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The existence and uniqueness of the solution for obstacle problems with variable growth

Lifeng Guo, Jing Wang and Jinzi Liu

School of Mathematics and Statistics, Northeast Petroleum University, Daqing 163318, China.
e-mail: lfguo1981@126.com

Abstract

In this paper, we introduce the variable exponent spaces of differential forms. After we obtain the existence and uniqueness of weak solution for obstacle problem with variable growth in the setting of these spaces.

Keywords: *Variable exponent; Differential form; Obstacle problem; Weak solution.*

1 Introduction

After spaces of differential forms were introduced in [1,2], the study of A-harmonic equations for differential forms has been developed rapidly. Many interesting results concerning A-harmonic equations have been established recently. Differential forms has many important applications in many fields, such as general relativity (see [3]), theory of elasticity (see [4]), electromagnetism (see [5-6]), and differential geometry (see [7]) etc. Hence, differential forms have become invaluable tools for many fields. With the in-depth study of nonlinear problems in natural science and engineering, constant exponent spaces express their limitations in applications. For example, constant exponent spaces are not adequate in studies on nonlinear problems with variable exponential growth. This is a new research field and reflects physical phenomena of a new kind. After Kováčik and Rákosnik first discussed the $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ spaces in [8], a lot of research has been done concerning these kinds of variable exponent spaces (see [9-12] and the references therein). In recent years, the theory of problems with variable exponential growth conditions has important applications in nonlinear elastic mechanics (see [13]), electrorheological fluids (see [14]) and image processing (see [15]).

In this paper, we are interested in the following obstacle problem:

$$\int_\Omega \langle A(x,du), d(v-u)\rangle + \langle u|u|^{p(x)-2}, v-u\rangle \mathrm{d}x \geq 0 \tag{1}$$

for v belonging to

$$\mathcal{K}_{\psi,\theta} = \{v \in W^{1,p(x)}(\Omega,\Lambda^{l-1}) : v \geq \psi \text{ a.e. in } \Omega, v-\theta \in W_0^{1,p(x)}(\Omega,\Lambda^{l-1})\}. \tag{2}$$

Here, $\psi(x) = \Sigma\psi_I(x)\mathrm{d}x_I \in \Lambda^{l-1}(\mathbb{R}^n)$, $\psi_I : \Omega \to [-\infty,+\infty]$, $\theta \in W^{1,p(x)}(\Omega,\Lambda^{l-1})$ and $p(x)$ satisfies

$$1 < p_* \leq p(x) \leq p^* < \infty \text{ for a.e. } x \in \Omega. \tag{3}$$

$v \geq \psi$ a.e. in Ω means that for any I we have $v_I \geq \psi_I$ a.e. in Ω.

We will study the solution $u \in \mathcal{K}_{\psi,\theta}$ for (1)−(2) while $A(x,\xi) : \Omega\times\Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ satisfies the following conditions:

(H1) (Continuity) $A(x,\xi)$ is measurable with respect to x for all ξ and continuous with respect to ξ for a.e. $x \in \Omega$,

(H2) (Growth) $|A(x,\xi)| \leq |g_1(x)| + C_1|\xi|^{p(x)-1}$,

(H3) (Coercivity) $\langle A(x,\xi),\xi\rangle \geq C_2|\xi|^{p(x)} - |h_1(x)|$,

(H4) (Monotonicity) $\langle A(x,\xi_1) - A(x,\xi_2), \xi_1 - \xi_2\rangle \geq 0 \quad$ for $\xi_1 \neq \xi_2$,
where$g_1(x), h_1(x) \in L^{p'(x)}(\Omega)$, C_i $(i = 1,2)$ are nonnegative constants.

The paper is organized as follows. In Section 2, we will first introduce the spaces of differential forms $L^{p(x)}(\Omega,\Lambda^l)$ and $W^{1,p(x)}(\Omega,\Lambda^l)$, which are the spaces $L^p(\Omega,\Lambda^l)$ and $W_d^p(\Omega,\Lambda^l)$ respectively (See [2]) when the variable exponent $p(x)$ reduced to a constant p $(1 < p < \infty)$. We also discuss the properties of such spaces, which will be needed later. In Section 3, we will prove our main results.

2 Preliminaries

Let $e_1, e_2, ..., e_n$ denote the standard orthogonal basis of $\mathbb{R}^n$. The space of all l-forms in $\mathbb{R}^n$ is denoted by $\Lambda^l(\mathbb{R}^n)$. The dual basis to $e_1, e_2, \cdots, e_n$ is denoted by $e^1, e^2, \cdots, e^n$ and referred to as the standard basis for 1-form $\Lambda^1(\mathbb{R}^n)$. The Grassman algebra $\Lambda(\mathbb{R}^n) = \oplus\Lambda^l(\mathbb{R}^n)$ is a graded algebra with respect to the exterior products. The standard ordered basis for $\Lambda(\mathbb{R}^n)$ consists of the forms

$$1, e^1, e^2, \cdots, e^n, e^1 \wedge e^2, \cdots, e^{n-1} \wedge e^n, \cdots, e^1 \wedge e^2 \cdots \wedge e^n.$$

For $u = \sum u_I e^I \in \Lambda^l(\mathbb{R}^n)$ and $v = \sum v_I e^I \in \Lambda^l(\mathbb{R}^n)$, the inner product is obtained by $\langle u,v\rangle = \sum u_I v_I$ with summation over all l–tuples $I = (i_1, \cdots i_l)$

and all integers $l = 0, 1, \cdots, n$. The Hodge star operator (see [16]) $\star : \Lambda(\mathbb{R}^n) \to \Lambda(\mathbb{R}^n)$ defined by the formulas:

$$\star 1 = e^1 \wedge \; e^2 \cdots \wedge e^n, \quad u \wedge \star v = v \wedge \star u = \langle u, v \rangle \; e^1 \wedge \; e^2 \cdots \wedge e^n.$$

Hence the norm of u is given by the formula $|u|^2 = \langle u, u \rangle = \star(u \wedge \star u) = \sum u_I^2 \in \Lambda^0(\mathbb{R}^n) = \mathbb{R}^n$. Notice, the Hodge star operator is an isometric isomorphism on $\Lambda(\mathbb{R}^n)$. Moreover

$$\star : \Lambda^l(\mathbb{R}^n) \to \Lambda^{n-l}(\mathbb{R}^n), \quad \star\star = (-\mathrm{I})^{l(n-l)} : \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$$

where I denotes the identity map.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The coordinate function $x_1, x_2, \cdots, x_n$ in $\Omega \subset \mathbb{R}^n$ are considered to be differential forms of degree 0. The 1-forms $dx_1, dx_2, ..., dx_n$ are constant functions from Ω into $\Lambda^1(\mathbb{R}^n)$. The value of dx_i is simply e^i, $i = 1, 2, \cdots, n$. Therefore every $l-$form $u : \Omega \to \Lambda^l(\mathbb{R}^n)$ may be written uniquely as

$$u(x) = \sum u_I(x) dx_I = \sum_{1 \le i_1 < \cdots < i_l \le n} u_{i_1, \cdots i_l}(x) \; dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

where the coefficients $u_{i_1, \cdots i_l}(x)$ are distributions from $\mathcal{D}'(\Omega)$. The exterior differential $d : \mathcal{D}'(\Omega, \Lambda^l) \to \mathcal{D}'(\Omega, \Lambda^{l+1})$ is expressed by

$$du(x) = \sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_l \le n} \frac{\partial u_{i_1, \cdots i_l}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l}.$$

The formal adjoint operator, called the Hodge codifferential, is given by

$$d^\star = (-\mathrm{I})^{nl-1} \star d \star : \mathcal{D}'(\Omega, \Lambda^{l+1}) \to \mathcal{D}'(\Omega, \Lambda^l).$$

$C^\infty(\Omega, \Lambda^l)$ denote the space of infinitely differentiable l-forms on Ω and $C_0^\infty(\Omega, \Lambda^l)$ denote the space $C^\infty(\Omega, \Lambda^l)$ with compact support on Ω.

Next we recall the following classes of differential forms with L^p-integrable coefficients.

$L^p(\Omega, \Lambda^l)$ is the space of differential l-forms $u(x)$ with coefficients in $L^p(\Omega)$, $1 \le p \le \infty$. The norm is given by

$$||u(x)||_{L^p(\Omega, \Lambda^l)} = \Big(\int_\Omega |u(x)|^p dx \Big)^{\frac{1}{p}}, \quad 1 \le p < \infty;$$

$$||u(x)||_{L^\infty(\Omega, \Lambda^l)} = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|.$$

$W^{1,p}(\Omega, \Lambda^l)$ is the space of differential l-forms $u(x) \in L^p(\Omega, \Lambda^l)$ such that $du \in L^p(\Omega, \Lambda^{l+1})$ with $l = 0, 1, \cdots, n-1$. For $W^{1,p}(\Omega, \Lambda^l)$ the norm is

$$||u(x)||_{W^{1,p}(\Omega, \Lambda^l)} = ||u(x)||_{L^p(\Omega, \Lambda^l)} + ||du(x)||_{L^p(\Omega, \Lambda^{l+1})}.$$

$W_0^{1,p}(\Omega,\Lambda^l)$ is the completion of $C_0^\infty(\Omega,\Lambda^l)$ in $W^{1,p}(\Omega,\Lambda^l)$ with respect to the norm $||u(x)||_{W^{1,p}(\Omega,\Lambda^l)}$.

For $u(x)=\sum u_I(x)dx_I\in L^p(\Omega,\Lambda^l)$ and $\varphi(x)=\sum\varphi_I(x)dx_I\in L^{p'}(\Omega,\Lambda^{n-l})$ we have the bilinear function

$$L(u,\varphi)=\int_\Omega u\wedge\varphi$$

satisfying $|L(u,\varphi)|\le ||u(x)||_{L^p(\Omega,\Lambda^l)}||\varphi(x)||_{L^{p'}(\Omega,\Lambda^{n-l})}$. For each form $\varphi(x)\in L^{p'}(\Omega,\Lambda^{n-l})$ it correspond to a functional L_φ on $L^{p'}(\Omega,\Lambda^{n-l})$ by setting

$$L_\varphi(u)=\int_\Omega u\wedge\varphi=\int_\Omega\sum u_I(x)\varphi_I(x)\mathrm{d}x.$$

$L^p(\Omega,\Lambda^l)$ and $W^{1,p}(\Omega,\Lambda^l)$ are two reflexive Banach spaces for $1<p<\infty$. The correspondence $\varphi\to L_\varphi$ is an isometric isomorphism form the Banach space $L^{p'}(\Omega,\Lambda^{n-l})$ to the space $[L^p(\Omega,\Lambda^l)]'$, which is dual to $L^p(\Omega,\Lambda^{n-l})$, where p' is the conjugate number of p .

Finally we recall some basic properties of variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and variable exponent Sobolev space $W^{k,p(x)}(\Omega)$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain.

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p:\Omega\to[1,\infty]$. For $p\in\mathcal{P}(\Omega)$ we put $\Omega_1=\{x\in\Omega:p(x)=1\}$, $\Omega_\infty=\{x\in\Omega:p(x)=\infty\}$, $\Omega_0=\Omega\backslash(\Omega_1\cup\Omega_\infty)$, $p_*=\mathrm{essinf}_{\Omega_0}\,p(x)$ and $p^*=\mathrm{esssup}_{\Omega_0}\,p(x)$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\int_{\Omega\backslash\Omega_\infty}|\lambda u(x)|^{p(x)}\mathrm{d}x+\mathrm{esssup}_{\Omega_\infty}|\lambda u(x)|<\infty$ for some $\lambda=\lambda(u)>0$, the space $L^{p(x)}(\Omega)$ is a reflexive Banach space equipped with the following norm

$$||u||_{L^{p(x)}(\Omega)}=\inf\big\{\lambda>0:\int_\Omega|\frac{u}{\lambda}|^{p(x)}\mathrm{d}x+\mathrm{esssup}_{\Omega_\infty}|\frac{u}{\lambda}|\le 1\big\}.$$

The variable exponent Sobolev space $W^{k,\,p(x)}(\Omega)$ is the class of all functions $u\in L^{p(x)}(\Omega)$ such that $\delta_k u=\{D^\alpha u:|\alpha|\le k\}\subset L^{p(x)}(\Omega)$, the space $W^{k,\,p(x)}(\Omega)$ is a reflexive Banach space equipped with the following norm

$$||u||_{W^{k,\,p(x)}(\Omega)}=\sum_{|\alpha|\le k}||D^\alpha u||_{L^{p(x)}(\Omega)}.$$

For a differential l-form $u(x)$ on Ω we define the functional $\rho_{p(x)}$ by

$$\rho_{p(x),\Lambda^l}(u)=\int_{\Omega\backslash\Omega_\infty}|u(x)|^{p(x)}\mathrm{d}x+\mathrm{esssup}_{\Omega_\infty}|u(x)|.$$

Definition 2.1. *The variable exponent Lebesgue spaces of differential l-forms $L^{p(x)}(\Omega,\Lambda^l)$ is the set of differential l-forms u such that $\rho_{p(x),\Lambda^l}(\lambda u)<\infty$ for some $\lambda=\lambda(u)>0$ and we endow it with the following norm:*

$$||u||_{L^{p(x)}(\Omega,\Lambda^l)}=\inf\{\lambda>0:\rho_{p(x),\Lambda^l}(\frac{u}{\lambda})\le 1\}.$$

Given $p \in \mathcal{P}(\Omega)$ we define the conjugate function $p\,' \in \mathcal{P}(\Omega)$ by

$$p\,'(x) = \begin{cases} \infty & \text{if } x \in \Omega_1, \\ 1 & \text{if } x \in \Omega_\infty, \\ \frac{p(x)}{p(x)-1} & \text{if } x \in \Omega_0. \end{cases}$$

Definition 2.2. *The variable exponent Sobolev spaces of differential l-forms* $W^{1,p(x)}(\Omega, \Lambda^l)$ *is the space of differential l-forms* $u \in L^{p(x)}(\Omega, \Lambda^l)$ *such that* $du \in L^{p(x)}(\Omega, \Lambda^{l+1})$ *with* $l = 0, 1, \cdots, n-1$. *For* $W^{1,p(x)}(\Omega, \Lambda^l)$ *the norm is defined as*

$$||u||_{W^{1,p(x)}(\Omega,\Lambda^l)} = ||u||_{L^{p(x)}(\Omega,\Lambda^l)} + ||du||_{L^{p(x)}(\Omega,\Lambda^{l+1})}.$$

$W_0^{1,p(x)}(\Omega, \Lambda^l)$ is the completion of $C_0^\infty(\Omega, \Lambda^l)$ in $W^{1,p(x)}(\Omega, \Lambda^l)$ with respect to the norm $||u||_{W^{1,p(x)}(\Omega,\Lambda^l)}$.

Theorem 2.1.(see [17]) *If* $p(x)$ *satisfies (3), then the inequality*

$$\int_\Omega \langle u(x), v(x)\rangle \mathrm{d}x \le C||u(x)||_{L^{p(x)}(\Omega,\Lambda^l)}||v(x)||_{L^{p'(x)}(\Omega,\Lambda^l)}$$

holds for every $u(x) \in L^{p(x)}(\Omega, \Lambda^l)$, $v(x) \in L^{p'(x)}(\Omega, \Lambda^l)$ *with the constant* C *dependent on* $p(x)$ *only.*

Theorem 2.2.(see [17]) *If* $p(x)$ *satisfies (3), then the spaces* $L^{p(x)}(\Omega, \Lambda^l)$ *and* $W^{1,p(x)}(\Omega, \Lambda^l)$ *are a reflexive Banach space.*

3 Main results

In this section, as an application of spaces of differential forms with variable growth, we shall obtain the existence and uniqueness of the solution for obstacle problems with variable growth. We first introduce a theorem of Kinderlehrer and Stampacchia.

Let X be a reflexive Banach space with dual X' and let $(\cdot, \cdot)$ denote a dual between X' and X. If $\mathcal{K} \subset X$ is a closed convex set, then a mapping $\mathcal{L} : \mathcal{K} \to X'$ is called monotone, if

$$(\mathcal{L}u - \mathcal{L}v, u - v) \ge 0$$

for all $u, v \in \mathcal{K}$. The mapping $\mathcal{L}$ is called coercive on $\mathcal{K}$, if there exists $\varphi \in \mathcal{K}$ such that

$$\frac{(\mathcal{L}u_j - \mathcal{L}\varphi, u_j - \varphi)}{||u_j - \varphi||_X} \to \infty,$$

whenever $\{u_j\}$ is a sequence in $\mathcal{K}$ with $||u_j - \varphi||_X \to \infty$. Further, $\mathcal{L}$ is called strongly-weakly continuous on $\mathcal{K}$, if $u \in \mathcal{K}$, $\{u_j\}$ is a sequence in $\mathcal{K}$ with $||u_j - u||_X \to 0$, then

$$(\mathcal{L}u_j, v) \to (\mathcal{L}u, v)$$

for all $v \in X$.

Theorem 3.1._(Kinderlehrer and Stampacchia [18]). Let $\mathcal{K}$ be a nonempty closed convex subset of X and let $\mathcal{L} : \mathcal{K} \to X'$ be monotone, coercive and strongly-weakly continuous on $\mathcal{K}$. Then there exists an element u in $\mathcal{K}$ such that_

$$(\mathcal{L}u, v - u) \geq 0$$

for all $v \in \mathcal{K}$.

Now we set $X = W^{1,p(x)}(\Omega, \Lambda^{l-1})$ and let $(\cdot, \cdot)$ be the usual dual between X and X', i.e.

$$(u, v) = \int_\Omega u \wedge v,$$

where $v \in X$ and $u \in X'$. We will take $\mathcal{K}_{\psi,\theta}$ as $\mathcal{K}$.

Lemma 3.1. _$\mathcal{K}_{\psi,\theta}$ is a closed convex set._

It is immediate to obtain the conclusion.

Next we define a mapping $\mathcal{L} : \mathcal{K}_{\psi,\theta} \to \left[W^{1,p(x)}(\Omega, \Lambda^{l-1})\right]'$ by

$$(\mathcal{L}v, u) = \int_\Omega \langle A(x, dv), du\rangle + \langle v|v|^{p(x)-2}, u\rangle \mathrm{d}x$$

for all $u \in W^{1,p(x)}(\Omega, \Lambda^{l-1})$.

Lemma 3.2. _For each $v \in \mathcal{K}_{\psi,\theta}$, $\mathcal{L}v \in \left[W^{1,p(x)}(\Omega, \Lambda^{l-1})\right]'$._

Proof. By (H2) and Theorem 2.1

$$\begin{aligned}
&|\int_\Omega \langle A(x, dv), du\rangle \mathrm{d}x| \\
\leq& C(|||dv|^{p(x)-1}||_{L^{p'(x)}(\Omega)} ||du||_{L^{p(x)}(\Omega,\Lambda^l)} + ||g_1||_{L^{p'(x)}(\Omega)} ||du||_{L^{p(x)}(\Omega,\Lambda^l)}) \\
\leq& C||du||_{L^{p(x)}(\Omega,\Lambda^l)},
\end{aligned}$$

because

$$\begin{aligned}
&|||dv|^{p(x)-1}||_{L^{p'(x)}(\Omega)} \\
=& \inf\{\lambda > 0 : \int_\Omega \frac{|dv|^{p(x)}}{\lambda^{p'(x)}} \mathrm{d}x \leq 1\} \\
=& \inf\{\lambda > 0 : \int_\Omega (\frac{|dv|}{\lambda^{\frac{1}{p(x)-1}}})^{p(x)} \mathrm{d}x \leq 1\} \\
\leq& \max\left\{||dv||^{p_*-1}_{L^{p(x)}(\Omega,\Lambda^l)}, ||dv||^{p^*-1}_{L^{p(x)}(\Omega,\Lambda^l)}\right\}.
\end{aligned}$$

Similarly, by Theorem 2.1, we also obtain

$$|\int_\Omega \langle v|v|^{p(x)-2}, u\rangle \mathrm{d}x| \leq C||u||_{L^{p(x)}(\Omega,\Lambda^{l-1})},$$

where $C = \max\left\{||v||^{p_*-1}_{L^{p(x)}(\Omega,\Lambda^l)}, ||v||^{p^*-1}_{L^{p(x)}(\Omega,\Lambda^l)}\right\}$. So we get $\mathcal{L}v \in \left[W^{1,p(x)}(\Omega,\Lambda^{l-1})\right]'$.

Lemma 3.3. *$\mathcal{L}$ is monotone and coercive on $\mathcal{K}_{\psi,\theta}$.*

Proof. In view of (H4), it is immediate that $\mathcal{L}$ is monotone.

Next we show that $\mathcal{L}$ is coercive. Fix $\varphi \in \mathcal{K}_{\psi,\theta}$. By (H2) and (H3)

$$\begin{aligned}
&(\mathcal{L}u - \mathcal{L}\varphi, u-\varphi)\\
\geq& C_2\int_\Omega |du|^{p(x)} + |d\varphi|^{p(x)}\mathrm{d}x - 2\int_\Omega |h_1|\mathrm{d}x - \int_\Omega |du||g_1|\mathrm{d}x\\
&-\int_\Omega |d\varphi||g_1|\mathrm{d}x - C_1\int_\Omega |du|^{p(x)-1}|d\varphi| + |du||d\varphi|^{p(x)-1}\mathrm{d}x\\
&+\int_\Omega |u|^{p(x)} + |\varphi|^{p(x)}\mathrm{d}x - \int_\Omega |u|^{p(x)-1}|\varphi| + |u||\varphi|^{p(x)-1}\mathrm{d}x\\
\geq& \min\{C_2,1\}\int_\Omega |du|^{p(x)} + |u|^{p(x)} + |d\varphi|^{p(x)} + |\varphi|^{p(x)}\mathrm{d}x - 2\int_\Omega |h_1|\mathrm{d}x\\
&-\int_\Omega |d\varphi||g_1|\mathrm{d}x - \int_\Omega \frac{\varepsilon}{p(x)}(|du|^{p(x)} + |u|^{p(x)}) + \frac{\varepsilon^{\frac{1}{1-p(x)}}}{p'(x)}(|g_1|^{p'(x)})\mathrm{d}x\\
&-(C_1+1)\int_\Omega \frac{\varepsilon}{p'(x)}(|du|^{p(x)} + |u|^{p(x)}) + \frac{\varepsilon^{\frac{1}{1-p'(x)}}}{p(x)}(|d\varphi|^{p(x)} + |\varphi|^{p(x)})\mathrm{d}x\\
&-(C_1+1)\int_\Omega \frac{\varepsilon}{p(x)}(|du|^{p(x)} + |u|^{p(x)}) + \frac{\varepsilon^{\frac{1}{1-p(x)}}}{p'(x)}(|d\varphi|^{p(x)} + |\varphi|^{p(x)})\mathrm{d}x\\
\geq&\Big(\min\{C_2,1\} - (\frac{1}{p_*} + C_1 + 1)\varepsilon\Big)\int_\Omega (|du|^{p(x)} + |u|^{p(x)})\mathrm{d}x - C(g_1,h_1,\varepsilon,\varphi,p(x)).
\end{aligned}$$

Taking $\varepsilon = \frac{\min\{C_2,1\}p_*}{2(1+(C_1+1)p_*)}$, we have

$$\begin{aligned}
&(\mathcal{L}u - \mathcal{L}\varphi, u-\varphi)\\
\geq& C\int_\Omega |du|^{p(x)} + |u|^{p(x)}\mathrm{d}x - C(g_1,h_1,\varepsilon,\varphi,p(x))\\
\geq& C\int_\Omega 2^{-p^*}(|du-d\varphi|^{p(x)} + |u-\varphi|^{p(x)}) - |d\varphi|^{p(x)} - |\varphi|^{p(x)}\mathrm{d}x - C(g_1,h_1,\varepsilon,\varphi,p(x))\\
\geq& 2^{-p^*}C\int_\Omega |du-d\varphi|^{p(x)} + |u-\varphi|^{p(x)}\mathrm{d}x - C(g_1,h_1,\varepsilon,\varphi,p(x)).
\end{aligned}$$

For a sufficiently small constant δ, we have

$$\begin{aligned}&\frac{\int_\Omega |du-d\varphi|^{p(x)}\mathrm{d}x}{||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}}\\=&\int_\Omega \Big(\frac{|du-d\varphi|}{||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}-\delta}\Big)^{p(x)}\frac{\big(||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}-\delta\big)^{p(x)}}{||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}}\mathrm{d}x\\ \geq&\frac{\big(||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}-\delta\big)^{p(x)}}{||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}}.\end{aligned}$$

Taking $\delta=\frac{1}{2}||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}$, we have

$$\frac{\int_\Omega |du-d\varphi|^{p(x)}\mathrm{d}x}{||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}}\to\infty$$

as $||du-d\varphi||_{L^{p(x)}(\Omega,\Lambda^l)}\to\infty$. Similarly, we also obtain

$$\frac{\int_\Omega |u-\varphi|^{p(x)}\mathrm{d}x}{||u-\varphi||_{L^{p(x)}(\Omega,\Lambda^{l-1})}}\to\infty$$

as $||u-\varphi||_{L^{p(x)}(\Omega,\Lambda^{l-1})}\to\infty$. Then it is immediate to obtain

$$\frac{(\mathcal{L}u-\mathcal{L}\varphi,u-\varphi)}{||u-\varphi||_{W^{1,p(x)}(\Omega,\Lambda^{l-1})}}\to\infty$$

as $||u-\varphi||_{W^{1,p(x)}(\Omega,\Lambda^{l-1})}\to\infty$. That is to say, $\mathcal{L}$ is coercive on $\mathcal{K}_{\psi,\theta}$.

Lemma 3.4. *$\mathcal{L}$ is strongly-weakly continuous.*

Proof. Let $\{v_n(x): v_n(x)=\sum v_{nI}(x)dx_I\}\subset\mathcal{K}_{\psi,\theta}$ be a sequence that converges to $v(x)=\sum v_I(x)dx_I\subset\mathcal{K}_{\psi,\theta}$ in $W^{1,p(x)}(\Omega,\Lambda^{l-1})$. Then $\{v_n(x)\}$ is uniformly bounded in $W^{1,p(x)}(\Omega,\Lambda^{l-1})$. Now in view of (H1) and (H2), we know that $A(x,dv_n)$ is uniformly bounded in $L^{p'(x)}(\Omega,\Lambda^l)$ and there exists a subsequence of $A(x,dv_n)$ (still denoted by $A(x,dv_n)$) such that $A(x,dv_n)\to A(x,dv)$ a.e. $x\in\Omega$.

Next let $A(x,dv_n)=\eta_n(x)=\Sigma\eta_{nI}(x)dx_I$, $A(x,dv)=\eta(x)=\Sigma\eta_I(x)dx_I$, we have $\eta_{nI}(x)\to\eta_I(x)$ a.e. $x\in\Omega$ and $\{\eta_{nI}(x)\}_n$ is uniformly bounded in $L^{p'(x)}(\Omega)$ for any I. Denote $du=\varphi(x)=\Sigma\varphi_I(x)dx_I$. Notice that $du\in L^{p(x)}(\Omega,\Lambda^l)$ for $u\in W^{1,p(x)}(\Omega,\Lambda^{l-1})$. Then there exists a subsequence of $\{\eta_{nI}(x)\}_n$ (still denoted by $\{\eta_{nI}(x)\}_n$) such that $\int_\Omega\eta_{nI}(x)\varphi_I(x)\mathrm{d}x\to\int_\Omega\eta_I(x)\varphi_I(x)\mathrm{d}x$ as $n\to\infty$ for each I. Therefore

$$\int_\Omega\langle A(x,dv_n),du\rangle\mathrm{d}x\to\int_\Omega\langle A(x,dv),du\rangle\mathrm{d}x$$

as $n \to \infty$. Then we have

$$\begin{aligned}
&(\mathcal{L}v_n, u)\\
=&\int_\Omega \langle A(x, dv_n), du\rangle + \langle v_n|v_n|^{p(x)-2}, u\rangle \mathrm{d}x\\
\to&\int_\Omega \langle A(x, dv), du\rangle + \langle v|v|^{p(x)-2}, u\rangle \mathrm{d}x\\
=&(\mathcal{L}v, u),
\end{aligned}$$

that is to say, $\mathcal{L}$ is strongly-weakly continuous.

Theorme 3.2. *Suppose $\mathcal{K}_{\psi,\theta} \neq \phi$ and $p(x)$ satisfies (3). Under conditions (H1)-(H4), there exists a unique solution $u \in \mathcal{K}_{\psi,\theta}$ to the obstacle problem (1)-(2). That is to say, there exists a unique $u \in \mathcal{K}_{\psi,\theta}$ such that*

$$\int_\Omega \langle A(x, du), d(v-u)\rangle + \langle u|u|^{p(x)-2}, v-u\rangle \mathrm{d}x \geq 0$$

for any $v \in \mathcal{K}_{\psi,\theta}$.

Proof. It is immediate to obtain the existence from the above lemmas. If there are two weak solutions $u_1, u_2 \in \mathcal{K}_{\psi,\theta}$ to the obstacle problem (1)-(2), then

$$\int_\Omega \langle A(x, du_1), d(u_2-u_1)\rangle + \langle u_1|u_1|^{p(x)-2}, u_2-u_1\rangle \mathrm{d}x \geq 0$$

and

$$\begin{aligned}
&-\int_\Omega \langle A(x, du_2), d(u_2-u_1)\rangle + \langle u_2|u_2|^{p(x)-2}, u_2-u_1\rangle \mathrm{d}x\\
=&\int_\Omega \langle A(x, du_2), d(u_1-u_2)\rangle + \langle u_2|u_2|^{p(x)-2}, u_1-u_2\rangle \mathrm{d}x \geq 0,
\end{aligned}$$

furthermore

$$\int_\Omega \langle A(x, du_1)-A(x, du_2), d(u_1-u_2)\rangle + \langle u_1|u_1|^{p(x)-2}-u_2|u_2|^{p(x)-2}, u_1-u_2\rangle \mathrm{d}x \leq 0.$$

In view of (H4), we can further infer that

$$\int_\Omega \langle A(x, du_1)-A(x, du_2), d(u_1-u_2)\rangle \mathrm{d}x = 0 \quad \text{on } \Omega,$$

and

$$\int_\Omega \langle u_1|u_1|^{p(x)-2}-u_2|u_2|^{p(x)-2}, u_1-u_2\rangle \mathrm{d}x = 0 \quad \text{on } \Omega,$$

that is to say, $u_1 = u_2$ a.e. on Ω.

Corollary 3.1. *Suppose that Ω is a bounded domain with a smooth boundary $\partial\Omega$, $\theta \in W^{1,p(x)}(\Omega, \Lambda^{l-1})$ and $p(x)$ satisfies (3). Under conditions (H1)-(H4), there is a differential form $u \in W^{1,p(x)}(\Omega, \Lambda^{l-1})$ such that*

$$d^\star A(x, du) + u|u|^{p(x)-2} = 0, \quad x \in \Omega, \tag{4}$$

$$u = \theta, \quad x \in \partial\Omega, \tag{5}$$

that is to say

$$\int_\Omega \langle A(x, du), d\varphi\rangle + \langle u|u|^{p(x)-2}, \varphi\rangle \mathrm{d}x = 0$$

for any $\varphi \in W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$.

Proof. Let $\psi_I = -\infty$, and u be a solution to the obstacle problem of (1.1)-(1.2) in $\mathcal{K}_{\psi,\theta}$. For any $\varphi \in W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$, we have that both $u+\varphi$ and $u-\varphi$ belong to $\mathcal{K}_{\psi,\theta}$. Then

$$\int_\Omega \langle A(x, du), d\varphi\rangle + \langle u|u|^{p(x)-2}, \varphi\rangle \mathrm{d}x \geq 0$$

and

$$-\int_\Omega \langle A(x, du), d\varphi\rangle + \langle u|u|^{p(x)-2}, \varphi\rangle \mathrm{d}x \geq 0,$$

from which we can further infer that

$$\int_\Omega \langle A(x, du), d\varphi\rangle + \langle u|u|^{p(x)-2}, \varphi\rangle \mathrm{d}x = 0.$$

Now we complete the proof.

Example 3.1. Suppose that Ω is a bounded domain with a smooth boundary $\partial\Omega$ and $p(x)$ satisfies (3), let $\theta \in W^{1,p(x)}(\Omega, \Lambda^{l-1})$, $A(x, \xi) = \xi|\xi|^{p(x)-2}$ and $f(x) \in L^{p'(x)}(\Omega, \wedge^{l-1})$. The equation (4)-(5) reduces to the following equation for differential forms

$$d^*(du|du|^{p(x)-2}) + u|u|^{p(x)-2} + f(x) = 0, \ x \in \Omega, \tag{6}$$

$$u = \theta, \quad x \in \partial\Omega, \tag{7}$$

Obviously, A, B satisfy the required conditions. Now by Corollary 3.1, we deduce that the equation (6)-(7) has a unique weak solution in $W^{1,p(x)}(\Omega, \wedge^{l-1})$.

4 Open Problem

The open problem here is to discuss the existence of the solutions for following A-harmonic equations with variable growth on Riemannian manifolds by this method:

$$d^*(du|du|^{p(m)-2}) + u|u|^{p(m)-2} + f(m) = 0, \ m \in M,$$
$$u = \theta(m), \quad m \in \partial M,$$

where M is a Riemannian manifold, differential forms f, θ satisfy suitable conditions.

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