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Multistage Adomian Decomposition Method for Solving Initial Value Problem of Bratu-Type

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Abstract

The aim of this study is applying the Multistage Adomian Decomposition Method (MADM), based on the Adomian Decomposition Method (ADM) to solve an initial value problem of Bratu-type. The approximat-ed solutions are computed and then the numerical results compared those of ADM, Finally comparisons of the results show that the pro-posed method is very effective and accurate.

Keywords: *Adomian Decomposition Method; Multistage Adomian Decomposition Method; Bratu-type problem.*

1 Introduction

The Adomian Decomposition method is a solution method with a wide range of applications, including the solution of algebraic, differential, integral and integro-differential equations. This method was first introduced by Adomian [1, 3] in the beginning of the 1980s. In this method the solution is considered in rapidly converging, infinite series. The convergence of the method was proved by Y. Cherrualt et al. [2, 8].

Consider the Liouville-Bratu-Gelfand equation [6, 13, 19, 20]

$$(1) \qquad \begin{aligned} &u + \lambda e^{u} = 0, x \in \Omega \\ &u = 0, \quad x \in \partial\Omega \end{aligned}$$

Where $\lambda > 0$ is a physical parameter and Ω is a bounded domain in R^N. We restrict our attention to the one-dimensional case, where Ω=(0,1). This is known in the literature as the classical Bratu's problem:

$$(2) \qquad u''(x) + \lambda e^{u} = 0, \quad 0 < x < 1$$
$$u(0) = u(1) = 0.$$

The exact solution of this boundary value problem (BVP) is given by:

$$(3) \qquad u(x) = -2\ln\left[\frac{\cosh\left((0.5)(x-0.5)\theta\right)}{\cosh(\theta/4)}\right],$$

where θ is the solution of $\theta = \sqrt{2\lambda}\cosh(\theta/4)$. The problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where $\lambda_c = 3.513830719$ [4,5,7,15].

Such two-point BVPs are widely used in science and engineering to describe complicated chemical models. Also Bratu's problem is used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of thermal reaction process [6,10,12,17,18,20,21,23], the Chandrasekhar model of the expansion of the universe, question theory, radiative heat transfer and nanotechnology [12].

Several analytical and numerical methods, such as finite difference method [6,20], decomposition method [9], Laplace transform decomposition method [16,21], Adomian decomposition method [11,23], Restarted Adomian decomposition method [22], Homotopy analysis method [18], He's variational method [12], and Non-polynomial spline method [14] have been used to obtain the approximate solution of the Bratu-type problem. However, in the most of previous studies the error of approximated solutions increase by distance from the initial value. Therefore, in this study, were applied the MADM to solve the initial value problem of Bratu-type and show that the MADM in identical conditions, results in more accurate and suitable solution for the Bratu-type problem than those for the ADM, Also more importantly error of the solution is uniformly in the whole of the time horizon.

The present paper is summarized as follows: In Section 2, we describe the ADM for initial value problem of the Bratu-type. Section 3 is devoted to the solution of the initial value problem of Bratu-type by using the MADM. Finally, in Section 4, the conclusion is briefly discussed. The computations associated with numerical results were performed using the mathematica 7.

2 The ADM for initial value problem of the bratu-type

In this section, we apply the ADM for initial value problem of the Bratu-type as follows:

(4) $u''(x) - 2e^{u(x)} = 0, \quad 0 < x < 1$

$u(0) = u'(0) = 0$

Whose exact solution is. Wazwaz in [23] used ADM $u(x) = -2\ln(\cos x)$ to solve this problem. We describe this procedure briefly. To do this, consider (4) in an operator form

(5) $Lu(x) = 2e^{u(x)}$

Denoting $\frac{d^2}{dt^2}$ by L, We have L^{-1} as two-fold integration given by $L^{-1}(.) = \int_0^x\int_0^x (.)\,dxdx$. Applying the inverse operator to both sides of Eq. (5) and using the initial conditions $u(0) = u'(0) = 0$ we find

(6) $\quad u(x) = 2L^{-1}e^{u(x)}.$

In order to use ADM, let $u(x) = \sum_{m=0}^{\infty} u_m(x)$ and $e^{u(x)} = \sum_{m=0}^{\infty} A_m(x)$ where and are $e^{u(x)}$'s are the Adomian polynomials that represent the nonlinear term A_m given by

(7) $\quad A_0(x) = e^{u_0(x)},$

(8) $\quad A_1(x) = u_1(x)e^{u_0(x)},$

(9) $\quad A_2(x) = \left(u_2(x) + \frac{1}{2}u_1^2(x)\right)e^{u_0(x)},$

(10) $\quad A_3(x) = \left(u_3(x) + u_1(x)u_2(x) + \frac{1}{6}u_1^3(x)\right)e^{u_0(x)},$

(11) $\quad A_4(x) = \begin{pmatrix} u_4(x) + u_1(x)u_3(x) + \frac{1}{2}u_2^2(x) + \\ \frac{1}{2}u_1^2(x)u_2(x) + \frac{1}{24}u_1^4(x) \end{pmatrix} e^{u_0(x)},$

$\vdots$

By identifying the zeroth component $u_0(x)$ by 0, the remaining components $u_m(x)$ for $m \geq 0$ can be determined by using the recurrence relation

(12) $u_0(x) = 0,$

(13) $u_{m+1}(x) = 2\int_0^x\int_0^x A_m(x)\,dxdx, \quad m \geq 0$

This in turn gives

(14) $u_1(x) = x^2,$

(15) $u_2(x) = \dfrac{x^4}{6},$

(16) $u_3(x) = \dfrac{2x^6}{45},$

(17) $u_4(x) = \dfrac{17x^8}{1260},$

$\vdots$

More components in the decomposition series can be calculated to enhance the accuracy of the approximation. By computing five terms of the series solution, we obtain

(18) $u_{ADM}(x) \cong \varphi_5(x) = \sum_{m=0}^{4} u_m(x) = x^2 + \dfrac{x^4}{6} + \dfrac{2x^6}{45} + \dfrac{17x^8}{1260}.$

3 The MADM for initial value problem of the Bratu-type

In this section, we introducethe MADM to solve the initial value problem of Bratu-type. Toward this end, consider the Bratu problem (4) defined on the time horizon [0, 1]. We divide the time horizon into n equal subintervals $\Delta T = T_{j+1} - T_j$, $j = 0, 1, \ldots, n-1$with $T_0 = 0$ and $T_n = 1$. The approximate solution $u_{j+1}(x) = \sum_{m=0}^{\infty} u_{j+1,m}(x)$ can be constructed by applying the ADM on $[T_j, T_{j+1}]$, $j = 0, 1, \ldots, n-1.$

ForTherefore . $u'_1(0) = u'(0)$ and $u_1(0) = u(0)$ we let $[0, T_1]$, over $j = 0$ we have

(19) $u_{1,0}(x) = 0,$

(20) $u_{1,m+1}(x) = 2\int_0^x\int_0^x A_m(x)\,dxdx, \quad m \geq 0$

and for $j = 1,\ldots,n-1$, over $[T_j, T_{j+1}]$ we let $u_{j+1}(T_j) = u_j(T_j)$ and $u'_{j+1}(T_j) = u_j'(T_j)$. Therefore we have

(21) $u_{j+1,0}(x) = u_j(T_j) + u_j'(T_j)(x - T_j),$

(22) $u_{j+1,m+1}(x) = 2\int_{T_j}^x\int_{T_j}^x A_m(x)\,dxdx, \quad m \geq 0.$

Applying MADM, the results obtained are illustrated in Table 1 for various ΔT. The results forof the with five terms $\Delta T = 0.1$and $\Delta T = 0.2$, $\Delta T = 0.25$ series solution which shows the improvement of the solution whenever the step size, ΔT gets smaller. We can see clearly that the obtained solution has high accuracy, and this method gives much better results.

Clearly the accuracy of the MADM can be increased either by choosing a small step size or by adding more terms of the series solution. Choosing a small step size increases the number of iterations required to reach the steady-state solution whilst adding more terms need more analysis to calculate the new terms.

Time	Error of ADM	Error of MADM for $\Delta T = 0.25$	Error of MADM for $\Delta T = 0.2$	Error of MADM for $\Delta T = 0.1$
0	0	0	0	0
0.1	4.3876E – 13	4.3876E – 13	4.3876E − 13	2.2196E − 9
0.2	4.5402E – 10	4.5402E – 10	4.3247E − 10	2.0723E − 9
0.3	2.6638E – 8	1.2838E – 8	2.7175E − 9	2.2034E − 9
0.4	4.8488E – 7	3.0300E – 8	5.7256E − 9	2.3808E − 9
0.5	4.6664E – 6	5.6003E – 8	1.1561E − 8	2.6062E − 9
0.6	3.0124E – 5	1.0703E – 7	1.9165E − 8	2.8867E − 9
0.7	1.4821E – 4	1.6389E – 7	3.3507E − 8	3.2300E − 9
0.8	6.0039E – 4	3.0843E – 7	5.4494E − 8	3.6389E − 9
0.9	2.1074E – 3	5.0111E – 7	1.0155E − 7	4.0897E − 9
1	6.6498E − 3	9.8068E − 7	1.9435E − 7	4.4329E − 9

Table 1. Errors of the ADM and the MADM

4 Conclusion

In this work, we successfully apply the Multistage Adomian decomposition method and compared with the Adomian decomposition method to solve the initial value problem of the Bratu-type. The results reveal that the MADM is very effective, accurate and simple and error of the solution is uniformly in the whole of the time horizon.

5 Open Problem

In this paper, the initial solution of the classical Bratu's problem:

$$u''(x) + \lambda e^{u} = 0, \quad 0 < x < 1 \tag{23}$$

is investigated by Multistage Adomian decomposition method. But the second solution of this problem is not approximated. The methods that we can use to find the second solution are the challenges that we face in this work.

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