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A logarithmic integral related to generalized harmonic numbers

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Abstract

In this paper, we calculate a logarithmic integral $I_{n,m} = \int_0^1 x^{n-1} \ln^m(1-x)dx$ ***related to generalized harmonic numbers by using binomial theorem.***

Keywords: *logarithmic integral; generalized harmonic numbers; Riemann zeta function.*

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1 Introduction

In 2015, H. F. Sandham proposed the quadratic series [[2]] $\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2$ named Au-Yeung series where H_n is the $n-$th harmonic number defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

In this paper, we will also use the donation

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} = 1 + \frac{1}{2^r} + \cdots + \frac{1}{n^r}.$$

So that $H_n^{(1)} = H_n$.Recently, C. I. Vălean[[3]] give a new proof for classical quadratic harmonic series

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3)$$

by combining a series of technique based on special logarithmic integral where $\zeta(z)$ is the Riemann zeta function defined by $\zeta(z) = \sum\limits_{n=1}^{\infty} \frac{1}{n^z}$. Later, R. Dutta give an interesting proof for the cubic series

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3 = \frac{93}{16}\zeta(6) - \frac{5}{2}\left(\zeta(3)\right)^2$$

in [4]. In their proofs of references [3],[4], the logarithmic integral plays a key role. motivated by methods of proofs, we establish an explicit formula about general logarithmic integral$I_{n,m} = \int_0^1 x^{n-1} \ln^m(1-x)dx$. Throughout this paper, we always use this donation $I_{n,m}$.

2 main results

Theorem 2.1 *Let $m, n > 1$ be two integers. Then the following equalities hold:*

(i) $I_{n,m} = (-1)^m m! \sum\limits_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{(k+1)^{m+1}}$;

(ii) $I_{n,m} = \frac{(-1)^{m+1} m!}{n} \sum\limits_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^m}$;

(iii) $I_{n,m} = \frac{(-1)^{m+1} m!}{n} \sum\limits_{1m_1+2m_2+3m_3+\cdots} \frac{1}{m_1! m_2! m_3! \cdots} \left(\frac{H_n^{(1)}}{1}\right)^{m_1} \left(\frac{H_n^{(2)}}{2}\right)^{m_2} \left(\frac{H_n^{(3)}}{3}\right)^{m_3} \cdots$.

Proof. (i) Using substitution $1 - x = t$ and binomial theorem, we have

$$\begin{aligned} I_{n,m} &= \int_0^1 x^{n-1} \ln^m(1-x)dx \\ &= \int_0^1 (1-t)^{n-1} \ln^m t dt \\ &= \int_0^1 \sum_{k=0}^{n-1} \binom{n-1}{k} (-t)^k \ln^m t dt \\ &= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^1 t^k \ln^m t dt. \end{aligned} \tag{2.1}$$

We consider the integral $J_{k,m}$ defined by

$$J_{k,m} = \int_0^1 t^k \ln^m t dt \tag{2.2}$$

Using integration by parts, we obtain

$$J_{k,m} = \frac{1}{k+1}\int_0^1 \ln^m t dt^{k+1} = J_{k,m-1}. \tag{2.3}$$

So, we easily obtain

$$J_{k,m} = (-1)^m \frac{m!}{(k+1)^{m+1}} \tag{2.4}$$

based on the recurrence relation (2.3). Combining (2.1) with (2.4), we complete the proof of (i).
(ii) It is obvious that

$$\begin{aligned} &\int_0^1 \tfrac{1-(1-t)^n}{t} \ln^{m-1} t dt \\ &= \tfrac{1}{m}\int_0^1 [1-(1-t)^n]\, d\ln^m t \\ &= -\tfrac{n}{m}\int_0^1 (1-t)^{n-1}\ln^m t dt \end{aligned}$$

by using integration by parts. So, we obtain the following identity

$$I_{n,m} = \int_0^1 (1-t)^{n-1}\ln^m t dt = -\frac{n}{m}\int_0^1 \frac{1-(1-t)^n}{t}\ln^{m-1} t dt. \tag{2.5}$$

Simple computation and binomial theorem yield

$$\begin{aligned} &\int_0^1 \tfrac{1-(1-t)^n}{t}\ln^{m-1} t dt \\ &= \int_0^1 \frac{1-\sum\limits_{k=0}^{n}\binom{n}{k}(-t)^k}{t}\ln^{m-1} t dt \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k \int_0^1 t^{k-1}\ln^{m-1} t dt \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{k+m}\tfrac{(m-1)!}{k^m}. \end{aligned}$$

Therefore, we easily complete the proof of (ii).
(iii) Using the known formula(See [1])

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k^m} = -\sum_{1m_1+2m_2+3m_3+\cdots}\frac{1}{m_1!m_2!m_3!\cdots}\left(\frac{H_n^{(1)}}{1}\right)^{m_1}\left(\frac{H_n^{(2)}}{2}\right)^{m_2}\left(\frac{H_n^{(3)}}{3}\right)^{m_3}\cdots$$

we easily obtain (iii).

Corollary 2.2 *Setting $m = 1, 2, 3$ in (iii) of Theorem 2.1, we respectively obtain*
(a) $I_{n,1} = \int_0^1 x^{n-1}\ln(1-x)dx = -\frac{H_n}{n}$;
(b) $I_{n,2} = \int_0^1 x^{n-1}\ln^2(1-x)dx = \frac{H_n^{(2)}}{n} + \frac{(H_n)^2}{n}$;
(c) $I_{n,3} = \int_0^1 x^{n-1}\ln^3(1-x)dx = -\left(\frac{H_n^{(3)}+3H_n(H_n)^2+2(H_n)^3}{n}\right)$.
In this way, we get Lemma 2 in [3] and Lemma 4.1 in [4] again.

3 Open Problem

Let $m > 1, n < 0$ be two integers.. Compute

$$I_{n,m} = \int_0^1 x^{n-1} \ln^m(1-x)dx. \tag{3.1}$$

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