2}\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|\prod_{j=1}^m(b_j(y)-(b_j)_{2B})||f(y)|dy\right)\\
\leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}k^m\nu(\tilde{x})^m||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(f)(\tilde{x})\\
\leq\ & C\nu(\tilde{x})^m||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(f)(\tilde{x}).
\end{aligned}$$

Case 2. $t > d$. In this case, we choose δ_0 such that $(n-1)/2 < \delta_0 < \min(\delta, (n+1)/2)$, notice that

$$|(\partial/\partial x)B_1^\delta(x)| \leq C(1+|x|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned}
&|\int_{(2B)^c}[\prod_{j=1}^m(b_j(y)-(b_j)_{2B})]f(y)(B_t^\delta(x-y)-B_t^\delta(x_0-y))dy| \\
\leq\ & Ct^{-n}\int_{(2B)^c}|\prod_{j=1}^m(b_j(y)-(b_j)_{2B})||f(y)||B^\delta((x-y)/t)-B^\delta((x_0-y)/t)|dy \\
\leq\ & Ct^{-n-1}\int_{(2B)^c}|\prod_{j=1}^m(b_j(y)-(b_j)_{2B})||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta+(n+1)/2)}dy \\
\leq\ & Ct^{-n-1}\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}|\prod_{j=1}^m(b_j(y)-(b_j)_{2B})||f(y)| \\
& \times|x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\
\leq\ & C(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^\infty 2^{k((n-1)/2-\delta_0)} \\
& \times\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|\prod_{j=1}^m(b_j(y)-(b_j)_{2B})||f(y)|dy\right) \\
\leq\ & C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta)}k^m\nu(\tilde{x})^m||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(f)(\tilde{x}) \\
\leq\ & C\nu(\tilde{x})^m||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(f)(\tilde{x}).
\end{aligned}$$

Thus,

$$\left(\frac{1}{|B|}\int_B|I_4(x)|^\varepsilon dx\right)^{1/\varepsilon} \leq C\nu(\tilde{x})^m||\vec{b}||_{Lip_{\beta,\nu}}M_{m\beta,\nu,s}(f)(\tilde{x}).$$

This completes the proof of the Theorem 1.

Proof of Theorem 2.2. From Lemma 3.1, since $\nu \in A_1(R^n)$, then $\nu^{1-q} \in A_q(R^n)$(see [7]). Then by Theorem 2.1 with $0 < \varepsilon < 1 < s < p$, we get, when $m = 1$,

$$\begin{aligned}
||B_{\delta,*}^{b_1}f(x)||_{L^q(\nu^{1-q})} &\leq ||M_\varepsilon(B_{\delta,*}^{b_1}f)||_{L^q(\nu^{1-q})} \leq C||M_\varepsilon^\#(B_{\delta,*}^{b_1}f)||_{L^q(\nu^{1-q})} \\
&\leq C||b_1||_{Lip_{\beta,\nu}}\left(||M_{\beta,\nu,s}(B_*^\delta f)||_{L^q(\nu)} + ||M_{\beta,\nu,s}(f)||_{L^q(\nu)}\right) \\
&\leq C||b_1||_{Lip_{\beta,\nu}}||f||_{L^p(\nu)}.
\end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 2.2 by induction.

5. Open problem

In this paper, the boundedness properties of the multilinear operators generated by the Bochner-Riesz operator and weighted Lipschitz functions. are obtained.

The open problem is to study the boundedness of the multilinear operators generated by the Bochner-Riesz operator and others locally integrable functions on others spaces.

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