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On the weighted variable exponent spaces of differential forms

Lifeng Guo, Jing Wang and Jinzi Liu

School of Mathematics and Statistics, Northeast Petroleum University
Daqing 163318, China.
e-mail: lfguo1981@126.com

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Abstract

In this paper, we introduce the weighted variable exponent spaces of differential forms. After discussing the properties of these spaces.

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1 Introduction

After spaces of differential forms were introduced in [1,2], the study of A-harmonic equations for differential forms has been developed rapidly. Many interesting results concerning A-harmonic equations have been established recently (see [3-5] and the references therein). Differential forms has many important applications in many fields, such as general relativity (see [6]), theory of elasticity (see [7]), electromagnetism (see [8-9]), and differential geometry (see [10]) etc. Hence, differential forms have become invaluable tools for many fields. With the in-depth study of nonlinear problems in natural science and engineering, constant exponent spaces express their limitations in applications. For example, constant exponent spaces are not adequate in studies on nonlinear problems with variable exponential growth. This is a new research field and reflects physical phenomena of a new kind. After Kováčik and Rákosnik first discussed the $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ spaces in [11], a lot of research has been done concerning these kinds of variable exponent spaces (see [12-15] and the references therein). In recent years, the theory of problems with variable exponential growth conditions has important applications in nonlinear elastic mechanics (see [16]), electrorheological fluids (see [17]) and image processing (see [18]).

The paper is organized as follows. In Section 2, we will first introduce the weighted spaces of differential forms $L^{p(x)}(\Omega, \Lambda^l, \omega)$ and $W^{1,p(x)}(\Omega, \Lambda^l, \omega)$, which are the spaces $L^p(\Omega, \Lambda^l)$ and $W^p_d(\Omega, \Lambda^l)$ respectively (See [2]) when the variable exponent $p(x)$ reduced to a constant p $(1 < p < \infty)$ and weighted $\omega(x) \equiv 1$. In Section 3, we will prove our main results. We will always assume $p(x)$ satisfies

$$1 < p_* \le p(x) \le p^* < \infty \text{ for a.e. } x \in \Omega. \tag{1}$$

2 Preliminaries

Let $e_1, e_2, ..., e_n$ denote the standard orthogonal basis of $\mathbb{R}^n$. The space of all l-forms in $\mathbb{R}^n$ is denoted by $\Lambda^l(\mathbb{R}^n)$. The dual basis to $e_1, e_2, \cdots, e_n$ is denoted by $e^1, e^2, \cdots, e^n$ and referred to as the standard basis for 1-form $\Lambda^1(\mathbb{R}^n)$. The Grassman algebra $\Lambda(\mathbb{R}^n) = \oplus\Lambda^l(\mathbb{R}^n)$ is a graded algebra with respect to the exterior products. The standard ordered basis for $\Lambda(\mathbb{R}^n)$ consists of the forms

$$1, e^1, e^2, \cdots, e^n, e^1 \wedge e^2, \cdots, e^{n-1} \wedge e^n, \cdots, e^1 \wedge e^2 \cdots \wedge e^n.$$

For $u = \sum u_I e^I \in \Lambda^l(\mathbb{R}^n)$ and $v = \sum v_I e^I \in \Lambda^l(\mathbb{R}^n)$, the inner product is obtained by $\langle u, v\rangle = \sum u_I v_I$ with summation over all l–tuples $I = (i_1, \cdots i_l)$ and all integers $l = 0, 1, \cdots, n$. The Hodge star operator (see [20]) $\star : \Lambda(\mathbb{R}^n) \to \Lambda(\mathbb{R}^n)$ defined by the formulas:

$$\star 1 = e^1 \wedge\ e^2 \cdots \wedge\ e^n, \quad u \wedge \star v = v \wedge \star u = \langle u, v\rangle\ e^1 \wedge\ e^2 \cdots \wedge\ e^n.$$

Hence the norm of u is given by the formula $|u|^2 = \langle u, u\rangle = \star(u \wedge \star u) = \sum u_I^2 \in \Lambda^0(\mathbb{R}^n) = \mathbb{R}^n$. Notice, the Hodge star operator is an isometric isomorphism on $\Lambda(\mathbb{R}^n)$. Moreover

$$\star : \Lambda^l(\mathbb{R}^n) \to \Lambda^{n-l}(\mathbb{R}^n), \quad \star\star = (-\mathrm{I})^{l(n-l)} : \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$$

where I denotes the identity map.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The coordinate function $x_1, x_2, \cdots, x_n$ in $\Omega \subset \mathbb{R}^n$ are considered to be differential forms of degree 0. The 1-forms $dx_1, dx_2, ..., dx_n$ are constant functions from Ω into $\Lambda^1(\mathbb{R}^n)$. The value of dx_i is simply e^i, $i = 1, 2, \cdots, n$. Therefore every l–form $u : \Omega \to \Lambda^l(\mathbb{R}^n)$ may be written uniquely as

$$u(x) = \sum u_I(x) dx_I = \sum_{1 \le i_1 < \cdots < i_l \le n} u_{i_1, \cdots i_l}(x)\ dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

where the coefficients $u_{i_1,\cdots i_l}(x)$ are distributions from $\mathcal{D}'(\Omega)$. The exterior differential $d : \mathcal{D}'(\Omega, \Lambda^l) \to \mathcal{D}'(\Omega, \Lambda^{l+1})$ is expressed by

$$du(x) = \sum_{k=1}^{n} \sum_{1\le i_1<\cdots<i_l\le n} \frac{\partial u_{i_1,\cdots i_l}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l}.$$

The formal adjoint operator, called the Hodge codifferential, is given by

$$d^\star = (-\mathrm{I})^{nl-1} \star d\star : \mathcal{D}'(\Omega, \Lambda^{l+1}) \to \mathcal{D}'(\Omega, \Lambda^l).$$

$C^\infty(\Omega, \Lambda^l)$ denote the space of infinitely differentiable l-forms on Ω and $C_0^\infty(\Omega, \Lambda^l)$ denote the space $C^\infty(\Omega, \Lambda^l)$ with compact support on Ω.

Next we recall the following classes of differential forms with L^p-integrable coefficients.

$L^p(\Omega, \Lambda^l)$ is the space of differential l-forms $u(x)$ with coefficients in $L^p(\Omega)$, $1 \le p \le \infty$. The norm is given by

$$||u(x)||_{L^p(\Omega,\Lambda^l)} = \big(\int_\Omega |u(x)|^p dx\big)^{\frac{1}{p}}, \quad 1 \le p < \infty;$$

$$||u(x)||_{L^\infty(\Omega,\Lambda^l)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

$W^{1,p}(\Omega, \Lambda^l)$ is the space of differential l-forms $u(x) \in L^p(\Omega, \Lambda^l)$ such that $du \in L^p(\Omega, \Lambda^{l+1})$ with $l = 0, 1, \cdots, n-1$. For $W^{1,p}(\Omega, \Lambda^l)$ the norm is

$$||u(x)||_{W^{1,p}(\Omega,\Lambda^l)} = ||u(x)||_{L^p(\Omega,\Lambda^l)} + ||du(x)||_{L^p(\Omega,\Lambda^{l+1})}.$$

$W_0^{1,p}(\Omega, \Lambda^l)$ is the completion of $C_0^\infty(\Omega, \Lambda^l)$ in $W^{1,p}(\Omega, \Lambda^l)$ with respect to the norm $||u(x)||_{W^{1,p}(\Omega,\Lambda^l)}$.

For $u(x) = \sum u_I(x)dx_I \in L^p(\Omega, \Lambda^l)$ and $\varphi(x) = \sum \varphi_I(x)dx_I \in L^{p'}(\Omega, \Lambda^{n-l})$ we have the bilinear function

$$L(u, \varphi) = \int_\Omega u \wedge \varphi$$

satisfying $|L(u, \varphi)| \le ||u(x)||_{L^p(\Omega,\Lambda^l)}||\varphi(x)||_{L^{p'}(\Omega,\Lambda^{n-l})}$. For each form $\varphi(x) \in L^{p'}(\Omega, \Lambda^{n-l})$ it correspond to a functional L_φ on $L^{p'}(\Omega, \Lambda^{n-l})$ by setting

$$L_\varphi(u) = \int_\Omega u \wedge \varphi = \int_\Omega \sum u_I(x)\varphi_I(x)\mathrm{d}x.$$

$L^p(\Omega, \Lambda^l)$ and $W^{1,p}(\Omega, \Lambda^l)$ are two reflexive Banach spaces for $1 < p < \infty$. The correspondence $\varphi \to L_\varphi$ is an isometric isomorphism form the Banach space $L^{p'}(\Omega, \Lambda^{n-l})$ to the space $[L^p(\Omega, \Lambda^l)]'$, which is dual to $L^p(\Omega, \Lambda^{n-l})$, where p' is the conjugate number of p .

Finally we recall some basic properties of weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ and weighted variable exponent Sobolev space $W^{k,p(x)}(\Omega,\omega)$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain.

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p:\Omega\to[1,\infty]$. For $p\in\mathcal{P}(\Omega)$ we put $\Omega_1=\{x\in\Omega:p(x)=1\}$, $\Omega_\infty=\{x\in\Omega:p(x)=\infty\}$, $\Omega_0=\Omega\backslash(\Omega_1\cup\Omega_\infty)$, $p_*=\operatorname{essinf}_{\Omega_0}p(x)$ and $p^*=\operatorname{esssup}_{\Omega_0}p(x)$. We denote by ω the Radon measure canonically associated with the weight $\omega(x)$ in the following way:

$$\omega(E)=\int_E\omega(x)\mathrm{d}x=\int_E\mathrm{d}\omega.$$

The weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ is the class of all functions u such that $\int_{\Omega\backslash\Omega_\infty}|\lambda u(x)|^{p(x)}\mathrm{d}\omega+\operatorname{esssup}_{\Omega_\infty}|\lambda u(x)|<\infty$ for some $\lambda=\lambda(u)>0$, the space $L^{p(x)}(\Omega,\omega)$ is a reflexive Banach space equipped with the following norm

$$||u||_{L^{p(x)}(\Omega,\omega)}=\inf\big\{\lambda>0:\int_\Omega|\frac{u}{\lambda}|^{p(x)}\mathrm{d}\omega+\operatorname{esssup}_{\Omega_\infty}|\frac{u}{\lambda}|\le 1\big\}.$$

The weighted variable exponent Sobolev space $W^{k,\,p(x)}(\Omega,\omega)$ is the class of all functions $u\in L^{p(x)}(\Omega,\omega)$ such that $\delta_k u=\{D^\alpha u:|\alpha|\le k\}\subset L^{p(x)}(\Omega,\omega)$, the space $W^{k,\,p(x)}(\Omega,\omega)$ is a reflexive Banach space equipped with the following norm

$$||u||_{W^{k,\,p(x)}(\Omega,\omega)}=\sum_{|\alpha|\le k}||D^\alpha u||_{L^{p(x)}(\Omega,\omega)}.$$

For a differential l-form $u(x)$ on Ω we define the functional $\rho_{p(x)}$ by

$$\rho_{p(x),\Lambda^l}(u)=\int_{\Omega\backslash\Omega_\infty}|u(x)|^{p(x)}\mathrm{d}\omega+\operatorname{esssup}_{\Omega_\infty}|u(x)|.$$

Definition 2.1. *The weighted variable exponent Lebesgue spaces of differential l-forms* $L^{p(x)}(\Omega,\Lambda^l,\omega)$ *is the set of differential l-forms u such that* $\rho_{p(x),\Lambda^l}(\lambda u)<\infty$ *for some* $\lambda=\lambda(u)>0$ *and we endow it with the following norm:*

$$||u||_{L^{p(x)}(\Omega,\Lambda^l,\omega)}=\inf\{\lambda>0:\rho_{p(x),\Lambda^l}(\frac{u}{\lambda})\le 1\}.$$

Given $p\in\mathcal{P}(\Omega)$ we define the conjugate function $p\,'\in\mathcal{P}(\Omega)$ by

$$p\,'(x)=\begin{cases}\infty & \text{if}\, x\in\Omega_1,\\ 1 & \text{if}\, x\in\Omega_\infty,\\ \frac{p(x)}{p(x)-1} & \text{if}\, x\in\Omega_0.\end{cases}$$

Definition 2.2. *The weighted variable exponent Sobolev spaces of differential l-forms* $W^{1,p(x)}(\Omega,\Lambda^l,\omega)$ *is the space of differential l-forms* $u\in L^{p(x)}(\Omega,\Lambda^l,\omega)$

such that $du \in L^{p(x)}(\Omega, \Lambda^{l+1}, \omega)$ *with* $l = 0, 1, \cdots, n-1$. *For* $W^{1,p(x)}(\Omega, \Lambda^l, \omega)$ *the norm is defined as*

$$||u||_{W^{1,p(x)}(\Omega,\Lambda^l,\omega)} = ||u||_{L^{p(x)}(\Omega,\Lambda^l,\omega)} + ||du||_{L^{p(x)}(\Omega,\Lambda^{l+1},\omega)}.$$

$W_0^{1,p(x)}(\Omega, \Lambda^l, \omega)$ is the completion of $C_0^\infty(\Omega, \Lambda^l, \omega)$ in $W^{1,p(x)}(\Omega, \Lambda^l, \omega)$ with respect to the norm $||u||_{W^{1,p(x)}(\Omega,\Lambda^l,\omega)}$.

3 Main results

Theorem 3.1. *If* $p(x)$ *satisfies (1), then the inequality*

$$\int_\Omega \langle u(x), v(x)\rangle \mathrm{d}\omega \le C||u(x)||_{L^{p(x)}(\Omega,\Lambda^l,\omega)}||v(x)||_{L^{p'(x)}(\Omega,\Lambda^l,\omega)}$$

holds for every $u(x) \in L^{p(x)}(\Omega, \Lambda^l, \omega)$, $v(x) \in L^{p'(x)}(\Omega, \Lambda^l, \omega)$ *with the constant* C *dependent on* $p(x)$ *only.*

Since the proof of Theorem 3.1 is similar to the proof of Theorem 2.1 in [19], we omit it here.

Theorem 3.2. *If* $p(x)$ *satisfies (1), then the space* $L^{p(x)}(\Omega, \Lambda^l, \omega)$ *is complete.*

Proof. Let $\{u_n : u_n(x) = \Sigma u_{nI}(x)dx_I\}$ be a Cauchy sequence in $L^{p(x)}(\Omega, \Lambda^l, \omega)$. Then $\{u_{nI}(x)\}$ is a Cauchy sequence in $L^{p(x)}(\Omega, \omega)$ for any I. In view of the completeness of $L^{p(x)}(\Omega, \omega)$, $\{u_{nI}(x)\}$ converges in $L^{p(x)}(\Omega, \omega)$. Suppose that $u_{nI}(x) \to u_I(x)$ in $L^{p(x)}(\Omega, \omega)$. Now let $u(x) = \Sigma u_I(x)dx_I \in L^{p(x)}(\Omega, \Lambda^l, \omega)$, we obtain that $u_n(x) \to u(x)$ in $L^{p(x)}(\Omega, \Lambda^l, \omega)$. Now we complete the proof.

Theorem 3.3. *If* $p(x)$ *satisfies (1), then the space* $L^{p(x)}(\Omega, \Lambda^l, \omega)$ *is reflexive.*

Proof. We will show that the dual of $L^{p(x)}(\Omega, \Lambda^l, \omega)$ is $L^{p'(x)}(\Omega, \Lambda^{n-l}, \omega)$ in steps.

(1) For fixed $v(x) = \sum v_I(x)dx_I \in L^{p'(x)}(\Omega, \Lambda^{n-l}, \omega)$ we define a linear functional

$$L_v(u) = \int_\Omega \omega(x) \cdot (u \wedge v) = \int_\Omega \sum u_I(x)v_I(x)\mathrm{d}\omega. \tag{2}$$

where $u(x) = \sum u_I(x)dx_I \in L^{p(x)}(\Omega, \Lambda^l, \omega)$. By Theorem 2.1 we have

$$|L_v(u)| \le C||u(x)||_{L^{p(x)}(\Omega,\Lambda^l,\omega)}||v(x)||_{L^{p'(x)}(\Omega,\Lambda^{n-l},\omega)},$$

that is to say, $L_v(\cdot)$ is a bounded linear functional on $L^{p(x)}(\Omega, \Lambda^l, \omega)$, i.e. $L_v(\cdot)$ belongs to $[L^{p(x)}(\Omega, \Lambda^l, \omega)]'$.

(2) By Theorem 2.3 in [19], we know that each continuous linear functional $L' \in [L^{p(x)}(\Omega,\omega)]'$ can be represented uniquely in the form $L'(u_I) = \int_\Omega u_I(x) v_{L'}(x) d\omega$ for some $v_{L'} \in L^{p'(x)}(\Omega,\omega)$. So we conclude that each continuous linear functional $L \in [L^{p(x)}(\Omega,\Lambda^l,\omega)]'$ can be represented uniquely in the form (2).

(3) We shall show $||v(x)||_{L^{p'(x)}(\Omega,\Lambda^{n-l},\omega)} \le C||L_v||$ with the constant C depending only on $p(x)$. We take

$$u(x) = \frac{|v(x)|^{p'(x)-2}}{||v(x)||^{\frac{1}{p(x)-1}}_{L^{p'(x)}(\Omega,\Lambda^l,\omega)}}\big(\star\, v(x)\big),$$

then

$$||u(x)||_{L^{p(x)}(\Omega,\Lambda^l,\omega)} = \inf\{\lambda > 0 : \int_\Omega \Big(\frac{|v(x)|}{\lambda^{p(x)-1}||v(x)||_{L^{p'(x)}(\Omega,\Lambda^{n-l},\omega)}}\Big)^{p'(x)} d\omega \le 1\} = 1.$$

Moreover

$$\begin{aligned}
|L_v(u)| =& \Big|\int_\Omega \omega(x)\cdot(u\wedge v)\Big| \\
=& \int_\Omega \Big(\frac{|v|}{||v(x)||_{L^{p'(x)}(\Omega,\Lambda^l,\omega)}}\Big)^{p'(x)} ||v(x)||_{L^{p'(x)}(\Omega,\Lambda^{n-l},\omega)} d\omega \\
\ge& \frac{||v(x)||_{L^{p'(x)}(\Omega,\Lambda^l,\omega)}}{2^{\frac{p_*}{p_*-1}}} \int_\Omega \Big(\frac{|v|}{\frac{1}{2}||v(x)||_{L^{p'(x)}(\Omega,\Lambda^{n-l},\omega)}}\Big)^{p'(x)} d\omega \\
\ge& \frac{||v(x)||_{L^{p'(x)}(\Omega,\Lambda^l,\omega)}}{2^{\frac{p_*}{p_*-1}}}.
\end{aligned}$$

Therefore

$$||v(x)||_{L^{p'(x)}(\Omega,\Lambda^l,\omega)} \le 2^{\frac{p_*}{p_*-1}} ||L_v||.$$

Now we reach the conclusion $[L^{p(x)}(\Omega,\Lambda^l,\omega)]' = L^{p'(x)}(\Omega,\Lambda^{n-l},\omega)$ and furthermore $L^{p(x)}(\Omega,\Lambda^l,\omega)$ is reflexive.

By Theorem 3.2 and Theorem 3.3, we have

Corollary 3.2. *If $p(x)$ satisfies (1), then the space $L^{p(x)}(\Omega,\Lambda^l,\omega)$ is a reflexive Banach space.*

Theorem 3.4. *If $p(x)$ satisfies (1), then the space $W^{1,p(x)}(\Omega,\Lambda^l,\omega)$ is a reflexive Banach space.*

Proof. We treat $W^{1,p(x)}(\Omega,\Lambda^l,\omega)$ in a natural way as a subspace of the product space $L^{p(x)}(\Omega,\Lambda^l,\omega) \times L^{p(x)}(\Omega,\Lambda^{l+1},\omega)$. Then we need only to show that $W^{1,p(x)}(\Omega,\Lambda^l,\omega)$ is a closed subspace of $L^{p(x)}(\Omega,\Lambda^l,\omega) \times L^{p(x)}(\Omega,\Lambda^{l+1},\omega)$. Let $\{u_n : u_n(x) = \sum u_{nI}(x) dx_I\} \subset W^{1,p(x)}(\Omega,\Lambda^l,\omega)$ be a convergent sequence. Then $\{u_{nI}\}$ is a convergent sequence in $L^{p(x)}(\Omega,\omega)$. In view of Theorem 2.2

in [19], there exists $u_I \in L^{p(x)}(\Omega, \omega)$ such that $u_{nI} \to u_I$ in $L^{p(x)}(\Omega, \omega)$ for any I. Hence we conclude that $u_n \to u = \sum u_I(x)dx_I$ in $L^{p(x)}(\Omega, \Lambda^l, \omega)$. Similarly there exists $\widetilde{u} \in L^{p(x)}(\Omega, \Lambda^{l+1}, \omega)$ such that $du_n \to \widetilde{u}$ in $L^{p(x)}(\Omega, \Lambda^{l+1}, \omega)$. For any $\varphi \in C_0^\infty(\Omega, \Lambda^{l+1}) \subset L^\infty(\Omega, \Lambda^{l+1})$

$$\int_\Omega \langle du_n, \varphi\rangle \mathrm{d}x = \int_\Omega \langle u_n, d^\star\varphi\rangle \mathrm{d}x,$$

we have

$$\int_\Omega \langle \widetilde{u}, \varphi\rangle \mathrm{d}x = \int_\Omega \langle u, d^\star\varphi\rangle \mathrm{d}x.$$

by letting $n \to \infty$. Since $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$, we obtain that $du = \widetilde{u}$. Then it is immediate that $W^{1,p(x)}(\Omega, \Lambda^l, \omega)$ is a closed subspace of $L^{p(x)}(\Omega, \Lambda^l, \omega) \times L^{p(x)}(\Omega, \Lambda^{l+1}, \omega)$. Now we complete the proof of Theorem 3.4.

Let weighted $\omega(x) = 1$, we have

Corollary 3.2. *If $p(x)$ satisfies (1), then the spaces $L^{p(x)}(\Omega, \Lambda^l)$ and $W^{1,p(x)}(\Omega, \Lambda^l)$ are a reflexive Banach space.*

4 Open Problem

With regards to the problems solved, the this work can also be applied to other spaces. For example, can discuss the properties on the weighted variable exponent spaces of differential forms on Riemannian manifold.

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