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Extension of Q-soft ideals in semigroups

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Abstract

In this paper the concept of extension of a Q-soft ideals in semigroups has been introduced and some important properties have been studied.

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1 Introduction

The formal study of semigroups began in the early 20th century [3]. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The concept of fuzzy sets was introduced by Zadeh [9] in his classic paper in 1965. The idea of fuzzy subsemigroup was also introduced by Kuroki [4, 5, 6]. To deal with the complicated problems involving uncertainties in economics, engineering, environmental science, medical science and social science, methods of classical mathematics can not be successfully used. Alternatively, mathematical theories such as probability theory, fuzzy set theory, rough set theory, vague set theory and the interval mathematics were established by researchers to deal with uncertainties appearing in the above fields. These methods also have some inherent difficulties. To over come these kinds of difficulties, Molodtsov [7] introduced the concept of soft sets. The works of the algebraic structure of soft sets was first started by H. Aktas and N. Cagman

[1].They presented the notion of the soft group and derived its some basic properties. This paper is organized as follows. In the preliminaries, we give the concept of semigroups and soft sets. In section three, we introduce the definition of Q-soft ideals, the extension of a Q-soft ideals in semigroups and give some fundamental properties of them.

2 Preliminaries

In this section we discuss some elementary definitions that we use in the sequel. For more details of the semigroups and soft subsets, we refer to the earlier studies [8, 2, 7].

Definition 2.1 *Let $(S, *)$ be a mathematical system such that $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$. Then $*$ is called a ssociative and $(S, *)$ is called a semigroup.*

Definition 2.2 *A semigroup $(S, *)$ is said to be commutative if $a * b = b * a$ for all $a, b \in S$.*

Definition 2.3 *A left (right) ideal of a semigroup S is a non-empty subset I of S such that $SI \subseteq I(IS \subseteq I)$. If I is both a left and a right ideal of a semigroup S, then we say that I is an ideal of S.*

Definition 2.4 *Let S be a semigroup. Then an ideal I of S is said to be*
(1) completely prime if $xy \in I$ implies that $x \in I$ or $y \in I$ for all $x, y \in S$,
(2) completely semiprime if $x^2 \in I$ implies that $x \in I$, for all $x \in S$.

Definition 2.5 *Let Q be a non-empty set. For any set A, a Q-soft set μ over U is a set, defined by a function μ, representing a mapping $\mu : A \times Q \to P(U)$, such that $\mu(x, q) = \emptyset$ if $x \notin A$. A Q-soft set over U can also be represented by the set of ordered pairs $\mu = \{(x, \mu(x, q)) \mid x \in A, \mu(x, q) \in P(U)\}$. From here on, Q-soft set will be used without over U.*

Definition 2.6 *Let μ and ν be Q-soft sets of set A.Then,*
(1) μ is called an empty Q-soft subset, if $\mu(x, q) = \emptyset$ for all $(x, q) \in A \times Q$,
(2) μ is called a $A \times Q$-universal soft set, if $\mu(x, q) = U$ for all $(x, q) \in A \times Q$,
(3) the set $Im(\mu) = \{\mu(x, q) : (x, q) \in A \times Q\}$ is called image of μ,
(4) μ is a Q-soft subset of ν, if $\mu(x, q) \subseteq \nu(x, q)$ for all $(x, q) \in A \times Q$,
(5) μ and ν are soft equal, if and only if $\mu(x, q) = \nu(x, q)$ for all $(x, q) \in A \times Q$,
(6) the set $(\mu \cup \nu)(x, q) = \mu(x, q) \cup \nu(x, q)$ for all $(x, q) \in A \times Q$ is called union of μ and ν,
(7) the set $(\mu \cap \nu)(x, q) = \mu(x, q) \cap \nu(x, q)$ for all $(x, q) \in A \times Q$ is called intersection of μ and ν.

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and let $Q = \{p, q\}$, $A = \{x_1, x_2, x_3, x_4, x_5\}$.
Let μ, ν, ξ and π be Q-soft sets of set A. Define

$$\mu = \{((x_1, p), \{u_1, u_2, u_3\}), ((x_1, q), \{u_3, u_4, u_5\}), ((x_2, p), \{u_1, u_5\}), ((x_2, q), \{u_3, u_4\})\},$$

$$\nu = \{((x_2, p), \{u_1, u_2\}), ((x_2, q), \{u_1, u_3\}), ((x_3, p), \{u_2, u_4\}), ((x_3, q), \{u_1, u_5\})\}$$

and

$$\xi = \{((x_4, p), U), ((x_4, q), U)\}, \pi = \{((x_5, p), \{\}), ((x_5, q), \{\})\}.$$

Then for all $(x, r) \in A \times Q$ we have

$$(\mu \cup \nu)(x, r) = \{((x_1, p), \{u_1, u_2, u_3\}), ((x_1, q), \{u_3, u_4, u_5\}), ((x_2, p), \{u_1, u_2, u_5\}), ((x_2, q), \{u_1, u_3, u_4\}), ((x_3, p), \{u_2, u_4\}), ((x_3, q), \{u_1, u_5\})\}$$

and $(\mu \cap \nu)(x, r) = \{((x_2, p), \{u_1\}), ((x_2, q), \{u_3\})\}$. Also $\xi = U$ and $\pi = \emptyset$.

Remark 2.7 *The difinition of classical subset is not valid for the Q-soft subset. For example, let* $U = \{u_1, u_2, u_3, u_4, u_5\}, Q = \{q\}, A = \{x_1, x_2, x_3\}$. *Let* μ *and* ν *be Q-soft sets of set A. If* $\mu = \{((x_1, q), \{u_1, u_2\}), ((x_2, q), \{u_4\})\}$ *and* $\nu = \{((x_1, q), \{u_1, u_2, u_3\}), ((x_2, q), \{u_4, u_5\}), ((x_3, q), \{u_1\})\}$, *then* $\mu \subseteq \nu$ *as Q-soft subset, but* $\mu\nu$ *as classical subset.*

Throughout this work, Q is a non-empty set, U refers to an initial universe set and $P(U)$ is the power set of U.

3 Main results

In this section, we introduce basic definitions of Q-soft ideal, Q-soft completely prime ideal, Q-soft completely semiprime ideal of a semigroup S and study their properties.

Definition 3.1 *A non-empty Q-soft subset* μ *of a semigroup S is called a Q-soft left(right) ideal of S if* $\mu(xy, q) \supseteq \mu(y, q)$*(resp.* $\mu(xy, q) \supseteq \mu(x, q)$*) for all* $x, y \in S$ *and* $q \in Q$.

Definition 3.2 *A non-empty Q-soft subset* μ *of a semigroup S is called a Q-soft two-sided ideal or a Q-soft ideal of S if it is both a Q-soft left and a Q-soft right idealof S.*

Definition 3.3 *A Q-soft ideal* μ *of a semigroup S is called a Q-soft completely prime ideal of S if* $\mu(xy, q) = \mu(x, q) \cup \mu(y, q)$ *for all* $x, y \in S$ *and* $q \in Q$.

Definition 3.4 *A Q-soft ideal μ of a semigroup S is called a Q-soft completely semiprime ideal of S if $\mu(x,q) \supseteq \mu(x^2,q)$ for all $x \in S$ and $q \in Q$.*

Proposition 3.5 *Let I be a non-empty sub set of a semigroup S and χ as*

$$\chi(x,q) = \begin{cases} U & \text{if } x \in I \\ \emptyset & \text{if } x \notin I \end{cases}$$

be the Q-soft of I. If I is a left ideal (right ideal, ideal, completely prime ideal, completely semiprime) of S, then χ is a Q-soft left ideal (resp. Q-soft right ideal, Q-soft ideal, Q-soft completely prime ideal, Q-soft completely semi prime ideal) of S.

Proof 3.6 *Let I be a left ideal of a semigroup S and $x, y \in S, q \in Q$.*
If $y \in I$, then $xy \in I$ and so $\chi(xy,q) = U = \chi(y,q)$.
If $y \notin I$, then $\chi(xy,q) \supseteq \emptyset = \chi(y,q)$. Hence χ is a Q-soft left ideal of S.
Similarly we can prove that the other parts of the proposition.

Proposition 3.7 *Let S be a semigroup and μ be a non-empty Q-soft subset of S, then μ is a Q-soft left ideal (Q-soft right ideal, Q-soft ideal, Q-soft completely prime ideal, Q-soft completely semiprime ideal) of S if and only if for all $\alpha \in Im\mu$ and $q \in Q$, the set $\mu_\alpha = \{x \in S \mid \mu(x,q) \supseteq \alpha\}$ be left ideal (resp. right ideal, ideal, completely prime ideal, completely semiprime ideal) of S.*

Proof 3.8 *Let μ be a Q-soft left ideal of S and $\alpha \in Im\mu$. Then there exist some $x \in S$ and $q \in Q$ such that $\mu(x,q) = \alpha$ and so $x \in \mu_\alpha$. Thus $\mu_\alpha \neq \emptyset$. If $x \in \mu_\alpha$ and $s \in S$, then $\mu(sx,q) \supseteq \mu(x,q) \supseteq \alpha$ and we obtain $sx \in \mu_\alpha$. Hence μ_α is a left ideal of S.*
Conversely, assume that μ_α be left ideal of S for all $\alpha \in Im\mu$. If $x, s \in S$ such that $\mu(x,q) = \alpha$ for all $q \in Q$, then $x \in \mu_\alpha$. Since μ_α is a left ideal of S so $sx \in \mu_\alpha$. Hence $\mu(sx,q) \supseteq \alpha = \mu(x,q)$ and then μ is a Q-soft left ideal of S.
The proof other parts of the proposition is similar.

In this section, we introduce the concept of extension of a Q-soft ideals in semigroups and investigate their properties.

Definition 3.9 *Let S be a semigroup, μ be a Q-soft subset of S and $x \in S$. Define*
$< x, \mu >: S \times Q \to P(U)$ the Q-soft extension of μ by x as : $< x, \mu > (y,q) = \mu(xy,q)$ for all $y \in S, q \in Q$.

We consider the following example as an illustration. Let $S = \{a, b, c\}$ and $*$ be a binary operation on S defined as : $a * a = a * b = a * c = a, b * a = b * b = b * c = b$ and $c * a = c * b = c * c = c$. Then S is a semigroup. Let $U =$

$\{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $Q = \{q\}$. Define the Q-soft subset $\mu : S\times Q \to P(U)$ as $\mu = \{((a, q), \{u_1, u_2, u_3\}), ((b, q), \{u_3, u_4, u_5\}), ((c, q), \{u_1, u_5\})\}$. If $x = a$, then the Q-soft extension of μ by $x = a$ is : $< a, \mu > (a, q) = \mu(a * a, q) = \mu(a, q) = \{((a, q), \{u_1, u_2, u_3\})\}, < a, \mu > (b, q) = \mu(a * b, q) = \mu(a, q) = \{((a, q), \{u_1, u_2, u_3\})\}$ and
$< a, \mu > (c, q) = \mu(a * c, q) = \mu(a, q) = \{((a, q), \{u_1, u_2, u_3\})\}$.
The Q-soft extension of μ by $x = b$ and $x = c$ is similar.

Proposition 3.10 *Let S be a commutative semigroup. If μ be a Q-soft ideal of S, then so does $< x, \mu >$ for all $x \in S$.*

Proof 3.11 *Let $y, z \in S$ and $q \in Q$. Then $< x, \mu > (yz, q) = \mu(xyz, q) \supseteq \mu(xy, q) =< x, \mu > (y, q)$. Also since S is commutative so $< x, \mu > (yz, q) = \mu(xyz, q) = \mu(xzy, q) \supseteq \mu(xz, q) =< x, \mu > (z, q)$. Thus $< x, \mu >$ will be Q-soft ideal of S for all $x \in S$.*

Definition 3.12 *Let S be a semigroup and μ be a Q-soft subset of S. Define $\mu = \{x \in S : \mu(x, q) \neq \emptyset\}$ for all $q \in Q$.*

Let $S = \{1, -1\}$ be a simigroup and $U = \{u_1, u_2, u_3, u_4\}, Q = \{p, q\}$. Let μ be a Q-soft subset of S as
$\mu = \{((1, p), \{u_1, u_2\}), ((1, q), \{u_1, u_3\}), ((-1, q), \{\})\}$, then $\mu = \{1\}$.

Proposition 3.13 *Let S be a semigroup, μ be a Q-soft ideal of S and $x \in S$. The following assertions are hold.*
(1) $< x, \mu >\supseteq \mu$.
(2) $< x^{n+1}, \mu >\supseteq< x^n, \mu >$ for all $n \in N = \{1, 2, ...\}$.
(3) If $\mu(x, q) \neq \emptyset$, then $< x, \mu >= S \times Q$.

Proof 3.14 *Let μ bea Q-soft ideal of S and let $y \in S, q \in Q$.*
(1) $< x, \mu > (y, q) = \mu(xy, q) \supseteq \mu(y, q)$. Thus $< x, \mu >\supseteq \mu$.
(2) Let $n \in N = \{1, 2, ...\}$. Then

$$< x^{n+1}, \mu > (y, q) = \mu(x^{n+1}y, q) = \mu(x^n xy, q) \supseteq \mu(x^n y, q) =< x^n, \mu > (y, q).$$

Hence $< x^{n+1}, \mu >\supseteq< x^n, \mu >$.
(3) Let $\mu(x, q) \neq \emptyset$ and $(y, q) \in S \times Q$. Now $< x, \mu > (y, q) = \mu(xy, q) \supseteq \mu(x, q) \neq \emptyset$ and so $< x, \mu > (y, q) \neq \emptyset$. Thus $S \times Q \subseteq< x, \mu >$. Also we have always $< x, \mu >\subseteq S \times Q$. Hence we conclude $< x, \mu >= S \times Q$.

Definition 3.15 *Let S be a semigroup, $A \subseteq S$ and $x \in S$. Define*
$< x, A \times Q >= \{(y, q) \in S \times Q \mid (xy, q) \in A \times Q\}$.

Let $S = \{1, -1, i, -i\}$ be a simigroup such that $i^2 = -1$ and $A = \{1, -1\}$. Let $Q = \{p, q\}$ and so $A \times Q = \{(1, p), (1, q), (-1, p), (-1, q)\}$. If $x = i$, then $< i, A \times Q >= \{(i, q), (-i, q), (i, p), (-i, p)\}$.

Proposition 3.16 *Let S be a semigroup and $\emptyset \neq A \subseteq S$. Define $\mu : A \times Q \to P(U)$ as*

$$\mu(y,q) = \begin{cases} U & \text{if } (y,q) \in A \times Q \\ \emptyset & \text{if } (y,q) \notin A \times Q \end{cases}$$

then $< x, \mu >= \mu_{<x,A\times Q>}$ for all $x \in S$.

Proof 3.17 *Let $x, y \in S$ and $q \in Q$.*
(1) If $(y,q) \in < x, \mu >$, then $(xy,q) \in A \times Q$ and $\mu(xy,q) = U$ and so $< x, \mu > (y,q) = U$. Also $\mu_{<x,A\times Q>}(y,q) = U$. Thus $< x, \mu >= \mu_{<x,A\times Q>}$.
(2) If $(y,q) \notin < x, \mu >$, then $(xy,q) \notin A \times Q$ so $\mu(xy,q) = \emptyset$ then $< x, \mu_{A\times Q} > (y,q) = \emptyset$. Also $\mu_{<x,A\times Q>}(y,q) = \emptyset$. Therefore $< x, \mu >= \mu_{<x,A\times Q>}$.

Proposition 3.18 *Let S be a commutative semigroup and μ be a Q-soft completely prime ideal of S. Then $< x, \mu >$ is a Q-soft completely prime ideal of S for all $x \in S$.*

Proof 3.19 *Let μ be a Q-soft completely prime ideal of S. Then by Proposition 3.11, we have that $< x, \mu >$ is a Q-soft ideal of S for all $x \in S$. If $x, z \in S$ and $q \in Q$, then*

$$< x, \mu > (yz,q) = \mu(xyz,q) = \mu(x,q) \cup \mu(yz,q)$$

$$= \mu(x,q) \cup \mu(y,q) \cup \mu(z,q) = \mu(x,q) \cup \mu(y,q) \cup \mu(x,q) \cup \mu(z,q)$$

$$=< x, \mu > (y,q) \cup < x, \mu > (z,q).$$

Thus $< x, \mu >$ is a Q-soft completely prime ideal of S for all $x \in S$.

Corollary 3.20 *Let S be a semigroup and μbe a Q-soft completely prime ideal of S. Then $< x, \mu >=< x^2, \mu >$.*

Proposition 3.21 *Let S be a semigroup and μ be a non-empty Q-soft subset of S. If $\mu_\alpha = \{(x,q) \mid \mu(x,q) \supseteq \alpha\}$, then $< x, \mu_\alpha >=< x, \mu >_\alpha$ for all $x \in S$ and $\alpha \in P(U)$.*

Proof 3.22 *Let $x, y \in S, q \in Q$ and $\alpha \in P(U)$. Then*

$$(y,q) \in < x, \mu >_\alpha \Longleftrightarrow < x, \mu > (y,q) \supseteq \alpha \Longleftrightarrow \mu(xy,q) \supseteq \alpha$$

$$\Longleftrightarrow (xy,q) \in \mu_\alpha \Longleftrightarrow (y,q) \in < x, \mu_\alpha > .$$

Hence $< x, \mu_\alpha >=< x, \mu >_\alpha$.

Proposition 3.23 *Let S be a commutative semigroup and μ be a Q-soft subset of S such that $< x, \mu >= \mu$ for every $x \in S$. Then μ is a constant function.*

Proof 3.24 *Let $x, y \in S$ and $q \in Q$. Then*
$\mu(x,q) =< y, \mu > (x,q) = \mu(yx,q) = \mu(xy,q) =< x, \mu > (y,q) = \mu(y,q)$.

Proposition 3.25 *Let S be a commutative semigroup and μ be a Q-soft completely semiprime ideal of S. Then $< x, \mu >$ is a Q-soft completely semiprime ideal of S for all $x \in S$.*

Proof 3.26 *since μ is Q-soft completely semiprime ideal of S so by Proposition 3.11, $< x, \mu >$ is a Q-soft ideal of S for all $x \in S$. If $y \in S$, then*

$$< x, \mu > (y^2, q) = \mu(xy^2, q) \subseteq \mu(xy^2x, q) = \mu(xyyx, q)$$

$$= \mu(xyxy, q) = \mu((xy)^2, q) \subseteq \mu(xy, q) =< x, \mu > (y, q).$$

Therefore $< x, \mu >$ is a Q-soft completely semiprime ideal of S for all $x \in S$.

Corollary 3.27 *Let S be a commutative semigroup and $\{\mu_i\}$ be a family of Q-soft completely semiprime ideals of S. Let $\mu = \cap_{i\in\Lambda}\mu_i \neq \emptyset$. Then $< x, \mu >$ is a Q-soft completely semiprime ideal of S for any $x \in S$.*

Proof 3.28 *From Proposition 3.28, it is enough to prove that μ is a Q-soft completely semiprime ideal of S. Let $x, y \in S$ and $q \in Q$. Then*

$$\mu(xy, q) = \cap_{i\in\Lambda}\mu_i(xy, q) \supseteq \cap_{i\in\Lambda}\mu_i(y, q) = \mu(y, q)$$

and

$$\mu(xy, q) = \cap_{i\in\Lambda}\mu_i(xy, q) \supseteq \cap_{i\in\Lambda}\mu_i(x, q) = \mu(x, q).$$

Hence μ is a Q-soft ideal of S.
Also $\mu(x, q) = \cap_{i\in\Lambda}\mu_i(x, q) \supseteq \cap_{i\in\Lambda}\mu_i(x^2, q) = \mu(x^2, q)$ and so μ is a Q-soft completely semiprime ideal of S.

4 Open Problem

One can define the concepts of Q-soft subsemigroups, Q-soft regular semigroups, Q-soft archimedean semigroups and Q-soft interior ideals and can obtain some new results as we obtained.

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