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Characterizations of weak relatively complemented almost distributive lattices

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Abstract

In this paper, we characterize the class of weak relatively complemented almost distributive lattices in terms of α-ideals.

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1 Introduction

Almost **D**istributive **L**attice is an algebra of type $(2, 2, 0)$ satisfying almost all conditions of a distributive lattice except commutativity of $\wedge$, $\vee$ and right distributivity of $\vee$ over $\wedge$. This structure is a both ring theoretic and lattice theoretic generalization of a Boolean algebra (Boolean ring). Since ADL is nether a distributive lattice nor a lattice, it is too difficult to deal with it. Moreover the associativity of $\vee$ is still under investigation. In [12], Swamy and Rao introduced ideals, prime ideals in ADLs analogously from distributive lattices. They have studied ADLs in both algebraical and topological aspects. In [3] [4] [8] [9] [10], the authors rigorously studied ideals, minimal prime ideals, maximal ideals, annihilator ideals, α-ideals in ADLs and explored several results on them. In [5] [6] [7], the authors extended the concepts like quasi complementation, relative complementation to the class of ADLs.

In second section, we present some preliminary and useful results in ADLs. In third section, we observe and explain some remarks on α- ideals with counter examples and we obtain an epimorphism between the set of ideals and the set of α-ideals in L. In fourth section, we discuss principal α-ideal in ADLs and obtain several properties on them. We observe that the class of principal α-ideals in ADLs forms a distributive lattice. In last section, we characterize weak relatively complemented ADLs, when an ADL has dense elements and every dense element is maximal. We deliberate the class of prime α-ideals in an ADL. We obtain a good number of equivalent conditions for an ADL to become weak relatively complemented in terms of prime α-ideals and principal α-ideals.

2 Preliminaries

Let us first recall the notion of almost distributive lattices and certain necessary properties which are required in the sequel.

Definition 2.1 *[12] By an almost distributive lattice (abbreviated: ADL), we mean an algebra* $(L, \wedge, \vee, 0)$ *of type* $(2, 2, 0)$*, if it satisfies the following;*

(i) $0 \wedge a = 0$

(ii) $a \vee 0 = a$

(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(iv) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(v) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(vi) $(a \vee b) \wedge b = b$

for all $a, b, c \in L$.

Throughout this paper L stands for an ADL $(L, \wedge, \vee, 0)$ unless otherwise mentioned. For any $a, b \in L$, we say that a is less than or equal to b and write $a \leq b$ if $a \wedge b = a$ or, equivalently $a \vee b = b$. It is easy to observe that $\leq$ is a partial ordering on L.

Lemma 2.2 *[12] For any* $a, b, c \in L$*, we have*

(i) $a \wedge 0 = 0$ *and* $0 \vee a = a$

(ii) $a \wedge a = a \vee a = a$

(iii) $a \vee (b \vee a) = a \vee b$

(iv) $\wedge$ *is associative*

(v) $a \wedge b \wedge c = b \wedge a \wedge c$

(vi) $a \wedge b = 0 \iff b \wedge a = 0$

(vii) $a \wedge b \leq b$ *and* $a \leq a \vee b$

(viii) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(ix) $a \vee b = b \vee a \Longleftrightarrow a \wedge b = b \wedge a.$

A non-empty subset I of L is said to be an ideal, if for any $a, b \in I$ and $x \in L$, $a \vee b$, $a \wedge x \in I$. The set $\mathcal{I}(L)$ of ideals in L forms a bounded distributive lattice, where $I \cap J$ is the infimum and $I \vee J = \{i \vee j \mid i \in I, j \in J\}$ is the supremum of I and J in $\mathcal{I}(L)$. For any non-empty subset S of L, $(S] = \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_1, s_2, ..., s_n \in S,\ x \in L$ and n is a positive integer$\}$ is the smallest ideal containing S. In particular, for any $a \in L$, $(a] = \{a \wedge x \mid x \in L\}$ is the principal ideal generated by a. The set $\mathcal{PI}(L)$ of principal ideals in L forms a sublattice of $\mathcal{I}(L)$, where $(a] \wedge (b] = (a \wedge b]$ and $(a] \vee (b] = (a \vee b]$. A proper ideal P is said to be prime, if for any $a, b \in L$, $a \wedge b \in P$, implies $a \in P$ or $b \in P$. A minimal prime ideal is a minimal among prime ideals. Similarly, we can define filters, prime filters and minimal prime filters.

For any non-empty subset A of L, the set $A^* = \{x \in L \mid a \wedge x = 0$, for all $a \in A\}$ is an annihilator ideal of L. In particular, for any $a \in L$, $\{a\}^* = (a)^*$, where $(a) = (a]$ is the principal ideal generated by a.

Lemma 2.3 *[10] For any $a, b \in L$, we have*

(i) $a \leq b \Longrightarrow (b)^* \subseteq (a)^*$

(ii) $(a)^{***} = (a)^*$

(iii) $(a \vee b)^* = (a)^* \cap (b)^*$

(iv) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$

(v) $(a)^* \subseteq (b)^* \Longleftrightarrow (b)^{**} \subseteq (a)^{**}$

(vi) $a \in (a)^{**}$.

An element d in L is said to be dense, if $(d)^* = \{0\}$. Let us denote D the set of dense elements in L. Then D is a filter (provided $D \neq \emptyset$). Moreover, if $d \in D$, then $d \vee x$, $x \vee d \in D$ for all $x \in L$. An element $m \in L$ is said to be maximal, if $m \wedge x = x$ for all $x \in L$. It is easy observe that every maximal element is dense. If M is the set of maximal elements in L, then M is also a filter (provided $M \neq \emptyset$).

Definition 2.4 *[9] For any ideal I of L, let us denote $I^+ = \{x \in L \mid (x)^* \supseteq (a)^*$, for some $a \in I\}$. In particular, for any $a \in L$, $(a]^+ = \{x \in L \mid (x)^* \supseteq (a)^*\}$, where $(a]$ is the principal ideal generated by a.*

Lemma 2.5 *[9] For any ideals I, J of L, we have*

(i) For any ideal I of L, $I \subseteq I^+$

(ii) $I^{++} = I^+$

(iii) $I^+ \cap J^+ = (I \cap J)^+$

(iv) $I \subseteq J$ implies $I^+ \subseteq J^+$.

3 α-ideals in ADLs

In this section, we present some counter examples for α-ideals in ADLs and obtain that there can be an epimorphism from set of ideals in L onto α-ideals in L.

Definition 3.1 *[9] An ideal I of L is said to be an α-ideal, if $I^+ = I$. It observed that, for any ideal I of L, I^+ is the smallest α-ideal containing I.*

Theorem 3.2 *[9] For any ideal I of L, the following are equivalent;*

(i) I is an α-ideal

(ii) For $x \in L$, $x \in I$ implies $(x]^+ \subseteq I$

(iii) For $x, y \in L$, $(x)^* = (y)^*$ and $x \in I$ implies $y \in I$

(iv) For $x, y \in L$, $(x]^+ = (y]^+$ and $x \in I$ implies $y \in I$

(v) $I = \bigcup_{x \in I} (x]^+$.

Remark 3.1 *We have*

(i) $L^+ = D^+ = M^+ = L$

(ii) $(0]^+ = \{0\}$

(iii) For any $d \in D$, $(d]^+ = L$.

Lemma 3.3 *For any ideals I, J of L, we have $(I^+ \vee J^+)^+ = (I \vee J)^+$.*

Proof: We have $I \vee J \subseteq I^+ \vee J^+$. Therefore $(I \vee J)^+ \subseteq (I^+ \vee J^+)^+$. Since $I^+ \vee J^+ \subseteq (I \vee J)^+$, $(I^+ \vee J^+)^+ \subseteq ((I \vee J)^+)^+ = (I \vee J)^{++} = (I \vee J)^+$. Thus $(I^+ \vee J^+)^+ = (I \vee J)^+$.

Example 3.4 *[5] Let $L = \{0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, d, m\}$ with the operations $\wedge$ and $\vee$ defined as follows;*

$\wedge$	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
0	0	0	0	0	0	0	0	0	0	0
b_1	0	b_1	0	b_1	b_1	0	b_1	0	b_1	b_1
b_2	0	0	b_2	b_2	b_2	0	0	b_2	b_2	b_2
b_3	0	b_1	b_2	b_3	b_3	0	b_1	b_2	b_3	b_3
b_4	0	b_1	b_2	b_3	b_4	0	b_1	b_2	b_3	b_4
b_5	0	0	0	0	0	b_5	b_5	b_5	b_5	b_5
b_6	0	b_1	0	b_1	b_1	b_5	b_6	b_5	b_6	b_6
b_7	0	0	b_2	b_2	b_2	b_5	b_5	b_7	b_7	b_7
d	0	b_1	b_2	b_3	b_3	b_5	b_6	b_7	d	d
m	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m

$\vee$	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
0	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
b_1	b_1	b_1	b_3	b_3	b_4	b_6	b_6	d	d	m
b_2	b_2	b_3	b_2	b_3	b_4	b_7	d	b_7	d	m
b_3	b_3	b_3	b_3	b_3	b_4	d	d	d	d	m
b_4	b_4	b_4	b_4	b_4	b_4	m	m	m	m	m
b_5	b_5	b_6	b_7	d	m	b_5	b_6	b_7	d	m
b_6	b_6	b_6	d	d	m	b_6	b_6	d	d	m
b_7	b_7	d	b_7	d	m	b_7	d	b_7	d	m
d	d	d	d	d	m	d	d	d	d	m
m	m	m	m	m	m	m	m	m	m	m

Then $(L, \wedge, \vee, 0)$ is an ADL.

From the above example, we have the following remarks.

Remark 3.2 *Every ideal need not be an α-ideal.*
For, see Example 3.4., $(b_3]$ is an ideal but not an α-ideal (because $(b_3] = \{0, b_1, b_2, b_3\} \neq \{0, b_1, b_2, b_3, b_4\} = (b_3]^+$).

Remark 3.3 *Every α-ideal need not be prime.*
For, see Example 3.4., $(b_1] = \{0, b_1\}$ is an α-ideal but not prime (because $b_4 \wedge b_6 = b_1 \in (b_1]$ but $b_4, b_6 \notin (b_1]$).

Remark 3.4 *Every prime ideal need not be an α-ideal.*
For, see Example 3.4., $(d]$ is a prime ideal but not an α-ideal (because $L = (d]^+ \neq (d]$).

Remark 3.5 *Every maximal ideal need not be an α-ideal.*
For, see Example 3.4., $(d]$ is a maximal ideal but not an α-ideal (because $L = (d]^+ \neq (d]$).

From [12], It is a well known fact that the set $\mathcal{I}(L)$ of ideals in L forms a distributive lattice. In this context, from [9], the set $\mathcal{NI}(L)$ of α-ideals in L forms a distributive lattice with the operations $I^+ \cap J^+ = (I \cap J)^+$ and $I \sqcup J = (I \vee J)^+$ for any $I, J \in \mathcal{NI}(L)$. which is not a sublattice of $\mathcal{I}(L)$.

Theorem 3.5 *There is an epimorphism from $\mathcal{I}(L)$ onto $\mathcal{NI}(L)$.*

Proof: Let $I, J \in \mathcal{NI}(L)$. Define a map $f : \mathcal{I}(L) \longrightarrow \mathcal{NI}(L)$ by $f(I) = I^+$. Then $f(I \cap J) = (I \cap J)^+ = I^+ \cap J^+ = f(I) \cap f(J)$ and $f(I \vee J) = (I \vee J)^+ = (I^+ \vee J^+)^+ = I^+ \sqcup J^+ = f(I) \sqcup f(J)$. Therefore f is a homomorphism. Since $\mathcal{NI}(L) \subseteq \mathcal{I}(L)$, f is an onto homomorphism.

Remark 3.6 *In the above theorem, the homomorphism f need not be one-one.*
For, see Example 3.4., $I = \{0, b_1, b_2, b_3\}$ and $J = \{0, b_1, b_2, b_3, b_4\}$ are two ideals in $\mathcal{I}(L)$. Then $f(I) = I^+ = \{0, b_1, b_2, b_3, b_4\} = J^+ = f(J)$ but $I \neq J$. Therefore f is not an one-one homomorphism.

4 Principal α-ideals in ADLs

In this section, we define principal α-ideal in ADLs and obtain some algebraic and order properties on them. Mainly we prove that the set of principal α-ideals in L forms a distributive lattice.

Definition 4.1 *An ideal I of L is said to be a principal α-ideal, if $I = (x]^+$, for some $x \in L$. It can be observe that every principal α-ideal is an α-ideal.*

Lemma 4.2 *For any $a, b \in L$, we have*

(i) $a \leq b \Longrightarrow (a]^+ \subseteq (b]^+$

(ii) $a \in (b]^+ \Longrightarrow (a]^+ \subseteq (b]^+$

(iii) $(a]^+ = \{0\} \Longleftrightarrow a = 0$

(iv) $(a]^+ = L \Longleftrightarrow a \in D$

(v) If a is maximal, then $(a]^+ = L$

(vi) $(a]^+ \cap (b]^+ = (a \wedge b]^+$.

Proof: (i) Suppose that $a \leq b$. Then $(a] \subseteq (b]$. Therefore $(a]^+ \subseteq (b]^+$.
(ii) It is clear from (i).
(iii) Suppose that $a \in L$ such that $(a]^+ = \{0\}$. Then $a \in (a]^+ = \{0\}$. Therefore $a = 0$. The converse is clear from Remark 3.1.
(iv) Suppose that $(a]^+ = L$. Choose a dense element $d \in L = (a]^+$. Then $\{0\} = (d)^* \supseteq (a)^*$. Therefore $(a)^* = \{0\}$ and hence a is dense. The converse is obvious from Remark 3.1.
(v) Since every maximal element is dense and from (iv) we have $(a]^+ = L$.
(vi) $(a]^+ \cap (b]^+ = ((a] \cap (b])^+ = (a \wedge b]^+$.

Lemma 4.3 *For any $a, b \in L$, we have*

(i) $a \wedge b = 0 \Longleftrightarrow (a]^+ \cap (b]^+ = \{0\}$

(ii) $a \vee b \in D \Longleftrightarrow (a]^+ \sqcup (b]^+ = L$

(iii) $(a)^* \cap (a]^+ = \{0\}$

(iv) $(a)^* = (b)^* \Longleftrightarrow (a]^+ = (b]^+$

(v) $(a]^+ = (b]^+ \Longrightarrow (a \wedge c]^+ = (b \wedge c]^+,$ *for all $c \in L$*

(vi) $(a]^+ = (b]^+ \Longrightarrow (a \vee c]^+ = (b \vee c]^+,$ *for all $c \in L$.*

Proof: (i) and (ii) are clear from Lemma 4.2.
(iii) Let $x \in (a)^* \cap (a]^+$. Then $x \in (a)^* \subseteq (x)^*$. Therefore $x = 0$. Hence $(a)^* \cap (a]^+ = \{0\}$.
(iv) Suppose that $(a)^* = (b)^*$. Then $a \in (b]^+$ and $b \in (a]^+$. Therefore $(a]^+ \subseteq (b]^+$ and $(b]^+ \subseteq (a]^+$ and hence $(a]^+ = (b]^+$. On the other hand, suppose $(a]^+ = (b]^+$, then $(a)^* \supseteq (b)^*$ and $(b)^* \supseteq (a)^*$. Therefore $(a)^* = (b)^*$.
(v) Suppose that $(a]^+ = (b]^+$. For $t \in L$,

$$\begin{aligned} t \in (a \wedge c)^* &\Longleftrightarrow t \wedge a \wedge c = 0 \\ &\Longleftrightarrow t \wedge c \in (a)^* = (b)^* \quad \text{(from } (iv)) \\ &\Longleftrightarrow t \wedge b \wedge c = 0 \\ &\Longleftrightarrow t \in (b \wedge c)^*. \end{aligned}$$

Therefore $(a \wedge c]^+ = (b \wedge c]^+$. Similarly, we can prove (vi).

Let us denote the set of principal α-ideals in L as $\mathcal{N}^+\mathcal{I}(L)$. Now, we have

Theorem 4.4 $(\mathcal{N}^+\mathcal{I}(L), \cap, \sqcup)$ *is a sublattice of $\mathcal{N}\mathcal{I}(L)$ with the least element $(0]^+$. Moreover L has a dense element if and only if $\mathcal{N}^+\mathcal{I}(L)$ has the greatest element.*

Proof: It is easy to very that $(\mathcal{N}^+\mathcal{I}(L), \cap, \sqcup)$ is a sublattice of the distributive lattice $(\mathcal{N}\mathcal{I}(L), \cap, \sqcup)$ in which $(0]^+$ is the least element. Suppose L has a dense element, say d, then $(d]^+ = L$. Therefore $(d]^+$ is the greatest element. Conversely, suppose $(a]^+$ is the greatest element in $\mathcal{N}^+\mathcal{I}(L)$ for some $a \in L$. Let $x \in (a)^*$. Then $a \wedge x = 0$. Therefore $(x \wedge a]^+ = (x]^+ \cap (a]^+ = (x]^+ = \{0\}$. So that $x = 0$. Hence a is a dense element in L.

Define a relation φ on L by $\varphi = \{(x, y) \in L \times L \mid (x]^+ = (y]^+\}$. By Lemma 4.3., the relation φ is a congruence relation on L. Now, we have the following;

Theorem 4.5 *The quotient L/φ forms a distributive lattice with the operations $x/\varphi \wedge y/\varphi = (x \wedge y)/\varphi$ and $x/\varphi \vee y/\varphi = (x \vee y)/\varphi$. Moreover the least element is $0/\varphi = \{0\}$ and the greatest element is $d/\varphi = D$.*

5 Characterizations of weak relatively complemented ADLs

In this section, we study the class of prime α-ideals in an ADL. We obtain necessary and sufficient conditions for an ADL to become weak relatively complemented ADL and the class of principal α-ideals to become a relatively complemented.

Lemma 5.1 *For any non-empty subset S of L, $(S]^+ \cap S^* = \{0\}$.*

Proof: Let $x \in (S]^+ \cap S^*$. Then $x \wedge s = 0$ for all $s \in S$ and there exists $(\bigvee_{i=1}^{n} s_i) \wedge t = \bigvee_{i=1}^{n} (s_i \wedge t) = y \in (S]$ such that $(x)^* \supseteq (y)^*$ where $t \in L$ and $s_i \in S$ for $i = 1, 2, ..., ..n$. Therefore $x \wedge y = x \wedge \{\bigvee_{i=1}^{n} (s_i \wedge t)\} = \bigvee_{i=1}^{n} (x \wedge s_i \wedge t) = 0$. Hence $x = 0$. Thus $(S]^+ \cap S^* = \{0\}$.

In [8], for any prime ideal P of L, they introduced an ideal $O(P) = \{x \in L \mid x \wedge y = 0$ for some $y \notin P\}$. Moreover they proved that P is minimal if and only if $O(P) = P$. In [12], the authors proved that an ADL is relatively complemented if and only if every prime ideal is maximal if and only if every prime ideal is minimal.

In [5], Ramesh and Rao introduced weak relatively complemented ADLs. That is, by a weak relatively complemented ADL we mean an ADL in which for any $a, b \in L$, there exists an element $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = a \vee b)^*$. In this context, we have the following.

Theorem 5.2 *The following are equivalent;*

(i) L is weak relatively complemented

(ii) Every prime α-ideal of L is maximal

(iii) Every prime α-ideal of L is minimal

(iv) $O(P) = P$ for any prime α-ideal P of L

Proof: $(i) \Longrightarrow (ii)$ Let P be a prime α-ideal of L. Suppose that Q is a prime α-ideal of L such that $P \subseteq Q$. Choose $a \in Q \setminus P$ and $b \in L$. By our assumption there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. Then $x \in P \subseteq Q$. Therefore $(a \vee x)^* = (a \vee b)^*$ and $a \vee x \in Q$. Since Q is an α-ideal, $a \vee b \in Q$. So that $b \in Q$ and hence $Q = L$. Thus P is maximal.
$(ii) \Longrightarrow (iii)$ Let P be a prime α-ideal of L. Suppose that Q is a prime α-ideal of L such that $Q \subseteq P$. By our assumption $P = Q$. Therefore P is minimal.
$(iii) \Longrightarrow (iv)$ Assume that every prime α-ideal is minimal. Let P be a prime α-ideal of L. Then P is minimal prime ideal. Hence $O(P) = P$ for any prime α-ideal P of L.
$(iv) \Longrightarrow (i)$ Assume that $O(P) = P$ for any prime α-ideal P of L.
For $a, b \in L$.
Case I. If $a = b$, take $x = 0$, then $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$.
Case II. If $a = 0(b = 0)$, take $x = b(x = a)$, then $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$.
Case III. If $a \neq 0$, $b \neq 0$ and $a \wedge b = 0$, take $x = b$, then $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$.
Case IV. If $a \neq 0$, $b \neq 0$ and $a \wedge b \neq 0$, suppose that $b \notin (a] \vee (a)^*$, then there exists a prime ideal P of L such that $b \notin P$ and $(a] \vee (a)^* \subseteq P$. For this $(a] \vee (a)^* \subseteq P$, there exists a minimal prime ideal (prime α- ideal) Q such that $(a] \vee (a)^* \subseteq Q \subseteq P$. By our assumption $a \in Q = O(Q)$. Therefore there exists $x \notin Q$ such that $a \wedge x = 0$. Now, $a \wedge x \wedge b = 0$. So that $x \wedge b \in (a)^* \subseteq Q$. Which is a contradiction. Hence $b \in (a] \vee (a)^*$. Thus $b = (a \wedge s) \vee t$ for some $s \in L,\ t \in (a)^*$. Now, for $y \in L$,

$$\begin{aligned}
y \in (a \vee t)^* &\Rightarrow y \wedge a = 0 \text{ and } y \wedge t = 0\\
&\Rightarrow y \wedge a \wedge s = 0 \text{ and } y \wedge t = 0\\
&\Rightarrow y \wedge \{(a \wedge s) \vee t\} = 0\\
&\Rightarrow y \wedge b = 0 \qquad (\text{since } b = (a \wedge s) \vee t)\\
&\Rightarrow y \wedge (a \vee b) = 0\\
&\Rightarrow y \in (a \vee b)^*.
\end{aligned}$$

$$\begin{aligned}
y \in (a \vee b)^* &\Rightarrow y \wedge a = 0 \text{ and } y \wedge b = 0\\
&\Rightarrow y \wedge a = 0 \text{ and } y \wedge \{(a \wedge s) \vee t\} = 0\\
&\Rightarrow y \wedge a = 0,\ y \wedge a \wedge s = 0 \text{ and } y \wedge t = 0\\
&\Rightarrow y \wedge (a \vee t) = 0\\
&\Rightarrow y \in (a \vee t)^*.
\end{aligned}$$

Therefore L is weak relatively complemented.

By Lemma 4.3(iv), for any elements $a, b \in L$, $(a)^* = (b)^* \iff (a]^+ = (b]^+$. In this regard, we have the following;

Lemma 5.3 *L is a weak relatively complemented ADL if and only if for any $a, b \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x]^+ = (a \vee b]^+$.*

Theorem 5.4 *L is a weak relatively complemented ADL if and only if $\mathcal{N}^+I(L)$ is a relatively complemented ADL.*

Proof: Suppose that L is a weak relatively complemented ADL. Let $I, J \in \mathcal{N}^+I(L)$. Then there exist $a, b \in L$ such that $I = (a]^+$ and $J = (b]^+$. For this $a, b \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. By Lemma 4.3., $(a \vee x]^+ = (a \vee b]^+$. Therefore $I \cap (x]^+ = (a]^+ \cap (x]^+ = (a \wedge x]^+ = (0]^+$. Also $I \sqcup (x]^+ = (I \vee (x]^+)^+ = ((a]^+ \vee (x]^+)^+ = (a \vee x]^+$ and $I \sqcup J = (I \vee J)^+ = ((a]^+ \vee (b]^+)^+ = (a \vee b]^+$. Hence $I \cap K = (0]^+$ and $(I \sqcup K) = (I \sqcup J)$, where $K = (x]^+$. Thus $\mathcal{N}^+I(L)$ is a relatively complemented ADL. On the other hand, let $a, b \in L$, then there exists $(x]^+ \in \mathcal{N}^+I(L)$ such that $(a]^+ \cap (x]^+ = \{0\}$ and $(a]^+ \vee (x]^+ = (a]^+ \vee (b]^+$. Therefore $a \wedge x = 0$ and $(a \vee x]^+ = (a \vee b]^+$ and hence $(a \vee x)^* = (a \vee b)^*$. Thus L is weak relatively complemented.

Theorem 5.5 *Every α-ideal of a weak relatively complemented ADL is weak relatively complemented.*

Proof: Let L be a weak relatively complemented ADL and I is an α-ideal of L. Let $a, b \in I$. Then there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. By Lemma 4.3., $(a \vee x]^+ = (a \vee b]^+$. Therefore $a \vee x \in I$ (since $a \vee b \in I$ and I is an α-ideal). Hence $x \in I$. Thus I is weak relatively complemented.

Corollary 5.6 *Every principal α-ideal of a weak relatively complemented ADL is weak relatively complemented.*

Theorem 5.7 *In an ADL L, the following are equivalent;*

(i) L is weak relatively complemented ADL

(ii) For any non-zero element a in L, the interval $[0, a]$ is weak relatively complemented ADL.

Proof: Suppose that L is a weak relatively complemented ADL. Let $a \in L$ and $a \neq 0$. Let $x, y \in [0, a]$. Then there exists $t \in L$ such that $x \wedge t = 0$ and $(x \vee t)^* = (x \vee y)^*$. Therefore $x \wedge t \wedge a = 0$ and $((x \vee t) \wedge a]^+ = ((x \vee y) \wedge a]^+$. So that $((x \wedge a) \vee (t \wedge a)]^+ = ((x \wedge a) \vee (y \wedge a)]^+$. We get that $x \wedge (t \wedge a) = 0$ and

$(x \vee (t \wedge a)]^+ = (x \vee y]^+$, where $t \wedge a \in [0, a]$. Hence $[0, a]$ is a weak relatively complemented ADL. Conversely suppose that for any non-zero element a in L, the interval $[0, a]$ is weak relatively complemented ADL. Let $x, y \in L$. Then $x, y \in [0, x \vee y]$. Therefore there exists $t \in [0, x \vee y]$ such that $x \wedge t = 0$ and $(x \vee t)^* = (x \vee y)^*$. Hence L is a weak relatively complemented ADL.

Theorem 5.8 *$\mathcal{PI}(L)$ is a weak relatively complemented if and only if $\mathcal{N}^+I(L)$ is a relatively complemented.*

Proof: Suppose that $\mathcal{PI}(L)$ is a weak relatively complemented ADL. Let $I, J \in \mathcal{N}^+I(L)$. Then there exists $a, b \in L$ such that $I = (a]^+$ and $J = (b]^+$. For this $a, b \in L$, there exists $(x] \in \mathcal{PI}(L)$ such that $(a] \wedge (x] = (0]$ and $((a] \vee (x])^* = ((a] \vee (b])^*$ for some $x \in L$. Therefore $(a \wedge x]^+ = (0]^+$ and $(a \vee x]^+ = (a \vee b]^+$. So that $(a]^+ \cap (x]^+ = (0]^+$ and $(a]^+ \sqcup (x]^+ = (a]^+ \sqcup (b]^+$. Hence $I \cap K = (0]$ and $(I \sqcup K) = (I \sqcup J)$, where $K = (x]^+$. Thus $\mathcal{N}^+I(L)$ is a relatively complemented ADL. On the other hand, let $(a], (b] \in \mathcal{PI}(L)$ for some $a, b \in L$, then there exists $(x]^+ \in \mathcal{N}^+I(L)$ such that $(a]^+ \cap (x]^+ = (0]^+$ and $(a]^+ \vee (x]^+ = (a]^+ \vee (b]^+$. Therefore $a \wedge x = 0$ and $(a \vee x]^+ = (a \vee b]^+$. So that $(a \vee x)^* = (a \vee b)^*$ (by Lemma 4.3). Hence $(a] \cap (x] = 0$ and $((a] \vee (x])^* = ((a] \vee (b])^*$. Thus $\mathcal{PI}(L)$ is a weak relatively complemented.

From [5], An ideal I of L is said to be a dense complemented, if there exists an ideal J such that $I \wedge J = \{0\}$ and $I \vee J$ is an ideal generated by a dense element in L. Now, we have the following.

Theorem 5.9 *The following are equivalent;*

(i) *L is weak relatively complemented*

(ii) *$(\mathcal{N}^+\mathcal{I}(L), \cap, \sqcup, \{0\}, L)$ is a Boolean algebra*

(iii) *$(L/\varphi, \wedge, \vee, 0/\varphi, d/\varphi)$ is a Boolean algebra*

(iv) *Every principal ideal is dense complemented.*

Proof: $(i) \Longrightarrow (ii)$: Assume (i). Let $x \in L$ and d is a dense in L. Then there exists $y \in L$ such that $x \wedge y = 0$ and $(x \vee y)^* = (x \vee d)^* = \{0\}$. Therefore $(x \wedge y]^+ = (x]^+ \cap (y]^+ = \{0\}$ and $(x \vee y]^+ = (x]^+ \sqcup (y]^+ = L$. Thus $\mathcal{N}^+\mathcal{I}(L)$ is a Boolean algebra.
$(ii) \Longrightarrow (iii)$: Assume (ii). Let $x \in L$. Then there exists $(y]^+ \in \mathcal{N}^+\mathcal{I}(L)$ such that $(x]^+ \cap (y]^+ = (x \wedge y]^+ = \{0\}$ and $(x]^+ \sqcup (y]^+ = (x \vee y]^+ = L$. Therefore $x \wedge y = 0$ and $x \vee y$ is dense. So that $x/\varphi \wedge y/\varphi = (x \wedge y)/\varphi = \{0\} = 0/\varphi$ and $x/\varphi \vee y/\varphi = (x \vee y)/\varphi = D = d/\varphi$. Thus L/φ is a Boolean algebra.
$(iii) \Longrightarrow (iv)$: Assume (iii). Let $x \in L$. Then there exists $y \in L$ such that $x/\varphi \wedge y/\varphi = (x \wedge y)/\varphi = \{0\}$ and $x/\varphi \vee y/\varphi = (x \vee y)/\varphi = D$. Therefore

$x \wedge y = 0$ and $x \vee y$ is dense. So that $(x \wedge y] = (x] \wedge (y] = \{0\}$ and $(x \vee y] = (x] \vee (y]$ is an ideal generated by a dense element $x \vee y$. Thus every principal ideal is dense complemented.
$(iv) \Longrightarrow (i)$: Assume (iv). Let $a, b \in L$. Then there exist $c, d \in L$ such that $(a] \wedge (c] = \{0\} = (b] \wedge (d]$ and $(a] \vee (c]$, $(b] \vee (d]$ are principal ideals generated by dense elements. Thus $a \wedge c = 0 = b \wedge d$ and $a \vee c$ and $b \vee d$ are dense elements. Take $x = c \wedge b$. Then $a \wedge x = a \wedge c \wedge b = 0$ (since $a \wedge c = 0$) and $(a \vee x) \wedge (a \vee b) = a \vee (x \wedge b) = a \vee (c \wedge b \wedge b) = a \vee x$. So that $(a \vee b)^* \subseteq (a \vee x)^*$. Now, for $t \in L$,

$$\begin{aligned} t \in (a \vee x)^* &\Rightarrow t \wedge (a \vee x) = 0 \\ &\Rightarrow t \wedge a = 0 \text{ and } t \wedge c \wedge b = 0 \\ &\Rightarrow t \wedge b \wedge (a \vee c) = 0 \\ &\Rightarrow t \wedge b = 0 \qquad \text{(since } a \vee c \text{ is dense)} \\ &\Rightarrow t \wedge (a \vee b) = 0 \\ &\Rightarrow t \in (a \vee b)^*. \end{aligned}$$

Therefore $(a \vee x)^* \subseteq (a \vee b)^*$ and hence $(a \vee x)^* = (a \vee b)^*$. Thus L is weak relatively complemented.

Lemma 5.10 *If every dense element is maximal in L, then the following are equivalent;*

(i) *L is relatively complemented*

(ii) $(\mathcal{N}^+\mathcal{I}(L), \cap, \sqcup, \{0\}, L)$ *is a Boolean algebra*

(iii) $(L/\varphi, \wedge, \vee, 0/\varphi, d/\varphi)$ *is a Boolean algebra*

(iv) *Every principal ideal is complemented.*

6 Open Problem

1. If one can study the weak relative complementation on ideals (sub ADLs) in a weak relatively complemented ADLs, then it may leads to get fruitful results in the ideal theory of almost distributive lattices.

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