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A Common Random Fixed Point Theorem for Contractive Type Mapping in Hilbert Space

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Abstract

The object of this paper is to obtain a common fixed point theorem for continuous random operators defined on a non-empty closed subset of a separable Hilbert space.

Keywords: Separable Hilbert Space, random operators, common random fixed point, Rational inequality.

1 Introduction

In recent years, the study of random fixed points have attracted much attention, some of the recent literatures in random fixed point may be noted in [1,3,4,6]. In this paper we construct a sequence of measurable functions and consider its convergence to the common random fixed point of two continuous random operators defined on a non empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of the two continuous random operators. We have used a rational inequality [from 5] and the parallelogram law.

Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets Ω, H stands for a separable Hilbert space, and C is a nonempty closed subset of H.

Definition 2.1 A function $f : \Omega \to C$ is said to be measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of H.

Definition 2.2 A function $F : \Omega \times C \to C$ is said to be a random operator if $F(.,x) : \Omega \to C$ is measurable for every $x \in C$.

Definition 2.3 A measurable function $g : \Omega \to C$ is said to be a random fixed point of the random operator $F : \Omega \times C \to C$ if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.

Definition 2.4 A random operator $F : \Omega \times C \to C$ is said to be continuous if for fixed $t \in \Omega$, $F(t, .) : C \to C$ is continuous.

Condition A) *Two mappings $S, T : C \to C$, where C is a non-empty subset of a Hilbert space H, is said to satisfy condition (A) if*

$$\| Sx - Ty \|^2 \leq \frac{a\big[\, \| x - Sx \|^2 \| x - y \|^2 + \| y - Ty \|^2 \| x - y \|^2 \,\big] + b \| x - Sx \|^2 \| y - Ty \|^2}{\| x - Sx \|^2 + \| y - Ty \|^2 + \| x - y \|^2} + \frac{c \| x - y \|^4}{\| x - Sx \|^2 + \| y - Ty \|^2 + \| x - y \|^2}$$

flusright

(2.1) *for each $x, y \in C$, $x \neq y$ and $\| x - Sx \|^2 + \| y - Ty \|^2 \neq 0$ where $a, b, c \geq 0$ and $2 < 2a + b + c < 3$.*

(2.2)

3 Main Result

Theorem 3.1 Let C be a non-empty closed subset of a separable Hilbert space H. Let S and T be two continuous random operators defined on C such that for $t \in \Omega$, $S(t, .)$, $T(t, .) : C \to C$ satisfy condition (A). Then S and T have a common random fixed point in C.

Proof: We construct a sequence of functions $\{g_n\}$ as $g_0 : \Omega \to C$ is arbitrary measurable function. For $t \in \Omega$, and $n = 0, 1, 2, 3, \ldots$

$$g_{2n+1}(t) = S(t, g_{2n}(t)), \quad g_{2n+2}(t) = T(t, g_{2n+1}(t))$$

If $g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t)$ for $t \in \Omega$, for some n then we see that $g_{2n}(t)$ is a random fixed point of S and T. Therefore we suppose that no two consecutive terms of sequence $\{g_n\}$ are equal.

Now consider for fixed $t \in \Omega$

$$\begin{aligned}\| g_{2n}(t) - g_{2n+1}(t) \|^2 &= \| T(t, g_{2n-1}(t)) - S(t, g_{2n}(t)) \|^2 \\ &= \| S(t, g_{2n}(t)) - T(t, g_{2n-1}(t)) \|^2\end{aligned}$$

$$\leq \frac{a\left[\|g_{2n}(t)-S(t,g_{2n}(t))\|^2\|g_{2n}(t)-g_{2n-1}(t)\|^2+\|g_{2n-1}(t)-T(t,g_{2n-1}(t))\|^2\|g_{2n}(t)-g_{2n-1}(t)\|^2\right]}{\|g_{2n}(t)-S(t,g_{2n}(t))\|^2+\|g_{2n-1}(t)-T(t,g_{2n-1}(t))\|^2+\|g_{2n}(t)-g_{2n-1}(t)\|^2}$$

$$+\frac{b \| g_{2n}(t) - S(t, g_{2n}(t)) \|^2 \| g_{2n-1}(t) - T(t, g_{2n-1}(t)) \|^2 + c \| g_{2n}(t) - g_{2n-1}(t) \|^4}{\| g_{2n}(t) - S(t, g_{2n}(t)) \|^2 + \| g_{2n-1}(t) - T(t, g_{2n-1}(t)) \|^2 + \| g_{2n}(t) - g_{2n-1}(t) \|^2}$$

$$\Rightarrow \| g_{2n}(t) - g_{2n+1}(t) \|^2 \left[\| g_{2n}(t) - g_{2n+1}(t) \|^2 + 2 \| g_{2n}(t) - g_{2n-1}(t) \|^2 \right]$$
$$\leq (2a+b+c) \| g_{2n}(t)-g_{2n-1}(t) \|^2 \max\left\{ \| g_{2n}(t)-g_{2n+1}(t) \|^2, \| g_{2n}(t)-g_{2n-1}(t) \|^2 \right\}$$
$$\Rightarrow \| g_{2n}(t) - g_{2n+1}(t) \|^4 \leq (2a + b + c - 2) \| g_{2n}(t) - g_{2n-1}(t) \|^2$$
$$\max\left\{ \| g_{2n}(t) - g_{2n+1}(t) \|^2, \| g_{2n}(t) - g_{2n-1}(t) \|^2 \right\}$$

Case I

$$\Rightarrow \| g_{2n}(t) - g_{2n+1}(t) \|^4 \leq (2a + b + c - 2) \| g_{2n}(t) - g_{2n-1}(t) \|^4$$

$$\Rightarrow \| g_{2n}(t) - g_{2n+1}(t) \| \leq (2a + b + c - 2)^{\frac{1}{4}} \| g_{2n}(t) - g_{2n-1}(t) \|$$

$$\Rightarrow \| g_{2n}(t) - g_{2n+1}(t) \| \leq k_1 \| g_{2n}(t) - g_{2n-1}(t) \| \tag{3.2}$$

where $k_1 = (2a + b + c - 2)^{\frac{1}{4}}$ $[by\ 2.2]$

Case II

$$\| g_{2n}(t) - g_{2n+1}(t) \|^4 \leq (2a+b+c-2) \| g_{2n}(t) - g_{2n-1}(t) \|^2 \| g_{2n}(t) - g_{2n+1}(t) \|^2$$

$$\Rightarrow \| g_{2n}(t) - g_{2n+1}(t) \| \leq (2a + b + c - 2)^{\frac{1}{2}} \| g_{2n-1}(t) - g_{2n}(t) \|$$

$$\Rightarrow \| g_{2n}(t) - g_{2n+1}(t) \| \leq k_2 \| g_{2n}(t) - g_{2n-1}(t) \| \tag{3.3}$$

where $k_2 = (2a + b + c - 2)^{\frac{1}{2}} < 1$ $[by\ 2.2]$

Again consider

$$\| g_{2n}(t) - g_{2n-1}(t) \|^2 = \| T(t, g_{2n-1}(t)) - S(t, g_{2n-2}(t)) \|^2$$
$$= \| S(t, g_{2n-2}(t)) - T(t, g_{2n-1}(t)) \|^2$$

$$\leq \frac{a\left[\|g_{2n-2}(t)-S(t,g_{2n-2}(t))\|^2\|g_{2n-2}(t)-g_{2n-1}(t)\|^2+\|g_{2n-1}(t)-T(t,g_{2n-1}(t))\|^2\|g_{2n-2}(t)-g_{2n-1}(t)\|^2\right]}{\|g_{2n-2}(t)-S(t,g_{2n-2}(t))\|^2+\|g_{2n-1}(t)-T(t,g_{2n-1}(t))\|^2+\|g_{2n-2}(t)-g_{2n-1}(t)\|^2}$$

$$+\frac{b \| g_{2n-2}(t) - S(t, g_{2n-2}(t)) \|^2 \| g_{2n-1}(t) - T(t, g_{2n-1}(t)) \|^2 + c \| g_{2n-2}(t) - g_{2n-1}(t) \|^4}{\| g_{2n-2}(t) - S(t, g_{2n-2}(t)) \|^2 + \| g_{2n-1}(t) - T(t, g_{2n-1}(t)) \|^2 + \| g_{2n-2}(t) - g_{2n-1}(t) \|^2}$$

$$= \frac{a \| g_{2n-2}(t) - g_{2n-1}(t) \|^2 \| g_{2n-2}(t) - g_{2n-1}(t) \|^2}{\| g_{2n-2}(t) - g_{2n-1}(t) \|^2 + \| g_{2n-1}(t) - g_{2n}(t) \|^2 + \| g_{2n-2}(t) - g_{2n-1}(t) \|^2}$$
$$+ \frac{a \| g_{2n-1}(t) - g_{2n}(t) \|^2 \| g_{2n-2}(t) - g_{2n-1}(t) \|^2}{\| g_{2n-2}(t) - g_{2n-1}(t) \|^2 + \| g_{2n-1}(t) - g_{2n}(t) \|^2 + \| g_{2n-2}(t) - g_{2n-1}(t) \|^2}$$
$$+ \frac{b \| g_{2n-2}(t) - g_{2n-1}(t) \|^2 \| g_{2n-1}(t) - g_{2n}(t) \|^2 + c \| g_{2n-2}(t) - g_{2n-1}(t) \|^4}{\| g_{2n-2}(t) - g_{2n-1}(t) \|^2 + \| g_{2n-1}(t) - g_{2n}(t) \|^2 + \| g_{2n-2}(t) - g_{2n-1}(t) \|^2}$$

$$\Rightarrow \| g_{2n}(t) - g_{2n-1}(t) \|^4 \leq (2a + b + c - 2) \| g_{2n-2}(t) - g_{2n-1}(t) \|^2$$
$$\max\left\{ \| g_{2n}(t) - g_{2n-1}(t) \|^2, \| g_{2n-1}(t) - g_{2n-2}(t) \|^2 \right\}$$

Case I:

$$\Rightarrow \| g_{2n}(t) - g_{2n-1}(t) \|^4 \leq (2a + b + c - 2) \| g_{2n-2}(t) - g_{2n-1}(t) \|^4$$

$$\Rightarrow \| g_{2n}(t) - g_{2n-1}(t) \| \leq (2a + b + c - 2)^{\frac{1}{4}} \| g_{2n-1}(t) - g_{2n-2}(t) \|$$

$$\Rightarrow \| g_{2n}(t) - g_{2n-1}(t) \| \leq k_3 \| g_{2n-1}(t) - g_{2n-2}(t) \| \tag{3.4}$$

where $k_3 = (2a + b + c - 2)^{\frac{1}{4}} < 1$ [*by* 2.2]

Case II:

$$\| g_{2n}(t) - g_{2n-1}(t) \|^4 \leq (2a+b+c-2) \| g_{2n-1}(t) - g_{2n-2}(t) \|^2 \| g_{2n}(t) - g_{2n-1}(t) \|^2$$

$$\Rightarrow \| g_{2n}(t) - g_{2n-1}(t) \| \leq (2a + b + c - 2)^{\frac{1}{2}} \| g_{2n-1}(t) - g_{2n-2}(t) \|$$

$$\Rightarrow \| g_{2n}(t) - g_{2n-1}(t) \| \leq k_4 \| g_{2n-1}(t) - g_{2n-2}(t) \| \tag{3.5}$$

where $k_4 = (2a + b + c - 2)^{\frac{1}{2}} < 1$ [by 2.2]

By (3.2),(3.3),(3.4) and (3.5) we have

$$\| g_n(t) - g_{n+1}(t) \| \leq k \| g_{n-1}(t) - g_n(t) \| \quad for \;\; n = 1, 2, 3......$$

where $k = \max\{k_1, k_2, k_3, k_4\} < 1$

$$\Rightarrow \| g_n(t) - g_{n+1}(t) \| \leq k^n \| g_0(t) - g_1(t) \| \quad for \; all \;\; t \in \Omega$$

Now we shall prove that for $t \in \Omega$, $\{g_n(t)\}$ is a Cauchy sequence. For this for every positive integer p we have, for $t \in \Omega$

$$\begin{aligned}
\| g_n(t) - g_{n+p}(t) \| &= \| g_n(t) - g_{n+1}(t) + g_{n+1}(t) - \ldots + g_{n+p-1}(t) - g_{n+p}(t) \| \\
&\leq \| g_n(t) - g_{n+1}(t) \| + \| g_{n+1}(t) - g_{n+2}(t) \| + \ldots + \| g_{n+p-1}(t) - g_{n+p}(t) \| \\
&\leq \left[k^n + k^{n+1} + \ldots + k^{n+p-1} \right] \| g_o(t) - g_1(t) \| \\
&= k^n \left[1 + k + k^2 + \ldots + k^{p-1} \right] \| g_o(t) - g_1(t) \| \\
&< \frac{k^n}{(1-k)} \| g_o(t) - g_1(t) \|
\end{aligned}$$

As $n \to \infty$, $\| g_n(t) - g_{n+p}(t) \| \to 0$, it follows that for $t \in \Omega$, $\{g_n(t)\}$ is a Cauchy sequence and hence is convergent in Hilbert space H.
For $t \in \Omega$, let

$$\{\mathrm{g}_n(t)\} \to g(t) \;\text{ as }\; n \to \infty$$

Since C is closed, g is a function from C to C.

Existence of random fixed point:
Consider for $t \in \Omega$,

$$\| g(t) - T(t, g(t)) \|^2 = \| g(t) - g_{2n+1}(t) + g_{2n+1}(t) - T(t, g(t)) \|^2$$
$$\leq 2 \| g(t) - g_{2n+1}(t) \|^2 + 2 \| g_{2n+1} - T(t, g(t)) \|^2$$

(by parallelogram law $\| x + y \|^2 \leq 2 \| x \|^2 + 2 \| y \|^2$)

$$= 2 \parallel g(t) - g_{2n+1}(t) \parallel^2 + 2 \parallel S(t, g_{2n}(t)) - T(t, g(t)) \parallel^2$$

$$\leq 2 \parallel g(t) - g_{2n+1}(t) \parallel^2 + \frac{2a \parallel g_{2n}(t) - S(t, g_{2n}(t)) \parallel^2 \parallel g_{2n}(t) - g(t) \parallel^2}{\parallel g_{2n}(t) - S(t, g_{2n}(t)) \parallel^2 + \parallel g(t) - T(t, g(t)) \parallel^2 + \parallel g_{2n}(t) - g(t) \parallel^2}$$

$$+ \frac{2a \parallel g(t) - T(t, g(t)) \parallel^2 \parallel g_{2n}(t) - g(t) \parallel^2}{\parallel g_{2n}(t) - S(t, g_{2n}(t)) \parallel^2 + \parallel g(t) - T(t, g(t)) \parallel^2 + \parallel g_{2n}(t) - g(t) \parallel^2}$$

$$+ \frac{2b \parallel g_{2n}(t) - S(t, g_{2n}(t)) \parallel^2 \parallel g(t) - T(t, g(t)) \parallel^2 + 2c \parallel g_{2n}(t) - g(t) \parallel^4}{\parallel g_{2n}(t) - S(t, g_{2n}(t)) \parallel^2 + \parallel g(t) - T(t, g(t)) \parallel^2 + \parallel g_{2n}(t) - g(t) \parallel^2}$$

$$= 2 \parallel g(t) - g_{2n+1}(t) \parallel^2 + \frac{2a \parallel g_{2n}(t) - g_{2n+1}(t) \parallel^2 \parallel g_{2n}(t) - g(t) \parallel^2}{\parallel g_{2n}(t) - g_{2n+1}(t) \parallel^2 + \parallel g(t) - T(t, g(t)) \parallel^2 + \parallel g_{2n}(t) - g(t) \parallel^2}$$

$$+ \frac{2a \parallel g(t) - T(t, g(t)) \parallel^2 \parallel g_{2n}(t) - g(t) \parallel^2}{\parallel g_{2n}(t) - g_{2n+1}(t) \parallel^2 + \parallel g(t) - T(t, g(t)) \parallel^2 + \parallel g_{2n}(t) - g(t) \parallel^2}$$

$$+ \frac{2b \parallel g_{2n}(t) - g_{2n+1}(t) \parallel^2 \parallel g(t) - T(t, g(t)) \parallel^2 + 2c \parallel g_{2n}(t) - g(t) \parallel^4}{\parallel g_{2n}(t) - g_{2n+1}(t) \parallel^2 + \parallel g(t) - T(t, g(t)) \parallel^2 + \parallel g_{2n}(t) - g(t) \parallel^2}$$

As $\{g_{2n}(t)\}$, $\{g_{2n+1}(t)\}$ are subsequences of $\{g_n(t)\}$, as $n \to \infty$, $\{g_{2n}(t)\} \to g(t)$ and $\{g_{2n+1}(t)\} \to g(t)$ for all $t \in \Omega$

Therefore,

$$\parallel g(t) - T(t, g(t)) \parallel^2 \leq 2 \parallel g(t) - g(t) \parallel^2 + \frac{2a \left[\|g(t)-g(t)\|^2 + \|g(t)-T(t,g(t))\|^2 \right] \|g(t)-g(t)\|^2}{\|g(t)-g(t)\|^2 + \|g(t)-T(t,g(t))\|^2 + \|g(t)-g(t)\|^2}$$

$$+ \frac{2b \parallel g(t) - g(t) \parallel^2 \parallel g(t) - T(t, g(t)) \parallel^2 + 2c \parallel g(t) - g(t) \parallel^4}{\parallel g(t) - g(t) \parallel^2 + \parallel g(t) - T(t, g(t)) \parallel^2 + \parallel g(t) - g(t) \parallel^2}$$

$$\Rightarrow \parallel g(t) - T(t, g(t)) \parallel^2 \leq 0$$

$$\Rightarrow \parallel g(t) - T(t, g(t)) \parallel^2 = 0$$

$$\Rightarrow T(t, g(t)) = g(t) \quad for\ all\ t \in \Omega \tag{3.7}$$

Again consider

$$\parallel g(t) - S(t, g(t)) \parallel^2 = \parallel g(t) - g_{2n+2}(t) + g_{2n+2}(t) - S(t, g(t)) \parallel^2$$

$$\leq 2 \parallel g(t) - g_{2n+2}(t) \parallel^2 + 2 \parallel T(t, g_{2n+1}(t)) - S(t, g(t)) \parallel^2$$

$$= 2 \parallel g(t) - g_{2n+2}(t) \parallel^2 + 2 \parallel S(t, g(t)) - T(t, g_{2n+1}(t)) \parallel^2$$

$$\leq 2 \parallel g(t) - g_{2n+1}(t) \parallel^2 + \frac{2a \left[\|g(t)-S(t,g(t))\|^2 + \|g_{2n+1}(t)-T(t,g_{2n+1}(t))\|^2 \right] \|g(t)-g_{2n+1}(t)\|^2}{\|g(t)-S(t,g(t))\|^2 + \|g_{2n+1}(t)-T(t,g_{2n+1}(t))\|^2 + \|g(t)-g_{2n+1}(t)\|^2}$$

$$+\frac{2b \parallel g(t)-S(t,g(t)) \parallel^2 \parallel g_{2n+1}(t)-T(t,g_{2n+1}(t)) \parallel^2 + c \parallel g(t)-g_{2n+1}(t) \parallel^4}{\parallel g(t)-S(t,g(t)) \parallel^2 + \parallel g_{2n+1}(t)-T(t,g_{2n+1}(t)) \parallel^2 + \parallel g(t)-g_{2n+1}(t) \parallel^2}$$

$$= 2 \parallel g(t)-g_{2n+1}(t) \parallel^2 + \frac{2a \parallel g(t)-S(t,g(t)) \parallel^2 \parallel g(t)-g_{2n+1}(t) \parallel^2}{\parallel g(t)-S(t,g(t)) \parallel^2 + \parallel g_{2n+1}(t)-g_{2n+2}(t) \parallel^2 + \parallel g(t)-g_{2n+1}(t) \parallel^2}$$

$$+\frac{2a \parallel g_{2n+1}(t)-g_{2n+2}(t) \parallel^2 \parallel g(t)-g_{2n+1}(t) \parallel^2}{\parallel g(t)-S(t,g(t)) \parallel^2 + \parallel g_{2n+1}(t)-g_{2n+2}(t) \parallel^2 + \parallel g(t)-g_{2n+1}(t) \parallel^2}$$

$$+\frac{2b \parallel g(t)-S(t,g(t)) \parallel^2 \parallel g_{2n+1}(t)-g_{2n+2}(t) \parallel^2 + c \parallel g(t)-g_{2n+1}(t) \parallel^4}{\parallel g(t)-S(t,g(t)) \parallel^2 + \parallel g_{2n+1}(t)-g_{2n+2}(t) \parallel^2 + \parallel g(t)-g_{2n+1}(t) \parallel^2}$$

As $\{g_{2n+1}(t)\}$, and $\{g_{2n+2}(t)\}$ are subsequences of $\{g_n(t)\}$,as $n \to \infty$,

$\{g_{2n+1}(t)\} \to g(t)$ and $\{g_{2n+2}(t)\} \to g(t)$ for all $t \in \Omega$

Therefore,

$$\parallel g(t)-S(t,g(t)) \parallel^2 \leq 2 \parallel g(t)-g(t) \parallel^2 + \frac{2a\left[\|g(t)-S(t,g(t))\|^2+\|g(t)-g(t)\|^2\right]\|g(t)-g(t)\|^2}{\|g(t)-S(t,g(t))\|^2+\|g(t)-g(t)\|^2+\|g(t)-g(t)\|^2}$$

$$+\frac{2b \parallel g(t)-S(t,g(t)) \parallel^2 \parallel g(t)-g(t) \parallel^2 + c \parallel g(t)-g(t) \parallel^2}{\parallel g(t)-S(t,g(t)) \parallel^2 + \parallel g(t)-g(t) \parallel^2 + \parallel g(t)-g(t) \parallel^2}$$

$$\parallel g(t)-S(t,g(t)) \parallel^2 \leq 0$$

$$\Rightarrow \parallel g(t)-S(t,g(t)) \parallel^2 = 0$$

$$\Rightarrow S(t,g(t)) = g(t) \quad for \;\; all \;\; t \in \Omega \tag{3.8}$$

Again, if $A : \Omega \times C \to C$ is a continuous random operator on a non-empty subset C of a separable Hilbert space H, then for any measurable function $f : \Omega \to C$, the function $h(t) = A(t, f(t))$ is also measurable .

It follows from the construction of $\{g_n\}$ (by (3.1)) and the above consideration that $\{g_n\}$ is a sequence of measurable functions. From (3.6), it follows that g is also a measurable function. This fact along with (3.7) and (3.8) shows that $g : \Omega \to C$ is a common random fixed point of S and T.

This completes the proof of the theorem (3.1).

4 open problem:

Are the above mentioned theorem true in a polish space?

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