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2^{nd} Meeting on Optimisation Modelization and Approximation

November 19-20-21, 2009, Casablanca, Morocco

Edited by M.N. Benbourhim
L. Ghannam
A. Hassouni
A. Ismail

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The Second international Meeting on Optimization Modelization and Approximation
(MOMA'09), November 19-21, 2009 - Casablanca, Morocco.

Editors

M.N. Benbourhim	Université Paul Sabatier, Toulouse, France.
L. Ghannam	Université Paul Sabatier, Toulouse, France.
A. Hassouni	Faculté des Sciences, Rabat, Morocco.
A. Ismail	Ecole Hassania des Travaux Publics, Casablanca, Morocco.

1 Introduction

This volume constitutes the proceedings of the second international Meeting on Optimization Modelization and Approximation (MOMA'09). The conference held at the Hassania School Public works (Ecole Hassania des Travaux Publics) in Casablanca, Morocco during November 19-20-21, 2009.

The scope of this meeting covers a range of major topics in Numerical Analysis, Optimization, also in Approximation and Engineering and related disciplines, ranging from theoretical developments to industrial applications and modeling of problems. It is intended that MOMA'2009 will provide a forum for moroccan and their foreigner colleagues, to discuss and exchange ideas, methods and results in contemporary topics in mathematics and engineering.

Our heartfelt thanks go to ours colleagues

- **A. BOUHAMIDI, LMPA, Calais, France.**
- **A. EL HAMI, INSA, Rouen, France.**
- **R. ELLAIA, Ecole Mohammadia d'Ingénieurs, Rabat, Morocco.**
- **A. LAHLOU, Faculté de Droit, Rabat, Morocco.**

who contributed to the organization and success of the meeting.

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- **INSA, Rouen, France.**
- **Université Paul Sabatier, Toulouse, France.**
- **IUT Paul Sabatier, Département Informatique, Toulouse, France.**
- **Centre international de Mathématiques Pures et Appliquées (CIMPA), France.**
- **Faculty of Science, King Faisal University, Kingdom of Saudi Arabia.**
- **Agence Universitaire de la Francophonie (AUF) France.**
- **AFA-SMAI, France.**

We would like to thank the authors for their scientific contributions and Pr. IQBAL. H. JEBRIL for his support. We also thank the referees for their very helpful reports and comments. Finally, we want to express our thanks to all attendees of the meeting, who made it pleasant and, we hope, useful.

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Optimization of the cost stock and transport in a multi-product one-level supply chain systems under a probabilistic demand

S. Achchab[1], K. Tikito and Y. Benadada

[1]Université Mohammed V - SOUISI, ENSIAS, Rabat, Morocco
e-mail: achchab@ensias.ma

Abstract

The awareness on the supply chain management has never been more essential, especially in such conjuncture as the economic crisis. The companies have started hunting waste, and optimizing their costs throughout all the process. Different approaches are available, but the centralized one is more interesting.
The cost of coordination of operations is generally difficult to quantify. Companies should test different options in terms of transport costs and gains and classify according on the complexity of coordination required. A manager can then make the decision appropriate: it must then consider the close relationship between the storage and transportation. Indeed, it is to find a satisfactory compromise between allowing stocks to savings in transport or stocks generating low losses in terms of transport relative to the quantities shipped. Under this thesis, we focus the problem of optimization of the combined costs stock and transport in a multi-product multi-level (multiple levels of storage). The studies on the subject are often limited to a level (a producer / supplier / retailer) and a deterministic demand. For this research, we assume that there are several items on each level of the chain and the demand for each of them is probabilistic.
At first step, we are looking for optimize the cost stock in a

multi-product one-level supply chain under a probabilistic demand. For that, we choose to model the problem under the Stochastic Model Predictive Control :

$$x_{t+1} = Ax_t + Bu_t + d_t$$

with:
x_t***: the state function at time*** t
u_t***: control at time*** t
d_t***: the disturbance at time*** t
The aim is to minimize the cost function under constraints on a Finite horizon, assuming that the control is quadratic.

1 Introduction

With over 2000 industrial installations, model predictive control (MPC) is currently the most widely implemented advanced process control technology for process plants (Qin and Badgwell, 1996). As is frequently the case, the idea of MPC appears to have been proposed long before MPC came to the forefront (Propoi, 1963; Rafal and Stevens, 1968; Nour-Eldin, 1971). Not unlike many technical inventions, MPC was first implemented in industry under various guises and names (see figure 1) long before a thorough understanding of its theoretical properties was available. Academic interest in MPC started growing in the mid eighties, particularly after two workshops organized by Shell (Prett and Morari, 1987; Prett et al., 1990). The understanding of MPC properties generated by pivotal academic investigations (Morari and Garcia, 1982; Rawlings and Muske, 1993) has now built a strong conceptual and practical framework for both practitioners and theoreticians. [3]

Company	Product name	Description
Aspen Tech	DMC	Dynamic Matrix Control
Adersa	IDCOM	Identification and Command
	HIECON	Hierarchical Constraint Control
	PFC	Predictive Functional Control
Honeywell Profimatics	RMPCT	Robust Model Predictive Control Technology
	PCT	Predictive Control Technology
Setpoint Inc.	SMCA	Setpoint Multivariable Control Architecture
	IDCOM-M	Multivariable
Treiber Controls	OPC	Optimum Predictive Control
Shell Global	SMOC-II	Shell Multivariable Optimizing Control
ABB	3dMPC	
Pavillion Technologies Inc.	PP	Process Perfecter
Simulation Sciences	Connoisseur	Control and Identification Package

Figure 1: Industrial Technology

Model Predictive Control (MPC) is recognized as the methodology of choice for optimizing closed loop performance in the presence of constraints and/or nonlinearity of multi-variable dynamical systems, and it is unique in providing computationally tractable optimal control laws by solving constrained receding horizon control problems online. Which why, MPC is widely used in (some) industries, typically for systems with slow dynamics (chemical process plant, supply chain, revenue management)

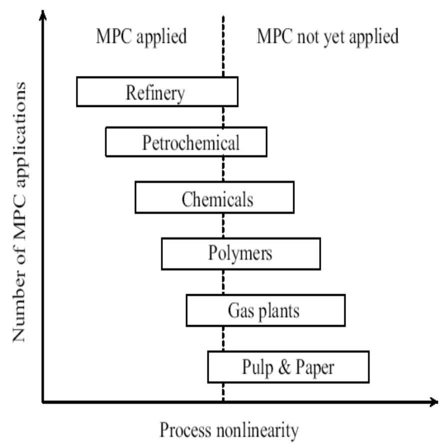

Figure 2: Industrial applications of model predictive control

The MPC is usually used in supply chain management, under constraints like buffer limits and shipping capacities limits, based on approximations which make the future values of disturbance exactly as predicted, thus no recourse is available in the future. However, most real life applications are not only subject to constraints but also involve multiplicative and/or additive stochastic uncertainty. Earlier work tended to ignore information on the distribution of model uncertainty, and as a result addressed control problems suboptimally using robust MPC strategies that employ only information on bounds on the uncertainty. Increasing demands for optimality in the presence of uncertainty motivate the development and application of MPC that takes explicit account of both omnipresent constraints and ubiquitous stochastic uncertainty [6] : which is the main idea of the Stochastic Model Predictive Control.

In this paper, we study the application of the Stochastic Model Predictive Control on a multi-product one-level Supply Chain to determine explicit control policies. Our assumptions are that the system works in discrete time under linear quadratic stochastic control.

2 Optimal policy by Stochastic Model Predictive Control

In general, we consider in a Stochastic Model Predictive Control, the linear dynamical system, over finite time horizon as:

$$x_{t+1} = Ax_t + Bu_t + d_t \qquad\qquad t = 0, \ldots, T-1 \tag{1}$$

with :
$x_t \in \mathbb{R}_n$ is the state of the system at time t,
$u_t \in \mathbb{R}_m$ is the input at time t (the control)
d_t is the process noise (or exogenous input) at time t

In our case Supply chain management with stochastic demand- we consider:

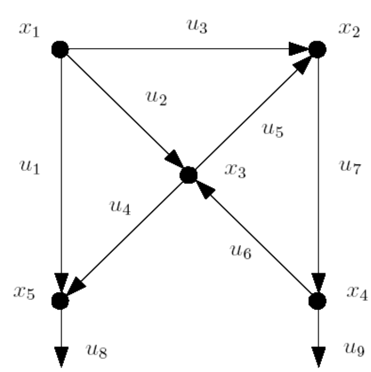

Figure 3: Example of a one-level Supply Chain

n nodes (warehouses/buffers); in the example at figure 3 $n = 5$
m unidirectional links between nodes, external world; in the example $m = 9$
$x_i(t)$ is amount of commodity at node i, in period t
$u_j(t)$ is amount of commodity transported along link j
$d_i(t)$ is amount of commodity demanded at node i, in period t

We express the incoming and outgoing node incidence matrices on this form:

$$B_{ij}^{in(out)} = \begin{cases} 1 & ;\text{ link } j \text{ enters (exits) node } i \\ 0 & ;\text{ otherwise} \end{cases} \tag{2}$$

Thus, the dynamics for this system could be expressed like:

$$x(t+1) = x(t) + B^{in}u(t) - B^{out}u(t) + d(t) \qquad\qquad t = 0, \ldots, T-1 \tag{3}$$

Let be X_t the state history up to time t, $X_t = (x_0, ..., x_t)$
Then the expression of the causal state-feedback control:

$$\begin{aligned} u_t &= \Phi_t(X_t) \\ &= \psi_t(x_0, d_0, \ldots, d_{t-1}) \qquad t = 0, \ldots, T-1 \end{aligned}$$

The function $\Phi_t : \mathbb{R}^{(t+1)n} \rightarrow \mathbb{R}^m$ is the control policy at time t. It gives the policy to use at the time t to reach to the next state of the system: x_{t+1}
Under these considerations , let now express **the objective function**:

$$J = \mathbb{E} \sum_{t=0}^{T-1} l_t(x_t, u_t) + l_T(x_T) \tag{4}$$

With:
$l_t : \mathbb{R}_n \mathrm{x} \mathbb{R}_m \rightarrow \mathbb{R}$, $t = 0, \ldots, T-1$: is the convex stage cost functions at time t
$l_T : \mathbb{R}_n \rightarrow \mathbb{R}$: convex terminal cost function. It's normal that this function depend only on the final state function, because at time T it will be no control.

For the constraints: $u_t \in \mathbb{U}_t$, $t = 0, \ldots, T-1$, we consider the convex input constraint sets $\mathbb{U}_0, \ldots, \mathbb{U}_{T-1}$

Thus the **stochastic control problem is to choose control policies Φ_0, . . . , Φ_{T-1} to minimize J, subject to the constraints**

In our case, we explicit the stage cost functions like:

$$l_t(x_t, u_t) = S(u(t)) + W(x(t)) \tag{5}$$

$S(u(t))$: Shipping/transportation cost (can also include sales revenue or manufacturing cost); wick depends of $u(t)$ the amount of commodity transported
$W(x(t))$: Warehousing/storage cost ; depending on the $x(t)$ amount of commodity at warehouses

Regarding the constraints:
The buffer limits: $0 \leq x_i(t) \leq x_{max}$ (could allow $x_i(t) < 0$, to represent backorder)
The link capacities:$0 \leq u_i(t) \leq u_{max}$
And $A^{out}u(t) \leq x(t)$ (can not ship out what is not on hand)

We consider the problem as linear quadratic stochastic control: And we assume: $\mathbb{U}_t = \mathbb{R}_m$
$x_0, d_0, \ldots, d_{T-1}$ are independent, with (to simplify the expressions)

$Ex_0 = 0,\ Ed_t = 0,\ Ex_0x_0^T = \Sigma,\ Ed_td_t^T = D_t$

Then, we express the cost functions in their convex quadratic form :

$$l_t(x_t, u_t) = x_t^T Q_t x_t + u_t^T R_t u_t, \ \ with Q_t \geq 0, R_t > 0 \tag{6}$$
$$l_T(x_T) = x_T^T Q_T x_T, \ \ with Q_T \geq 0 \tag{7}$$

Let $V_t(x_t)$ be the optimal value of our objective (quadratic):

$$V_t(x_t) = x_t^T P_t x_t + q_t; \qquad\qquad t = 0, \ldots, T \tag{8}$$

Using Bellman recursion:$P_T = Q_T, q_T = 0$; for $t = T-1, \ldots, 0$, we get

$$V_t(z) = \inf_v z^T Q_t z + v^T R_t v + \mathbf{E}((Az + Bv + d_t)^T P_{t+1}(Az + Bv + d_t) + q_{t+1}) \tag{9}$$

And it works out to [7]

$$P_t = A^T P_{t+1} A - A^T P_{t+1} B(B^T P_{t+1} B + R_t)^{-1} B^T P_{t+1} A + Q_t \tag{10}$$

and

$$q_t = q_{t+1} + \mathbf{Tr}(D_t P_{t+1}) \tag{11}$$

which define all the variables to express $V_t(x_t)$
And for the optimal policy, we found it as a linear state feedback:

$$\Phi_t^* = (x_t) = K_t x_t \tag{12}$$

with :

$$K_t = -(B^T P_{t+1} B + R_t)^{-1} B^T P_{t+1} A \tag{13}$$

Finally, the expression of the optimal cost is :

$$J^* = \mathbf{E} V_0(x_0) \tag{14}$$
$$= \mathbf{Tr}(\Sigma P_0) + q_0 \tag{15}$$
$$= \mathbf{Tr}(\Sigma P_0) + \Sigma_{t=0}^{T-1} \mathbf{Tr}(D_t P_{t+1}) \tag{16}$$

3 Conclusion

We proposed the Stochastic Model Predictive Control to optimize the costs of storage and transportation in a multi-product, one-level supply chain under uncertainty of the demand. This model is very sufficient in cases of control with constraints and multi-variable dynamical systems, which could be interesting on dealing with multi-level supply chain, as the constraints will be more, to take into account the relation between every product on each level. We saw

how the aim of the stochastic control problem is to choose the control policies to minimize the objective function J, subject to the constraints of shippement and storage limits. We also consider, the convex quadratic form to express the cost functions.
The non-linearity of the problem will be expressed when the levels of the supply chain will become multiples, which need more work on it.

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Revisiting quasi-convexity

Marc ATTEIA

IREM Université Paul Sabatier Bt 1R2.
118, route de Narbonne. 31062 Toulouse Cedex 9 France.
e-mail: marcatteia@orange.fr

Abstract

Following C.B. Morrey we say that a function $f : \mathbb{R}^{mn} \longrightarrow \mathbb{R} \cup \{+\infty\}$ ***is quasi-convex if***

$$\int_\Omega f\left(A + D\varphi(x)\right) dx \geq \int_\Omega f\left(A\right) dx = (meas(\Omega)) f\left(A\right)$$

for any bounded open subset Ω ***of*** $\mathbb{R}^n$***, for any*** $A \in \mathbb{R}^{mn}$ ***and for any*** $\varphi \in W_0^{1,k}\left(\Omega, \mathbb{R}^m\right)$***. In this paper, we study some properties of the convex cone of the quasi-convex functions. Especially we give definition and properties of reasonable tensorial products of two quasi-convex functions.***

Keywords: *Quasi-convexity, Young measure, Tensorial products.*

1 Quasi-convexity, convexity, multiconvexity and Legendre-Hadamard (L.-H.) condition.

1.1 Quasi-convexity.

Notations : Let $n, p \in \mathbb{N}^*$.

If $\mu \in \mathbb{R}^{np}$, we set: $\mu = \begin{pmatrix} \mu_1^1 & \dots & \mu_n^1 \\ \vdots & & \vdots \\ \mu_1^p & \dots & \mu_n^p \end{pmatrix}$

Let Ω be a bounded open subset of $\mathbb{R}^n$. For $\varphi = (\varphi^1, ..., \varphi^p) \in (D(\Omega))^p$, we set:

$$\forall t = (t_1, ..., t_n) \in \Omega \subset \mathbb{R}^n, \quad D\varphi(t) = \begin{pmatrix} D_1\varphi^1(t) & \dots & D_n\varphi^1(t) \\ \vdots & & \vdots \\ D_1\varphi^p(t) & \dots & D_n\varphi^p(t) \end{pmatrix},$$

where

$$\forall j \in (1, ..., n), \forall k \in (1, ..., p) \ , \ D_j\varphi^k(t) = \frac{\partial \varphi^k(t_1, ...t_n)}{\partial t_j}.$$

Definition 1.1 *Let $n, p \in \mathbb{N}^*$ and $f \in \mathcal{C}^0(\mathbb{R}^{np})$. We will say that f is a quasi-convex function, if :*
for any (regular) bounded open subset Ω of $\mathbb{R}^n$, for any $\varphi \in (D(\Omega))^p$ and for any $\mu \in \mathbb{R}^{np}$,

$$\int_\Omega f(\mu + D\varphi(t))dt - \int_\Omega f(\mu)dt \geq 0. \tag{1}$$

It can be easily proved that (1) is equivalent to

$$\forall \varepsilon \in \mathbb{R}^*_+, \int_\Omega f(\mu + \varepsilon D\varphi(t))dt - \int_\Omega f(\mu)dt \geq 0. \tag{2}$$

Now, let us suppose that $f \in \mathcal{C}^3(\mathbb{R}^{np})$ and that Ω is a "regular" open set. Then:
$\forall \varepsilon \in \mathbb{R}^*_+$, $\forall \varphi \in (\mathcal{D}(\Omega))^p$, $\forall \mu \in \mathbb{R}^{np}$ and $\forall t \in \Omega$, we have

$$\begin{aligned} f(\mu + \varepsilon D\varphi(t)) - f(\mu) &= \varepsilon \left[\sum_{j,l} \frac{\partial f(\mu)}{\partial \mu^l_j} D_j\varphi^l(t) \right] \\ &+ \frac{\varepsilon^2}{2!} \left[\sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu^l_j \ \partial \mu^m_k} D_j\varphi^l(t) D_k\varphi^m(t) \right] + \varepsilon^3 \varrho_3(\varphi, \mu, t) \end{aligned}$$

As $\varphi^l \in \mathcal{D}(\Omega)$, we get:

$$\int_\Omega \sum_{j,l} \frac{\partial f(\mu)}{\partial \mu^l_j} D_j\varphi^l(t) \ dt = 0.$$

Let us denote by $\widetilde{\varphi^l}$ the extension of φ^l from Ω to the whole space $\mathbb{R}^n$ obtained by setting $\widetilde{\varphi^l}(t) = 0$ if $t \notin \Omega$. Then $\widetilde{\varphi^l} \in \mathcal{S}(\mathbb{R}^n)$.
For any $\theta \in \mathcal{S}(\mathbb{R}^n)$, we denote by $\widehat{\theta}$ its Fourier transform. We can prove easily that:

$$\int_\Omega \left[\sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu^l_j \ \partial \mu^m_k} D_j\varphi^l(t) D_k\varphi^m(t) \right] dt =$$

$$\int_{\mathbb{R}^n} \left[\sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu^l_j \ \partial \mu^m_k} (2i\pi\zeta_j) \overline{(2i\pi\zeta_k)} \ \widehat{\widetilde{\varphi^l}(\zeta)} \overline{\widehat{\widetilde{\varphi^m}(\zeta)}} \right] d\zeta.$$

Let us suppose that f is quasi-convex. Then $\forall \varphi \in (\mathcal{D}(\Omega))^p$, $\forall \mu \in \mathbb{R}^{np}$, $\forall \varepsilon \in \mathbb{R}_+^*$,

$$\int_\Omega \varepsilon^{-2}(f(\mu + \varepsilon D\varphi(t)) - f(\mu))dt \geq 0$$

So, we deduce that : $\forall \varphi \in (\mathcal{D}(\Omega))^p, \forall \mu \in \mathbb{R}^{np}, \forall \varepsilon \in \mathbb{R}_+^*$,

$$\int_\Omega \left[\sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu_j^l \, \partial \mu_k^m} D_j \varphi^l(t) D_k \varphi^m(t) \right] \, dt + \varepsilon \int_\Omega \varrho_3(\varphi, \mu, t) \, dt \geq 0$$

Let :

$$F(\varphi, \mu) = \int_\Omega \left[\sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu_j^l \, \partial \mu_k^m} D_j \varphi^l(t) D_k \varphi^m(t) \right]$$

Let us suppose that there exist $\underline{\varphi} \in (\mathcal{D}(\Omega))^p$ and $\underline{\mu} \in \mathbb{R}^{np}$ such that: $F(\underline{\varphi}, \underline{\mu}) < 0$. Then there exists $\underline{\varepsilon} \in \mathbb{R}_+^*$ such that $F(\underline{\varphi}, \underline{\mu}) + \underline{\varepsilon} \int_\Omega \varrho_3(\underline{\varphi}, \underline{\mu}, t) \, dt < 0$ which is impossible.
Therefore, we deduce that $\forall \varphi \in (\mathcal{D}(\Omega))^p$ and $\forall \mu \in \mathbb{R}^{np}$, $F(\varphi, \mu) \geqslant 0$ and then :

$$\forall \varphi \in (\mathcal{D}(\Omega))^p \text{ and } \forall \mu \in \mathbb{R}^{np},$$

$$\int_{\mathbb{R}^n} \left[\sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu_j^l \, \partial \mu_k^m} (2i\pi\zeta_j)\overline{(2i\pi\zeta_k)} \widehat{\varphi^l(\zeta)} \overline{\widehat{\varphi^m(\zeta)}} \right] d\zeta \geqslant 0.$$

Consequently

$$\forall \psi^k \in \mathcal{S}(\mathbb{R}^n) \ , \quad \forall \mu \in \mathbb{R}^{np} \ , \quad \int_{\mathbb{R}^n} \left[\sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu_j^l \, \partial \mu_k^m} \zeta_j \ \zeta_k \ \psi^l(\varsigma) \ \overline{\psi^m(\varsigma)} \right] d\zeta \geqslant 0$$

Proposition 1.1 *If $p = 1$ or $n = 1$, then any quasi-convex function is convex.*

Proof If $p = 1$, then $\varphi^l = \varphi$ and

$$\forall \varphi \in (\mathcal{D}(\Omega)) \quad \text{and} \quad \forall \mu \in \mathbb{R}^n \ , \quad \int_\Omega \left[\sum_{j,k} \frac{\partial^2 f(\mu)}{\partial \mu_j \, \partial \mu_k} D_j \varphi(t) \ D_k \varphi(t) \right] \, dt \geq 0,$$

which is equivalent to

$$\forall \psi \in \mathcal{S}(\mathbb{R}^n) \ , \quad \forall \mu \in \mathbb{R}^n \ , \quad \int_{\mathbb{R}^n} \left[\sum_{j,k} \frac{\partial^2 f(\mu)}{\partial \mu_j \, \partial \mu_k} \zeta_j \ \zeta_k \ |\psi(\varsigma)|^2 \right] \, d\zeta \geqslant 0.$$

So, we deduce that:

$$\forall \zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{R}^n \ , \ \ \sum_{j,k} \frac{\partial^2 f(\mu)}{\partial \mu_j \ \partial \mu_k} \ \zeta_j \ \zeta_k \ \geqslant 0$$

As $f \in \mathcal{C}^3(\mathbb{R}^n)$, $D^2 f$ is a positive semi-definite matrix and f is convex on $\mathbb{R}^n$.
If $n = 1$, then $D_j = D$ and

$$\forall \varphi \in (\mathcal{D}(\Omega))^p \quad et \ \ \forall \mu \in \mathbb{R}^p, \qquad \int_\Omega \left[\sum_{l,m} \frac{\partial^2 f(\mu)}{\partial \mu^l \ \partial \mu^m} D\varphi^l(t) D\varphi^m(t) \right] \ dt \geqslant 0.$$

Which is equivalent to

$$\forall \psi = (\psi^1, \ldots, \psi^p) \in (\mathcal{S}(\mathbb{R}^n))^p \ , \quad \forall \mu \in \mathbb{R}^p \ , \ \ \int_{\mathbb{R}^n} \left[\sum_{l,m} \frac{\partial^2 f(\mu)}{\partial \mu^l \ \partial \mu^m} \psi^l(\varsigma) \overline{\psi^m(\varsigma)} \left|\varsigma\right|^2 \right] d\zeta \geqslant 0.$$

Using a convenient ψ, we conclude that f is convex, as done above. ■

Proposition 1.2 *Any quasi-convex function is multiconvex.*

Proof (With above hypotheses and notations).
let us suppose that f is quasi-convex and that $\forall m$, $\psi^m = \psi^1$.
So, it can be proved easily that

$$\forall \zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{R}^n \ , \ \ \sum_{j,k} \frac{\partial^2 f(\mu)}{\partial \mu_j^1 \ \partial \mu_k^1} \ \zeta_j \ \zeta_k \ \geqslant 0.$$

We deduce that f is convex for its first variable. By iteration we can prove the above property for others variables. ■

Proposition 1.3 *A quasi-convex function is of Legendre-Hadamard (L-H) type.*

Proof (With above hypotheses and notations), let us suppose that :
$\forall l \in \{1, ..., p\}$, $\forall \zeta \in \mathbb{R}^n$, $\psi^l(\zeta) = \rho^l \varphi(\zeta)$ where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\rho^l \in \mathbb{R}$. Then

$$\int_{\mathbb{R}^n} \left[\sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu_j^l \ \partial \mu_k^m} \zeta_j \ \zeta_k \ \varrho^l \varrho^m \ \ |\varphi(\zeta)|^2 \right] d\zeta \geqslant 0$$

We deduce (as $f \in \mathcal{C}^3(\mathbb{R}^{np})$) that

$$\forall \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n \ , \ \ \forall \varrho = (\varrho^1, \ldots, \varrho^p) \in \mathbb{R}^p \ , \qquad \sum_{j,k,l,m} \frac{\partial^2 f(\mu)}{\partial \mu_j^l \ \partial \mu_k^m} \zeta_j \ \zeta_k \ \varrho^l \varrho^m \geqslant 0.$$

So, f is of L-H type. ■

2 Quasi-convexity and Young measure

2.1 Definition

Let U (resp. V) be an open subset of a Banach space $(E,\|\cdot\|)$ (resp. $(F, [\cdot])$.
Let φ be a mapping from V into U , $\varphi \in U^V$ and f a mapping from U into a vectorial space G.
Set $\varphi^* = f \circ \varphi$: $V \to U \to G$
As:

$$\begin{aligned} \forall f_1 , \ f_2 , \ f &\in G^U , \forall \lambda \in \mathbb{R} , \\ \varphi^*(f_1 + f_2) &= (f_1 + f_2) \circ \varphi = (f_1 + f_2)(\varphi) = (f_1)(\varphi) + (f_2)(\varphi) = \varphi^*(f_1) + \varphi^*(f_2) \\ \text{and } \varphi^*(\lambda f) &= (\lambda f) \circ \varphi = (\lambda f)(\varphi) = \lambda f(\varphi) = \lambda \ \varphi^*(f) \end{aligned}$$

φ^* is a linear mapping from G^U into G^V.

2.2 Duality

We set: $\mathbf{E}= G^U$ and $\mathbf{F}=G^V$; $\mathbf{E}$ and $\mathbf{F}$ are linear spaces.
Let us denote by: $\mathbf{E}^\natural$ (resp. $\mathbf{F}^\natural$) the algebraïc dual of $\mathbf{E}$ (resp. $\mathbf{F}$) and $\langle \cdot, \cdot \rangle$ (resp. $(\cdot, \cdot)$) the duality bracket between $\mathbf{E}$ and $\mathbf{E}^\natural$ (resp. $\mathbf{F}$ and $\mathbf{F}^\natural$). Then :

$$\forall \omega^\natural \in \mathbf{F}^\natural , \quad \left(f \circ \varphi, \omega^\natural\right) = \left(\varphi^* \circ f, \omega^\natural\right) = \left\langle f,^t \varphi^*(\omega^\natural)\right\rangle ,$$

where ${}^t\varphi^*(\omega^\natural) \in \mathbf{E}^\natural$ (as $\varphi^\natural$ is a linear mapping from $\mathbf{E}$ into $\mathbf{F}$).

2.3 Examples

Example 2.1 Young measure
Let μ be a positive measure onto V .
The mapping which associates to any $\psi \in \mathbf{F}$ the scalar $\int_{\mathrm{V}} \psi(y) \ d\mu(y)$ is linear.
Hence, we can set: $\int_{\mathrm{V}} \psi(y) \ d\mu(y) = (\psi, \mu)$.
With the same notations as above, we have:

$$(f \circ \varphi, \mu) = \int_{\mathrm{V}} (f \circ \varphi)(y) \ d\mu(y) = \int_{\mathrm{U}} f(x) \ d\nu(x; \varphi) = \left\langle f,^t \varphi^*(\mu)\right\rangle ,$$

where ν is such that, for any measurable space B, $\nu(\mathrm{B}, \mu) = \mu \ (\overset{-1}{\varphi}(\mathrm{B}))$.

Example 2.2 Quasi-convex functions
Let $n, p, k \in \mathbb{N}^*$, Ω be a bounded open subset of $\mathbb{R}^n$ and f a mapping from $\mathbb{R}^{np}$ into $\mathbb{R}$.
If $u \in (W_0^{1,k}(\Omega))^p$, then: $Du \in (L^k(\Omega))^{np}$ with: $Du = (D_j u)_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant p}$.
If $\theta \in \mathrm{D}(\Omega)$ and $\omega \in \Omega$, we denote by φ_θ the mapping which, at any $t \in \Omega$ associates $\omega + D\theta(t) \in \mathbb{R}^{np}$.

(i) *Let us suppose that* $f \in \mathrm{C}^0(\mathbb{R}^{np})$. *Then:*

$$\forall t \in \Omega,\ f(\varphi_\theta(t)) = f(\omega + D\theta(t)) = \left\langle f, \delta_{\omega+D\theta(t)} \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ *is the duality bracket between* $\mathrm{C}^0(\mathbb{R}^{np})$ *and its topological dual,* δ_α *is the Dirac measure at the point* $\alpha \in \mathbb{R}^{np}$.
Hence, we can identify ${}^t\varphi^*$ *and* $\delta_{\omega+D\theta(t)}$ *and we can write:*

$$\begin{aligned} \int_\Omega f(\omega + D\theta(t))\ d\mu(t) &= (f \circ \varphi, \mu) = (\varphi^* \circ f, \mu) \\ &= \left\langle f, {}^t\varphi^*(\mu) \right\rangle = \int_{\mathbb{R}^{np}} f(x)\ d\nu(x; \varphi_\theta) \end{aligned}$$

(ii) *Let* $\mathrm{Hilb}\left(\mathcal{C}^0\left(\mathbb{R}^{np}\right)\right)$ *be the set of all hilbertian subspaces of* $\mathcal{C}^0\left(\mathbb{R}^{np}\right)$, $(\boldsymbol{H}, \langle \cdot, \cdot \rangle) \in \mathrm{Hilb}\left(\mathcal{C}^0\left(\mathbb{R}^{np}\right)\right)$ *and* H *the hilbertian kernel of* $(\boldsymbol{H}, \langle \cdot, \cdot \rangle)$. *Then :*

$$\forall t \in \Omega, \quad f(\varphi_\theta(t)) = f(\omega + D\theta(t)) = \langle f \mid \mathrm{H}(\cdot, \omega + D\theta(t)) \rangle .$$

So, we can identify ${}^t\varphi^*$ *and* $\mathrm{H}(\cdot, \omega + D\theta(t))$.

3 Quasi-convexity and duality

3.1 Duality.

3.1.1 Definition

Let Ω be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}^*$.
Let us denote by $(\cdot \mid \cdot)$ the canonical euclidean scalar product of $\mathbb{R}^n$.
Let $\theta \in \mathcal{C}^1(\Omega)$ and the function θ^*_Ω such that:

$$\theta^*_\Omega(t^*) = (t \mid D\ \theta(t)) - \theta(t) \qquad \text{where} \qquad t^* = D\ \theta(t), \quad t \in \Omega.$$

We shall say that θ^*_Ω is dual of θ relatively to Ω.
Below, we shall write θ^* instead of θ^*_Ω .

Remark 3.1 **(i)** *We point out that* θ^* *is parametrically defined. More explicitly, we have:*

$$\begin{aligned} t^* &= (t_1^*, \ldots, t_n^*) \ , \ t_j^* = D_j\ \theta(t) \text{ where } D_j\ \theta = \frac{\partial \theta}{\partial t_j}\ j = 1, \ldots, n \\ \theta^*(t^*) &= \sum_{j=1}^n t_j . D_j\ \theta(t) - \theta(t),\ t = (t_1, \ldots, t_n) \in \Omega. \end{aligned}$$

(ii) *In the definition of θ^*, we could use, in place of the scalar product $(\cdot \mid \cdot)$, any duality bracket between $\mathbb{R}^n$ and its algebraïc dual.*
Now, let us suppose that: $\theta \in \mathcal{C}^2(\Omega)$.
We set:

$$D_{jk}(\theta) = \frac{\partial^2 \theta}{\partial t_j \; \partial t_k} \quad \textit{and} \quad D^2 = (D_{jk}).$$

So :
$dt^* = (D^2\theta(t))dt$, *or more explicitly :*

$$\begin{bmatrix} dt_1^* \\ \vdots \\ dt_n^* \end{bmatrix} = \begin{bmatrix} D_{11}\theta(t) & \dots & D_{1n}\theta(t) \\ \vdots & & \vdots \\ D_{n1}\theta(t) & \dots & D_{nn}\theta(t) \end{bmatrix} . \begin{bmatrix} dt_1 \\ \vdots \\ dt_n \end{bmatrix}$$

and :

$$\begin{array}{rcl} (dt_1^* \wedge \ldots \wedge dt_n^*) & = & \det\ ((D^2\theta(t))(dt_1 \wedge \ldots \wedge dt_n) \\ (dt_1^* \ldots dt_n^*) & = & \left|\det\ ((D^2\theta(t))\right| (dt_1 \ldots dt_n). \end{array}$$

3.1.2 When $D\theta$ is a diffeomorphism

Let us suppose that: $\theta \in \mathcal{C}^2(\Omega)$ and that the mapping from Ω into $\mathbb{R}^n$ which associates to any $t \in \Omega$, $D\theta(t)$ be a $\mathcal{C}^1$-*diffeomorphism* which maps the bounded open subset Ω of $\mathbb{R}^n$ onto the open subset Ω^* of $\mathbb{R}^n$.
Then: $\forall t \in \Omega$, $D^2\theta(t)$ is a square non singular matrix. But:

$$\forall j \in \{1, \ldots, n\} \ , \ \frac{\partial}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial}{\partial t_k^*} . \frac{\partial t_k^*}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial}{\partial t_k^*} . D_{kj}\theta$$

It can be proved easily that:

$$\frac{\partial}{\partial t} = \begin{pmatrix} \frac{\partial}{\partial t_1} \\ \vdots \\ \frac{\partial}{\partial t_{nj}} \end{pmatrix} = {}^t\left(D^2\theta(t)\right) \begin{pmatrix} \frac{\partial}{\partial t_1^*} \\ \vdots \\ \frac{\partial}{\partial t_{nj}^*} \end{pmatrix} = {}^t\left(D^2\theta(t)\right) \frac{\partial}{\partial t^*}$$

So,

$$\begin{array}{rcl} \dfrac{\partial \theta^*(t^*)}{\partial t^*} & = & \left[{}^t\left(D^2\theta(t)\right)\right]^{-1} \dfrac{\partial \theta^*(t^*)}{\partial t} \\ & = & \left[{}^t\left(D^2\theta(t)\right)\right]^{-1} \begin{bmatrix} D_1\ \theta(t) + (\sum_{j=1}^n t_j . D_{j1}\ \theta(t)) - D_1\ \theta(t) \\ \vdots \\ D_n\ \theta(t) + (\sum_{j=1}^n t_j . D_{jn}\ \theta(t)) - D_n\ \theta(t) \end{bmatrix} \\ & = & \left[{}^t\left(D^2\theta(t)\right)\right]^{-1} \left[{}^t\left(D^2\theta(t)\right)\right] t \end{array}$$

Then: $t_j = D_j^*\theta^*(t^*)$ where $D_j^* = \frac{\partial}{\partial t_j^*}$, $j \in \{1,\ldots,n\}$. We deduce, with straightforward notations, that:

$$\forall\, j \ \in \ \{1,\ldots,n\}\ ,\ \ dt_j = \sum_{k=1}^{n} .\left[D_{kj}^*\theta^*(\theta^*(t^*)\right]\ dt_k^* \qquad \text{hence :}$$

$$\begin{aligned}(dt_1\wedge\ldots\wedge dt_n) &= \left(\sum_{k=1}^{n}.\left[D_{1k}^*\ \theta^*(t^*)\right]\ dt_k^*\right)\wedge\ldots\wedge\left(\sum_{k=1}^{n}.\left[D_{nk}^*\ (\theta^*(t^*)\right]\ dt_k^*\right)\\ &= \det\ \left[\ {}^t\left(D^{*2}\theta^*(t^*)\right)\right](dt_1^*\wedge\ldots\wedge dt_n^*)\\ &= \det\ \left[\ \left(D^{*2}\theta^*(t^*)\right)\right](dt_1^*\wedge\ldots\wedge dt_n^*)\end{aligned}$$

And we deduce that: $\det\ [\ D^{*2}\theta^*(t^*)] = \left(\det\ [\ D^2\theta(t)]\right)^{-1}$

3.2 Young measure and duality

With the same hypotheses as above, let $f \in \mathcal{C}^0(\mathbb{R}^n)$ and $g \in \mathcal{C}^0(\mathbb{R}^n)$.
We set: $\varphi_\theta = D\theta$ and $\varphi_\theta^*\circ f = f\circ\varphi_\theta$; φ_θ is a mapping from Ω into Ω^*
We know that φ_θ^* is a linear mapping from $\mathbb{R}^{\Omega^*}$ into $\mathbb{R}^{\Omega}$ and that ${}^t\varphi_\theta^*$ is a linear mapping from the dual of $\mathbb{R}^{\Omega}$ into the dual of $\mathbb{R}^{\Omega^*}$. Then:

$$\int_\Omega (f\circ\ \varphi_\theta)(t).g(t)\ dt = \int_\Omega(\varphi_\theta^*\circ f)(t).g(t)\ dt = (\varphi_\theta^*(f),g) = \left\langle f,{}^t\,\varphi_\theta^* g\right\rangle$$

But,

$$\begin{aligned}\int_\Omega (f(D\theta(t))(t).g(t)\ dt &= \int_{\Omega^*} f(t^*)\,|\det|\,(D^{*2}\theta^*(t^*)).g((D\theta)^{-1}(\theta^*))\ dt^*\\ &= \int_{\Omega^*} f(t^*)\,|\det|\,(D^{*2}\theta^*(t^*)).\left[(((D\theta)^{-1})^\natural g)(t^*)\right]\ dt^*\end{aligned}$$

So,

$${}^t\varphi_\theta^* = |\det|\,(D^{*2}\theta^*(t^*)).((D\theta)^{-1})^\natural \quad \text{where} \qquad ((D\theta)^{-1})^\natural = (\overset{-1}{\varphi_\theta})^\natural$$

3.3 An application. The general case

We suppose, as above, that Ω is a bounded open subset of $\mathbb{R}^n$.
Let θ be a $\mathcal{C}^2$-mapping from Ω into $\mathbb{R}^p$, which associates to any $t \in \Omega$, $(\theta^1(t),\ldots,\theta^p(t)) \in \mathbb{R}^p$.
Then:

$$D\theta = (D_j\theta^k)_{1\leqslant j\leqslant n, 1\leqslant k\leqslant p,}.$$

Let us suppose that for any $l \in \{1,\ldots,p\}$, $D\theta^l$ is a $\mathcal{C}^1$-diffeomorphism which maps Ω onto Ω_l^* . Then:

$$\forall t\in\Omega\ ,\ \forall l\in\{1,\ldots,p\}\ ,\quad (D_{jk}^2\ \theta^l)(t) \text{ is a non singular } \ (n\times n)\ \text{ matrix}$$

Let $(\theta^l)^*$ be the dual function of θ^l such that

$$\begin{aligned} \forall t &= (t_1, \ldots, t_n) \in \Omega \ , \ \ t_l^* = D\theta^l(t) \ \ , \ \ (\theta^l)^*(t_l^*) = (t \mid D\theta^l(t)) - \theta^l(t) \\ \text{where } t_l^* &= (t_{l,1}^*, \ldots, t_{l,n}^*) \in \mathbb{R}^n, \ l \in \{1, \ldots, p\} \, . \end{aligned}$$

From the above hypotheses, we deduce that

$$\begin{aligned} \forall \ l &\in \{1, \ldots, p\} \, , \ t = D^*((\theta^l)^*(t_l^*) \ \text{ or more explicitly :} \\ \forall \ l &\in \{1, \ldots, p\} \ \ , \forall \ j \in \{1, \ldots, n\} \, , \ t_j = D_j^*((\theta^l)^*(t_l^*) \text{ and} \\ \forall \ l &\in \{1, \ldots, p\} \ \ , (dt_{l,1}^* \wedge \ldots \wedge dt_{l,n}^*) = \left[\det \ (D^2\theta^l(t))\right] \ (dt_1 \wedge \ldots \wedge dt_n) \end{aligned}$$

So:

$$\begin{aligned} \forall \ l &\in \{1, \ldots, p\} \ \ , (dt_1 \wedge \ldots \wedge dt_n) = \left[\det \ (D^2\theta^l(t))\right]^{-1} (dt_{l,1}^* \wedge \ldots \wedge dt_{l,n}^* \ ) \\ &= \left[\det \ (D^{*^2}(\theta^l)^*(t_l^*)\right] (dt_{l,1}^* \wedge \ldots \wedge dt_{l,n}^* \ ) \end{aligned}$$

We remark that:

$$\forall \ k, l \in \ \{1, \ldots, p\} \ \ , \ \ \forall t \in \Omega \ , \ \ t = (D^*(\theta^k)^*)(t_k^*) = (D^*(\theta^l)^*)(t_l^*),$$

$t_k^* \in \Omega_k^*$, $t_l^* \in \Omega_l^*$

Hence: $t_k^* = (D^*(\theta^k)^*)^{-1}.(D^*(\theta^l)^*)(t_l^*) = \zeta_k(t_l^*)$. Moreover, we have:

$$\forall t \in \Omega \ , \quad D\theta(t) = \begin{bmatrix} t_{11}^* & \ldots & t_{1n}^* \\ \vdots & & \vdots \\ t_{p1}^* & \ldots & t_{pn}^* \end{bmatrix} = t^*$$

Hence, for any $l \in \{1, \ldots, p\}$, there exists a matrix M_l such that:
$t^* = M_l \ (t_l^*)$.
Now, let $a \in \mathbb{R}^{np}$ and $f : \mathbb{R}^{np} \mapsto \widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.
We set, when the following integral is defined:

$$I(f; \theta; \Omega) = \int_\Omega f(a + D\theta(t)) \ dt = \int_\Omega \widetilde{f}(a_1 + D\theta^1(t), \ldots, a_p + D\theta^p(t)) \ dt_1 \ldots dt_n$$

Then:

$$\begin{aligned} I(f; \theta; \Omega) &= \int_{\Omega_l^*} f(a + M_l \ (t_l^*)) \ \left[\det \ (D^{*^2}(\theta^l)^*(t_l^*)\right] \ dt_{l,1}^* \ldots dt_{l,n}^* \\ &= \int_\Omega \widetilde{f}(a_1 + \zeta_1(t_l^*), \ldots, a_p + \zeta_p(t_l^*)) \ \left|\det \ (D^{*^2}(\theta^l)^*(t_l^*)\right| \ dt_{l,1}^* \ldots dt_{l,n}^* \end{aligned}$$

4 The cone $\Gamma^k_{n,p}$ and its dual

4.1 Definition

Let $k, n, p \in \mathbb{N}^*$. We shall say that $f \in \mathcal{C}^0(\mathbb{R}^{np})$ is a *quasi-convex function of order* k, if:

For any bounded subset Ω of $\mathbb{R}^n$, for any $\mu \in \mathbb{R}^{np}$ and for any $\varphi \in (W_0^{1,k})^p$,

$$\int_\Omega (f(\mu + D\varphi(t)) - f(\mu))\, dt \geqslant 0$$

We shall denote by $\Gamma^k_{n,p}$, the set of all quasi-convex functions of order k .

Proposition 4.1 $\Gamma^k_{n,p}$ *is a convex cone contained in* $\mathcal{C}^0(\mathbb{R}^{np})$.

4.2 The cone $G^k_{n,p}$ dual of $\Gamma^k_{n,p}$

Let $BV(\mathbb{R}^{np})$ be the linear space of real functions on $\mathbb{R}^{np}$ with (normalized) bounded variation. $BV(\mathbb{R}^{np})$ is the topological dual of $\mathcal{C}^0(\mathbb{R}^{np})$.
Let $\langle \cdot, \cdot \rangle$ the duality bracket between $\mathcal{C}^0(\mathbb{R}^{np})$ and $BV(\mathbb{R}^{np})$.
Then $f \in \Gamma^k_{n,p}$ if and only if:
For any bounded subset Ω of $\mathbb{R}^n$, for any $\mu \in \mathbb{R}^{np}$ and for any $\varphi \in (W_0^{1,k})^p$,

$$\int_\Omega \left\langle f, \delta_{\mu + D\varphi(t)} - \delta_\mu \right\rangle\, dt \geqslant 0,$$

where δ_ω is the Dirac measure at ω ; $\delta_\omega \in BV(\mathbb{R}^{np})$.
Now, let us suppose that

$$\int_\Omega \left\langle f, \delta_{\mu + D\varphi(t)} - \delta_\mu \right\rangle\, dt = \left\langle f, \int_\Omega (\delta_{\mu + D\varphi(t)} - \delta_\mu)\, dt \right\rangle$$

We set:

$$G^k_{n,p} = \left\{ \int_\Omega (\delta_{\mu + D\varphi(t)} - \delta_\mu)\, dt;\, \mu \in \mathbb{R}^{np},\, \Omega \in \mathcal{O}_b(\mathbb{R}^n),\, \varphi \in (W_0^{1,k})^p \right\}$$

where $\mathcal{O}_b(\mathbb{R}^n)$ is the set of all bounded open subset of $\mathbb{R}^n$.

Definition 4.1 *Let* $\mathcal{A} \in \mathcal{C}^0(\mathbb{R}^{np})$. *The set:*

$$\mathcal{A}^o = \{\beta \in \mathrm{BV}(\mathbb{R}^{np}) \ ; \forall \alpha \in \mathcal{A} \ , \ \langle \alpha, \beta \rangle \geqslant 0\}$$

is called the polar of $\mathcal{A}$ *relatively to the duality between* $\mathcal{C}^0(\mathbb{R}^{np})$ *and* $\mathrm{BV}(\mathbb{R}^{np})$

From this definition, we deduce that

$$\Gamma^k_{n,p} = (G^k_{n,p}\ )^o\ .$$

If $\mathcal{B} \in BV(\mathbb{R}^{np})$, we denote by $\mathcal{B}^\diamond$ the polar of $\mathcal{B}$ relatively to the duality between $\mathcal{C}^0(\mathbb{R}^{np})$ and $BV(\mathbb{R}^{np})$ such that

$$\mathcal{B}^\diamond = \left\{\alpha \in C^0(\mathbb{R}^{np})\ ;\ \forall\beta \in \mathcal{B}\ ,\ \langle\alpha,\beta\rangle \geqslant 0\right\}.$$

Then

$$(\Gamma^k_{n,p})^\diamond = (G^k_{n,p}\ )^{o\diamond} = \overline{co}(G^k_{n,p}),$$

where $\overline{co}$ denotes the weakly (and strongly) closed convex hull in $BV(\mathbb{R}^{np})$.

4.3 A special case

Let $(\mathbf{H},(\cdot\mid\cdot)) \in Hilb(C^0(\mathbb{R}^{np}))$ and H its hilbertian kernel. We set

$$\widetilde{\Gamma^k_{n,p}} = \Gamma^k_{n,p} \cap \mathbf{H}$$

Then: $f \in \widetilde{\Gamma^k_{n,p}}$ if and only if

$$\begin{array}{c}\forall\Omega \ \in\ \mathcal{O}_b(\mathbb{R}^n)\ \ ,\forall\mu \in \mathbb{R}^{np}\ ,\forall\varphi \in (W_0^{1,k})^p\\ \displaystyle\int_\Omega (f \quad|\quad H(\cdot,\mu + D\varphi(t)) - H(\cdot,\mu))\ dt \geqslant 0\end{array}$$

We set

$$\widetilde{G^k_{n,p}} = \left\{\int_\Omega (H(\cdot,\mu + D\varphi(t)) - H(\cdot,\mu))\, dt\right\};\ \Omega \in \mathcal{O}_b(\mathbb{R}^n),\mu \in \mathbb{R}^{np},\ \varphi \in (W_0^{1,k})^p.$$

With the same above hypotheses and notations, we have

$$\widetilde{\Gamma^k_{n,p}} = (\widetilde{G^k_{n,p}})^o \quad \text{et} \qquad (\widetilde{\mathbf{\Gamma^k_{n,p}}})^o = (\widetilde{\mathbf{G^k_{n,p}}})^{oo} = \overline{co}(\widetilde{G^k_{n,p}}),$$

where $\overline{co}$ denotes the weakly (and strongly) closed convex envelop in the topological dual of $(\mathbf{H},(\cdot\mid\cdot))$ that we shall identify to $(\mathbf{H},(\cdot\mid\cdot))$.

5 Tensorial products of quasi-convex functions

5.1 Introduction.

If Ω is a bounded open subset of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, we shall denote by $\mathcal{C}^0(\ \Omega)$ the linear space of continuous function on Ω.

Let $n_k \in \mathbb{N}^*$ and Ω_k be an open subset of $\mathbb{R}^{n_k}$, $k = 1, 2$.
In the theory of topological tensorial products (T.T.P.), it is proved that if ε is the inductive tensorial norm, the completion of $\mathcal{C}^0(\Omega_1) \otimes \mathcal{C}^0(\Omega_2)$, embedded with the norm ε can be identified to the linear space $\mathcal{C}^0(\Omega_1 \times \Omega_2)$.
If A is a subset of $\mathbb{R}^n$, $n \in \mathbb{N}^*$ we denote, in the following, by $E(A)$, any locally convex space of functions on A .
Let A_k be a subset of $\mathbb{R}^{n_k}$, $k = 1, 2$.
We suppose that there exists a reasonable tensorial norm γ on $E(A_1) \otimes E(A_2)$ such that the completion of $E(A_1) \otimes E(A_2)$ embedded with the norm γ and denoted by $E(A_1) \,\widehat{\gamma}\, E(A_2)$ can be identified to $E(A_1 \times A_2)$. Then

$$\forall h \in E(A_1 \times A_2) \ , \ \ h = \sum_j (f_j \otimes g_j) \ \ , \ \ f_j \in E(A_1) \ \ , \ \ g_j \in E(A_2),$$

where

$$\sum_j (f_j \otimes g_j) \ \text{ is convergent to } \ h \ \text{ in } E(A_1) \,\widehat{\gamma}\, E(A_2),$$

and

$$\forall s \in A_1 \ \ , \ \ \forall t \in A_2 \ \ , \ \ h(s,t) = \sum_j (f_j(s).g_j(t)).$$

Now, let $k, l, p, q \in \mathbb{N}^*$, U (resp. V) an open subset of $\mathbb{R}^p$ (resp. $\mathbb{R}^q$) , and x (resp. y) a mapping from U into $\mathbb{R}^k$ (resp. V into $\mathbb{R}^l$) .

$$\begin{array}{rcl} \text{If} \quad h & \in & E(x(U)) \,\widehat{\gamma}\, E(y(V)) = E(x(U) \times y(V)) \\ h & \in & \displaystyle\sum_j (f_j \otimes g_j) \quad \text{where} \ \ f_j \in E(x(U)) \ \ , \ \ g_j \in E(y(V)) \end{array}$$

Then

$$\begin{aligned} \forall j, \ \int\int_{U \times V} (f_j \otimes g_j)(x(s), y(t)) \, ds \, dt &= \int\int_{U \times V} (f_j(x(s)).g_j(y(t)) \, ds \, dt \\ &= (\int_U (f_j(x(s)) \, ds).(\int_V g_j(y(t)) \, dt) \end{aligned}$$

when the above integrals are defined.
We deduce that

$$\int\int_{U \times V} h \ (x(s), y(t)) \ ds \, dt = \sum_j (\int_U (f_j(x(s)) \ ds).(\int_V g_j(y(t)) \ dt)$$

We set, below

$$\int\int_{U \times V} h \ (x(s), y(t)) \ ds \, dt = I(h \ ; x, y \ ; U, V)$$

5.2 Tensorial products

5.2.1 Definition

Let $k, l, p, q \in \mathbb{N}^*$, U (resp. V) an open subset of $\mathbb{R}^p$ (resp. $\mathbb{R}^q$) ,

$$\begin{array}{rcl} x & \in & \mathcal{C}^1(U\ ;\mathbb{R}^p) \quad \text{and} \quad y \in \mathcal{C}^1(V\ ;\mathbb{R}^l) \quad . \\ \text{Then} & : & Dx \in \mathcal{C}^0(U\ ;\mathbb{R}^{kp}) \quad \text{and} \quad Dy \in \mathcal{C}^0(V\ ;\mathbb{R}^{lq}) \end{array}$$

Let:

$$f \in \mathcal{C}^0(\mathbb{R}^{kp}) \quad \text{and} \quad g \in \mathcal{C}^0(\ \mathbb{R}^{lq})$$

It can be proved easily that:
$\forall \alpha \in \mathbb{R}^{kp}$, $\forall \beta \in \mathbb{R}^{lq}$,

$$I(f \otimes g\ ;\ \alpha + Dx\ ,\ \beta + Dy\ \ ;\ \ U, V) \geqslant\ I(f \otimes g\ ;\ \alpha\ ,\ \beta\ ;\ \ U, V)$$

More generally, let

$$\begin{array}{rcl} h & \in & \displaystyle\sum_j (f_j \otimes g_j) \quad \text{with} \quad f_j \in \mathcal{C}^0(\mathbb{R}^{kp}) \text{ and } g_j \in \mathcal{C}^0(\ \mathbb{R}^{lq}) \\ \text{such that } h & \in & \mathcal{C}^0(\mathbb{R}^{kp})\ \widehat{\varepsilon}\ \ \mathcal{C}^0(\ \mathbb{R}^{lq}) = \mathcal{C}^0(\mathbb{R}^{kp} \times\ \mathbb{R}^{lq}) \end{array}$$

If for any j, f_j and g_j are quasi-convex and positive, we have:

$$\forall \alpha \in \mathbb{R}^{kp}\ \ ,\ \ \forall \beta \in \mathbb{R}^{lq}\ \ ,\ \ I(h\ ;\ \alpha{+}Dx\ ,\ \beta{+}Dy\ \ ;\ \ U, V) \geqslant\ I(h\ ;\ \alpha\ ,\ \beta\ ;\ \ U, V)$$

Then, we shall say, that h is tensorially convex (positive) function on $U \times V$.
Let E_j be a locally convex space, $E_j^\natural$ its topological dual and $\langle\cdot,\cdot\rangle_j$ the duality bracket between E_j and $E_j^\natural$, $j = 1, 2$.
Let γ be a reasonable tensorial norm.
Let us denote by $E_1\ \gamma\ E_2$ the linear space $E_1 \otimes E_2$ embedded with the norm γ and by $E_1\ \widehat{\gamma}\ E_2$ its completion.
Let $n_j, p_j, k_j \in \mathbb{N}^*$, $j = 1, 2$.
Let us suppose that: $\Gamma^{k_j}_{n_j,p_j} \subset E_j$, $j = 1, 2$. Then:

$$\Gamma^{k_1}_{n_1,p_1} \otimes \Gamma^{k_2}_{n,p_2} \subset E_1 \otimes E_2 \text{ and}$$

$$(h \in \Gamma^{k_1}_{n_1,p_1} {\otimes} \Gamma^{k_2}_{n,p_2}) \Leftrightarrow (h = \sum_{j \in J \text{ finite}} (f_j {\otimes} g_j) \ \ \text{where} \ \ \ f_j \in \Gamma^{k_1}_{n_1,p_1} \ \ \text{and} \ \ g_j \in \Gamma^{k_2}_{n,p_2})$$

Let us denote by $(\ \Gamma^{k_1}_{n_1,p_1}\ \widehat{\gamma}\ \Gamma^{k_2}_{n,p_2})$ the closure of $(\Gamma^{k_1}_{n_1,p_1} \otimes \Gamma^{k_2}_{n,p_2})$ in $E_1\ \widehat{\gamma}\ E_2$.
It can be proved easily that $\Gamma^{k_1}_{n_1,p_1}\ \widehat{\gamma}\ \Gamma^{k_2}_{n,p_2}$ is a convex cone; then, we say that $\Gamma^{k_1}_{n_1,p_1}\ \widehat{\gamma}\ \Gamma^{k_2}_{n,p_2}$ is the set of *tensorial γ-products of quasi-convex functions* contained in $\Gamma^{k_1}_{n_1,p_1}$ and $\Gamma^{k_2}_{n,p_2}$.

Let us denote by $\gamma^\natural$ (resp. $\gamma^{\natural\natural}$) the dual (resp. bidual) tensorial norm of γ. Generally $(\gamma^\natural)^\natural \neq \gamma$.
Let $\langle \cdot, \cdot \rangle_\otimes$ the canonical duality bracket between $E_1^\natural \; \gamma^\natural \; E_2^\natural$ and $E_1 \; \widehat{\gamma} \; E_2$.
We denote by $(\cdot)_1^0$ (resp. $(\cdot)_2^0$, $(\cdot)_\otimes^0$) the polarity relatively to the duality bracket $\langle \cdot, \cdot \rangle_1$ (resp. $\langle \cdot, \cdot \rangle_2$ et $\langle \cdot, \cdot \rangle_\otimes$).
So we set:

$$\Gamma_{n_1,p_1}^{k_1} \; \widehat{\gamma^{\natural\natural}} \; \Gamma_{n,p_2}^{k_2} = \left(\left(\Gamma_{n_1,p_1}^{k_1} \right)^0 \widehat{\gamma^\natural} \left(\Gamma_{n,p_2}^{k_2} \right)^0 \right)_\otimes^0 = \left(G_{n_1,p_1}^{k_1} \widehat{\gamma^\natural} \; G_{n,p_2}^{k_2} \right)_\otimes^0$$

Example 5.1 *Let $n_k \in \mathbb{N}^*$ and Ω_k an open subset of $\mathbb{R}^{n_k}$, $k = 1, 2$. Let us suppose that: $E_k = \mathcal{C}^0(\Omega_k), k = 1, 2$. We can consider:*

$(\Gamma_{n_1,p_1}^{k_1} \; \widehat{\varepsilon} \; \Gamma_{n,p_2}^{k_2})$ *the closure of* $(\Gamma_{n_1,p_1}^{k_1} \otimes \Gamma_{n,p_2}^{k_2})$ *in* $\mathcal{C}^0(\Omega_1) \; \widehat{\varepsilon} \; \mathcal{C}^0(\Omega_2) = \mathcal{C}^0(\Omega_1 \times \Omega_2)$

and

$(\Gamma_{n_1,p_1}^{k_1} \; \widehat{\varepsilon^{\natural\natural}} \; \Gamma_{n,p_2}^{k_2}) = \left(G_{n_1,p_1}^{k_1} \; \widehat{\varepsilon^\natural} \; G_{n,p_2}^{k_2} \right)$ *which is contained in* $\mathrm{BV}(\Omega_1) \; \widehat{\varepsilon^\natural} \; \mathrm{BV}(\Omega_2)$

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Estimation of pollution term in Petrowski system with incomplete data

Amel BERHAIL[1] and Abdelhamid AYADI[2]

[1]University of 8 May 1945, Guelma, Algeria
e-mail: berhailamel @yahoo.fr
[2]University of Larbi Ben M'hidi, Oum El Bouaghi, Algeria
e-mail: hamidmath@hotmail.com

Abstract

In this paper, we want to estimate the pollution term in the Petrowski system of incomplete data. For this aim we use the notion of Sentinels introduced by J.L.Lions, which is a linear functional sensitive to some parameters we are trying to evaluate, and insensitive to others which do not interest us. We show that establishing the existence of such sentinels is reduced to the solution of a null-controllability problem with constraint on the control.

Keywords: *Null-Controllability, Control optimal, Petrowsky system, Distributed Sentinel.*

1 Introduction

Let Ω be a bounded open in R^N with sufficiently smooth boundary Γ, ω is a nonempty subdomain of Ω. For a fixed time $T > 0$, we define by $Q = \Omega \times]0, T[$, $\sum = \Gamma \times]0, T[$, $\sum_0 = \Gamma_0 \times]0, T[$ and $\sum_1 = \Gamma_1 \times]0, T[$ where Γ_0 and Γ_1 are a non empty open subset of Γ with $\Gamma_0 \cap \Gamma_1 = \phi$.

We consider here the Petrowsky system where initial conditions and some boundary conditions are not well determined in general (boundary conditions are known only on one part of the boundary). This problem can be formulated as follows:

$$\frac{\partial^2 y}{\partial t^2} + \triangle^2 y + f(y) = 0 \quad \text{in} \quad Q, \tag{1}$$

where $f : R \to R$ is a nonlinear C^1 function.

We add to (1) the initial conditions

$$\begin{cases} y(0) = y_0 + \tau_0 \widehat{y}_0 & \text{in } \Omega, \\ y'(0) = y_1 + \tau_1 \widehat{y}_1 & \text{in } \Omega, \end{cases} \tag{2}$$

and the boundary conditions

$$y = \begin{cases} g_0 + \lambda \widehat{g}_0 & \text{on } \Sigma_0, \\ g_1 + \tau_2 \widehat{g}_1 & \text{on } \Sigma_1, \\ 0 & \text{on } \Sigma / \left(\Sigma_0 \cup \Sigma_1\right), \end{cases} \tag{3}$$

where y_0, y_1, g_0 and g_1 are given, but the terms $\tau_0 \widehat{y}_0, \tau_1 \widehat{y}_1, \tau_2 \widehat{g}_1$ and $\lambda \widehat{g}_0$ are unknown functions. The parameters λ and $\tau = (\tau_0, \tau_1, \tau_2)$ are reals numbers chosen sufficiently small.

The term $(\tau_0 \widehat{y}_0, \tau_1 \widehat{y}_1, \tau_2 \widehat{g}_1)$ design the missing data and $\lambda \widehat{g}_0$ the pollution term. We also use the notation y^0 for the solution of the problem $(1) - (3)$ when $\lambda = 0$ and $\tau = 0$:

$$\begin{cases} \dfrac{\partial^2 y^0}{\partial t^2} + \triangle^2 y^0 + f(y^0) = 0 & \text{in } Q, \\ y^0(0) = y_0 & \text{in } \Omega, \\ y^{0\prime}(0) = y_1 & \text{in } \Omega, \\ y^0 = \begin{cases} g_0 & \text{on } \Sigma_0, \\ g_1 & \text{on } \Sigma_1, \\ 0 & \text{on } \Sigma / \left(\Sigma_0 \cup \Sigma_1\right) \end{cases} \end{cases} \tag{4}$$

The aim of this work is to obtain information on the pollution term $\lambda \widehat{g}_0$ not affected by missing term.

There are two possible approches to this problem, one is more classical and uses the least square method (see G.Chavent [02]), but the problem in this method that the pollution and the missing terms play the same role, so we can not separate them.

The other is the sentinel method introduced by J.L.Lions [06] which is used to study systems of incomplete data.

The notion permits to distinguish and to analyse two types of incomplete data, the pollution term and the missing terms.

So we show that this functional can be associated to our system and allows to characterize the pollution term.

2 Problem Formulations

The Lions sentinel is a linear functional sensitive to certain parameters we are trying to evaluate, and insensitive to others which do not interest us, it lies on three considerations:

- A state equation represented here by (1) and where the solution

$$y(\lambda,\tau) = y(x,t;\lambda,\tau) = y(x,t;\lambda,\tau_0,\tau_1,\tau_2),$$

depends on two parameters λ and τ.

- An observation

$$y(\lambda,\tau) = y_{obs}, \quad \text{on} \ \ O\times\left]0,T\right[\,, \tag{5}$$

of the state solution $y(\lambda,\tau)$ on a non empty open subset $O\subset\Omega$ called observatory, during the interval $]0,T[$.

- A function S associated to the function

$$h\in L^2(O\times\left]0,T\right[), \tag{6}$$

and to a control function

$$u\in L^2(\omega\times\left]0,T\right[) \tag{7}$$

is defined by

$$S(\lambda,\tau) = \int\limits_{O\times]0,T[} hy(x,t,\lambda,\tau)dxdt + \int\limits_{\omega\times]0,T[} uy(x,t,\lambda,\tau)dxdt, \tag{8}$$

we can write

$$S(\lambda,\tau) = \int\limits_{\Omega\times]0,T[} (h\chi_{_O} + u\chi_\omega)y(\lambda,\tau)dxdt, \tag{9}$$

where $\chi_{_O}$ and χ_ω are the characteristic functions for the open sets O and ω respectively such that $O\cap\,\omega\neq\phi$ [11]. We given now some definitions

Definition 2.1 *S is said insensitive respect to the data τ if the following condition is satisfied:*

$$\frac{\partial S}{\partial\tau}(\lambda,\tau)\bigg|_{\lambda=0,\tau=0} = 0; \tag{10}$$

i.e.

$$\frac{\partial S}{\partial\tau_0}(0,0)=0, \frac{\partial S}{\partial\tau_1}(0,0)=0, \frac{\partial S}{\partial\tau_2}(0,0)=0;$$

Definition 2.2 *S is said a sentinel defined by h if S satisfies to definition 1 and the control $u \in L^2(\omega \times]0,T[)$ satisfies:*

$$\|u\|_{L^2(O)} = \min_{w \in U} \|w\| ; \tag{11}$$

where $U = \left\{ w \in L^2(\omega \times]0,T[), \text{ such that } \left.\frac{\partial S}{\partial \tau}(\lambda,\tau)\right|_{\lambda=0,\tau=0} = 0 \right\}.$

Remark 2.3 *Condition (10) express insensitivity of S with respect to small variations of τ and assume the existence of the derivative.*

3 The equivalent controllability problem

Here it will be show that the existence of such control function satisfying (10) and (11), is equivalent to the null controllability property for a system with constrained control.

We write the derivatives of y at $(0,0)$ with respect to τ as:

$$y_{\tau_i} = \frac{\partial}{\partial \tau_i} y((\lambda,\tau)_{\lambda=0,\tau=0} \text{ with } \tau = (\tau_0,\tau_1,\tau_2) \text{ and } i = 0,1,2 \tag{12}$$

where y_{τ_0} is the solution of

$$\begin{cases} \dfrac{\partial^2 y_{\tau_0}}{\partial t^2} + \triangle^2 y_{\tau_0} + f'(y^0) y_{\tau_0} = 0 & \text{in } Q, \\ y_{\tau_0}(0) = \widehat{y}_0 & \text{in } \Omega, \\ y'_{\tau_0}(0) = 0 & \text{in } \Omega, \\ y_{\tau_0} = 0 & \text{on } \sum, \end{cases} \tag{13}$$

y_{τ_1} is the solution of

$$\begin{cases} \dfrac{\partial^2 y_{\tau_1}}{\partial t^2} + \triangle^2 y_{\tau_1} + f'(y^0) y_{\tau_1} = 0 & \text{in } Q, \\ y_{\tau_1}(0) = 0 & \text{in } \Omega, \\ y'_{\tau_1}(0) = \widehat{y}_1 & \text{in } \Omega, \\ y_{\tau_1} = 0 & \text{on } \sum, \end{cases} \tag{14}$$

and y_{τ_2} is the solution of

$$\begin{cases} \dfrac{\partial^2 y_{\tau_2}}{\partial t^2} + \triangle^2 y_{\tau_2} + f'(y^0) y_{\tau_2} = 0 & \text{in } Q, \\ y_{\tau_2}(0) = 0 & \text{in } \Omega, \\ y'_{\tau_2}(0) = 0 & \text{in } \Omega, \\ y_{\tau_2} = \begin{cases} \widehat{g}_1 \text{ on } \sum_1 \\ 0 \text{ on } \sum / \sum_1 \end{cases} \end{cases} \tag{15}$$

where $f'(y^0)$ is the derivative of the function f at the solution y^0.

Remark 3.1 *It is important to observe that if f has smooth regularity, insensitivity condition (10) becomes:*

$$\int_{\Omega\times]0,T[} (h\chi_O + u\chi_\omega) y_\tau dxdt = 0. \tag{16}$$

In order to transform the equation (16), we introduce now the classical adjoint state.

Theorem 3.2 *Let q be the solution to the following backward problem:*

$$\begin{cases} \dfrac{\partial^2 q}{\partial t^2} + \triangle^2 q + f'(y^0)q = \ (h\chi_O + u\chi_\omega) & in \ \ Q, \\ q(T) = q'(T) = 0 & in \ \ \Omega, \\ q = 0 & in \ \ \sum, \end{cases} \tag{17}$$

Then the existence of a distributed sentinel insensitive to the missing data is equivalent to the null-controllability problem with:

$$q'(0) = q(0) = 0 \ \ in \ \Omega, \frac{\partial q}{\partial \nu} = 0 \ \ on \ \sum, \tag{18}$$

where ν is the unit exterior normal to Γ; $\dfrac{\partial q}{\partial \nu}$ is the derivative of q with respect to the normal ν.

Proof Multiplying both members of the differential equation in (17) by y_τ and integrating by parts over Q, we obtain:

$$-\int_\Omega (q(0)y'_\tau(0) + q'(0)y_\tau(0))\, dx + \int_\Sigma \left(\frac{\partial q}{\partial \nu} y_\tau - q\frac{\partial y_\tau}{\partial \nu} \right) d\Sigma$$

$$= \int_{\Omega\times]0,T[} (h\chi_O + u\chi_\omega) y_\tau dxdt.$$

for more detail,

for $\tau = \tau_0$, we obtain

$$\int_\Omega q'(u)(0)\widehat{y}_0 dx = 0, \ \ \forall \widehat{y}_0 \ \ \Longrightarrow \ \ q'(0) = 0$$

for $\tau = \tau_1$, we obtain

$$\int_\Omega q(u)(0)\widehat{y}_1 dx = 0, \ \ \forall \widehat{y}_1 \ \ \Longrightarrow \ \ q(0) = 0$$

for $\tau = \tau_2$, we obtain

$$\int\limits_{\Sigma_1} \frac{\partial q}{\partial \nu} \widehat{g}_1 d\Sigma_1 = 0, \quad \forall \widehat{g}_1 \quad \Longrightarrow \quad \frac{\partial q}{\partial \nu} = 0 \quad \text{on } \Sigma_1$$

Thus, condition (10) holds if and only if we have (18) which is a null controllability problem. ■

4 Characterization of optimal control

The existence of the sentinel insensitive to the missing terms is equivalent to the null controllability which is equivalent to the existence of a unique pair (w, q) such that we have (17) and (18), so, we are interested in the problem of optimal control:

$$\min_{(w,q)\in M} \|w\|_{L^2(\omega)}, \tag{19}$$

with

$$M = \{(w, q) \text{ such that we have (17) and (18)}\}.$$

Let us introduce p by

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2} + \triangle^2 p + f'(y^0)p = h\chi_O & \text{in } Q, \\ p(T) = p'(T) = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \sum, \end{cases} \tag{20}$$

and let us define $z = z(u)$ the solution of

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} + \triangle^2 z + f'(y^0)z = u\chi_\omega & \text{in } Q, \\ z(T) = z'(T) = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \sum, \end{cases} \tag{21}$$

Then

$$q = p + z = p + z(u).$$

We want to find u such that

$$\begin{cases} z(0; u) = -p(0), \\ z'(0; u) = -p'(0), \\ \dfrac{\partial z}{\partial \nu} = -\dfrac{\partial p}{\partial \nu} \text{ on } \sum_1. \end{cases} \tag{22}$$

We define ρ as the solution of

$$\begin{cases} \dfrac{\partial^2 \rho}{\partial t^2} + \triangle^2 \rho + f'(y^0)\rho = 0 & \text{in } Q, \\ \rho(0) = \rho^0 \ , \quad \rho'(0) = \ \rho^1 & \text{in } \ \Omega \\ \rho = 0 & \text{on } \ \sum, \end{cases} \tag{23}$$

where $\{\rho^0, \rho^1\}$ is not determined. Let z is the solution of the system

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} + \triangle^2 z + f'(y^0)z = \rho\chi_\omega & \text{in } \ Q, \\ z(T) = \quad z'(T) = 0 & \text{in } \ \Omega, , \\ z = 0 & \text{on } \ \sum . \end{cases} \tag{24}$$

We introduce a linear operator Λ and Ψ by

$$\Lambda\{\rho^0, \rho^1\} = \{-z'(0), \ z(0)\}\,, \tag{25}$$

$$\Psi h = \{-p'(0), \ p(0)\}\,,$$

we obtain

$$\Lambda\{\rho^0, \rho^1\} = -\Psi h. \tag{26}$$

Multiplying (24) by $\{\rho^0, \rho^1\}$ and integrating by parts, we obtain

$$\left\langle \Lambda\{\rho^0, \rho^1\}, \{\rho^0, \rho^1\} \right\rangle = \int\limits_{\Omega\times]0,T[} \rho^2 dxdt. \tag{27}$$

Let as now set

$$\left\| \{\rho^0, \rho^1\} \right\|_F = \left(\int\limits_{\Omega\times]0,T[} \rho^2 dxdt \right)^{1/2} . \tag{28}$$

We define in this way a norm on the space of the functions $\{\rho^0, \rho^1\}$, where the Hilbert space F is the completion of smooth functions for the norm (28). Then if F' denotes the dual of F, we have

$$\Lambda : F \longrightarrow F' \text{ is an isomorphism,}$$

and

$$\left\langle \Psi h, \{\rho^0, \rho^1\} \right\rangle = \int\limits_{\Omega\times]0,T[} h.\rho dxdt. \tag{29}$$

Then the equation (26) has an unique solution given by

$$\{\rho^0, \rho^1\} = -\Lambda^{-1}\Psi h, \tag{30}$$

we get

$$u = \rho_{\chi_\omega} = \Psi^* \Lambda^{-1} \Psi h, \tag{31}$$

then the sentinel is given by

$$S(\lambda, \tau) = \int\limits_{\Omega\times]0,T[} (h - \Psi^* \Lambda^{-1} \Psi h) y(x, t, \lambda, \tau) dxdt$$

4.1 Estimation of the pollution term

To show how the sentinel defined above permits to estimate the pollution term, we consider y_{obs} be the measured state of the system on the observatory O during the interval $[0, T]$, then the measured sentinel associate to y_{obs} is given by :

$$S_{obs}(\lambda, \tau) = \int\limits_{\Omega\times]0,T[} (h\chi_O + u\chi_\omega) y_{obs}(x, t, \lambda, \tau) dxdt. \tag{32}$$

Theorem 4.1 *The pollution term is identified as follows:*

$$\int_{\Sigma_0} \frac{\partial q}{\partial \nu}(h) \widehat{g}_0 d\Sigma_0 = S_{obs}(\lambda, \tau) - S(0, 0). \tag{33}$$

Proof We have

$$S_{obs}(\lambda, \tau) = S(0, 0) + \lambda \frac{\partial S}{\partial \lambda}(\lambda, \tau_i)\bigg|_{\lambda=0, \tau_i=0} + O(\lambda, \tau_i); \tag{34}$$

with

$$\frac{\partial S}{\partial \lambda}(\lambda, \tau) = \int\limits_{\Omega\times[0,T]} (h_{\chi_O} + w_{\chi_\omega}) y_\lambda dxdt \tag{35}$$

where y_λ is the solution of

$$\begin{cases} \dfrac{\partial^2 y_\lambda}{\partial t^2} + \triangle^2 y_\lambda + f'(y^0) y_\lambda = 0 & \text{in } Q, \\ y_\lambda(0) = 0 & \text{in } \Omega, \\ y'_\lambda(0) = 0 & \text{in } \Omega, \\ y_\lambda = \begin{cases} \widehat{g}_0 & \text{on } \sum_0 \\ 0 & \text{on } \sum / \sum_0 \end{cases} \end{cases} \tag{36}$$

and

$$\lambda \frac{\partial S}{\partial \lambda}(\lambda, \tau)\bigg|_{\lambda=0, \tau=0} = S_{obs}(\lambda, \tau) - S(0, 0). \tag{37}$$

Let q be the solution of (17), multiplying (15) par q, it follows that

$$\frac{\partial S}{\partial \lambda}(0,0) = \int_{\Sigma_0} \frac{\partial q}{\partial \nu}(h)\widehat{g}_0 d\Sigma_0 \tag{38}$$

hence

$$\int_{\Sigma_0} \frac{\partial q}{\partial \nu}(h)\widehat{g}_0 d\Sigma_0 = S_{obs}(\lambda, \tau) - S(0,0). \quad \blacksquare$$

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Numerical study of two-dimensional incompressible Navier-Stokes equations in natural convection with nanofluids

I. EL BOUIHI[1], and R. SEHAQUI[2],

Hassan II University, Faculty of Sciences
Mechanical Laboratory, Casablanca, Morocco
e-mail[1]: i.elbouihi@hotmail.com
e-mail[2]: sehaqui@hotmail.com

Abstract

Nanofluids are considered to offer important advantages over conventional heat transfer fluids. A model is developed to analyze the behaviour of nanofluids taking into account the solid fraction χ***. The Navier-Stokes equations are solved numerically with ADI method. Copper-water nanofluid is used with*** $Pr = 6.2$ ***and solid volume fraction*** χ ***is varied as*** 0.0%, 5%, 10%, 15% ***and*** 20%***.***

Keywords: *numerical study, Nanofluids, solid volume fraction, convective heat transfer.*

1 Introduction

Buoyancy induced flow and heat transfer is an important phenomenon in engineering systems due to its wide applications in electronic cooling, heat exchangers, double pane windows etc. Enhancement of heat transfer in these systems is an essential topic from an energy saving perspective. The low thermal conductivity of convectional heat transfer fluids such as water and oils is a primary limitation in enhancing the performance and the compactness of such systems. Nanotechnology has been widely used in industry since materials with sizes of nanometers possess unique physical and chemical properties.

C_p	specific heat at constant pressure	μ	dynamic viscosity
$\overrightarrow{g}$	gravitational acceleration vector	ν	kinematic viscosity
Gr	Grashof number, $\frac{g\beta_f \Delta T H^3}{\nu_f^2}$	ω	vorticity
H	cavity height	ρ	density
k	thermal conductivity	ψ	stream function
L	cavity width	χ	solid volume fraction
$\overline{Nu}$	average Nusselt number	*Subscripts*	
Pr	Prandtl number, $\frac{\nu_f}{\alpha_f}$	c	cold
t	time	*eff*	effective
T	temperature	f	fluid
u,v	velocity components	h	hot
x,y	cartesian coordinates	nf	nanofluid
Greek symbols		s	solid
α	thermal diffusivity	*Superscripts*	
β	thermal expansion coefficient	$*$	dimensional term

Table 1: Nomenclature

Nano-scale particle added fluids are called as nanofluid. Yang and al.[1] conclude experimentally that the type of nanoparticles, particle loading, base fluid chemistry, and process temperature are all important factors to be considered while developing nanofluids for high heat transfer coefficients. Experimental work by Wen and al.[2] investigate into convective heat transfer of nanofluids at the entrance region under laminar flow conditions. Some numerical studies on nanofluids include thermal conductivity[3]. Studies on natural convection using nanofluids are very limited and they are related with differentially heated enclosures. Khanafer and al.[4] tested different models for nanofluid density, viscosity and thermal expansion coefficients.
In the present work, we simulate the flow features of nanofluids for a range of solid volume fraction χ.

2 Problem Formulations

The problem considered is a two-dimensional heat transfer in a square cavity (fig.1). The vertical walls are differentially heated, the left is maintained at hot condition (T_h) when the right one is cold (T_c). The horizontal walls are assumed to be insulated, non conducting and impermeable to mass transfer. The nanofluid in the enclosure is Newtonian, incompressible and laminar. The nanoparticles are assumed to have a uniform shape and size.

Moreover, it is assumed that both the fluid phase and nanoparticles are in thermal equilibrium state and they flow at the same velocity. The thermophysical properties (Table1) of the nanofluid are assumed to be constant except for the density variation in the buoyancy force, which is based on the Boussinesq approximation.

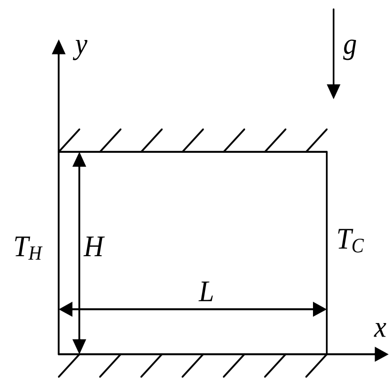

Figure 1: Schematic for the physical model.

Property	Fluid phase(water)	Solid phase(copper)
$C_p(J/kgK)$	4179	383
$\rho(kg/m^3)$	997.1	8954
$k(W/mK)$	2.1×10^{-4}	1.67×10^{-5}

Table 2: Thermophysical properties of different phases

Under the assumption of constant thermal properties, the Navier-Stokes equation for an unsteady, incompressible, two-dimensional flow are:

- Continuity equation:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \quad (1)$$

- x-momentum equation:

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho_{nf,0}}\frac{\partial p^*}{\partial x^*} + \frac{\mu_{eff}}{\rho_{nf,0}}(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}) \quad (2)$$

- y-momentum equation:

$$\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{1}{\rho_{nf,0}}\frac{\partial p^*}{\partial y^*} + \frac{\mu_{eff}}{\rho_{nf,0}}(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}) +$$

$$\frac{1}{\rho_{nf,0}}[\chi\rho_{s,0}\beta_s + (1-\chi)\rho_{f,0}\beta_f]g(T - T_c) \quad (3)$$

- Energy equation:

$$\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \alpha_{nf} (\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}) \tag{4}$$

where

$$\alpha_{nf} = \frac{k_{eff}}{(\rho C_p)_{nf,0}}$$

The effective viscosity of fluid containing a dilute suspension of small rigid spherical particles is given by Brinkman[5] as

$$\mu_{eff} = \frac{\mu_f}{(1-\chi)^{2.5}} \tag{5}$$

The effective density of the nanofluid at reference temperature is

$$\rho_{nf,0} = (1-\chi)\rho_{f,0} + \chi\rho_{s,0} \tag{6}$$

and the heat capacitance of nanofluid is

$$(\rho C_p)_{nf,0} = (1-\chi)(\rho C_p)_{f,0} + \chi(\rho C_p)_{s,0} \tag{7}$$

The effective thermal conductivity of fluid can be determined by Maxwell-Garnett's (MG model) self-consistent approximation model. For the two-component entity of spherical-particle suspension,the MG model gives

$$\frac{k_{eff}}{k_f} = \frac{(k_s + 2k_f) - 2\chi(k_f - k_s)}{(k_s + 2k_f) + \chi(k_f - k_s)} \tag{8}$$

The above equations can be converted to non-dimensional form, using the following dimensionless parameters:
$x = \frac{x^*}{H}$, $y = \frac{y^*}{H}$, $u = \frac{u^*}{V_0}$, $v = \frac{v^*}{V_0}$, $p = \frac{p^*}{(\rho V_0^2)}$, $T = \frac{T^* - T_c}{\Delta T}$
where $\Delta T = T_h - T_c$, $Gr = \frac{g\beta H^3 \Delta T}{\nu_{f^2}}$, $Pr = \frac{\nu_f}{\alpha_f}$
The governing equations can be writing in dimensionless form as follows:

- Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

- x-momentum equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{10}$$

- y-momentum equation:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + (\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + \frac{[\chi\rho_s\beta_s + (1-\chi)\rho_f\beta_f]\ \rho_{nf,0}\ \nu_f^2}{\beta_f\ \mu_{eff}^2} Gr\, T \tag{11}$$

- Energy equation:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Pr_{nf}}(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) \tag{12}$$

where $Pr_{nf} = \frac{\mu_{eff}}{\alpha_{nf}\ \rho_{nf,0}}$

Boundary conditions are:

- for $x = 0$ and $0 \leq y \leq 1$, $u = v = 0$ and $T = 1$,
- for $x = 1$ and $0 \leq y \leq 1$, $u = v = 0$ and $T = 0$,
- for $y = 0$ or $y = 1$ and $0 \leq x \leq 1$, $u = v = \frac{\partial T}{\partial y} = 0$.

The governing equations for the present study taking into the account the above mentioned assumptions are written in dimensionless form as:

- Kinematics equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \tag{13}$$

- Vorticity equation

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = (\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}) + \frac{[\chi\rho_s\beta_s + (1-\chi)\rho_f\beta_f]\ \rho_{nf,0}\ \nu_f^2}{\beta_f\ \mu_{eff}^2} Gr\, \frac{\partial T}{\partial x} \tag{14}$$

- Energy equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Pr_{nf}}(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) \tag{15}$$

The Nusselt number of the nanofluids is expected to depend on a number of factors such as thermal conductivity and heat capacitance of both the pure fluid and the ultrafine particles, the volume fraction of the suspended particles, the flow structure and the viscosity of the nanofluid. The local variation of the Nusselt number of the nanofluid can be expressed as

$$Nu = -\frac{k_{eff}}{k_f}\frac{\partial T}{\partial x} \tag{16}$$

Ra/Nu	present work(a)	Khanafer and al.[4]	Tiwari and al.[7]	De Vahl Davis [8](b)	$\frac{b-a}{b} \times 100$
10^3	1.042	1.118	1.087	1.118	6.79
10^4	2.024	2.245	2.197	2.243	9.76
10^5	4.520	4.522	4.450	4.519	0.02
10^6	8.978	8.826	8.803	8.799	2.03%

Table 3: Comparison of laminar solution with previous works

3 Numerical method

The governing equations are solved numerically by ADI method (Alternating Direct Implicit).

The developed numerical code is validated (Table2) for natural convection heat transfer by comparing the results a laminar heat transfer in a square cavity with air for Rayleigh numbers between 10^3 and 10^6.

The natural convection problem in a differentially heated square enclosure using nanofluids has been solved and compared the results with those of Santra and Sen [6] and Tiwari and Das [7] (fig.2). A very good agreement has been obtained.

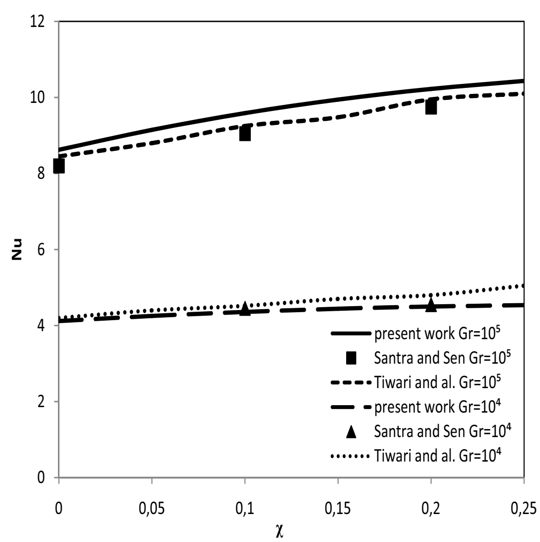

Figure 2: Validation of the present code with the results of Santra and Sen [6] and Tiwari and Das [7].

The grid independence test is performed using successively sized grids, $31 \times 31,\ 41 \times 41,\ 61 \times 61$ and 81×81 for $Gr = 10^5,\ Pr = 6.2$ and $\chi = 5\%$. Uniform grid has been used for all the computations. The distribution of the *u-velocity* in the vertical mid-plane and *temperature* and *v-velocity* in the horizontal mid-plane are shown in fig.3. It is observed that the curves overlap with each other for 61×61 and 81×81. So a grid number of 61×61 is chosen for further computation.

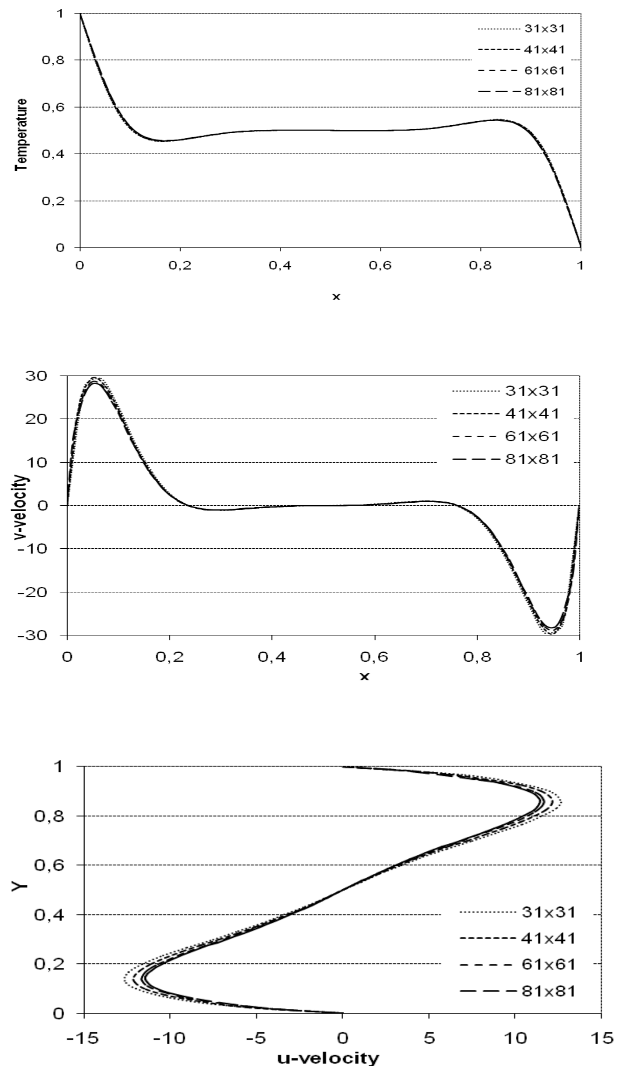

Figure 3: Temperature and velocity profiles at mid-sections of the cavity for various mesh sizes ($Gr = 10^5$, $Pr = 6.2$ and $\chi = 5\%$).

4 Discussion

The numerical code developed in the present investigation is used to carry out a number of simulations for a wide range of controlling parameters such as Grashof number and the volume fraction of particles. A comparison of the temperature and the velocity profiles is conducted inside a thermal cavity with isothermal vertical walls at various Grashof numbers and volume fractions as shown in fig.4. This figure shows that the nanofluid behaves like a fluid and it illustrates the effect of Grashof number and the volume fraction on the temperature and the velocity profiles at the mid-sections of the cavity for water with a Prandtl number of 6.2.

The numerical results of the present study indicate that the heat transfer feature of a nanofluid increases remarkably with the volume fraction of nanoparticles. As the volume fraction increases, irregular and random movements of particles increases energy exchange rates in the fluid and consequently enhances the thermal dispersion in the flow of nanofluid.

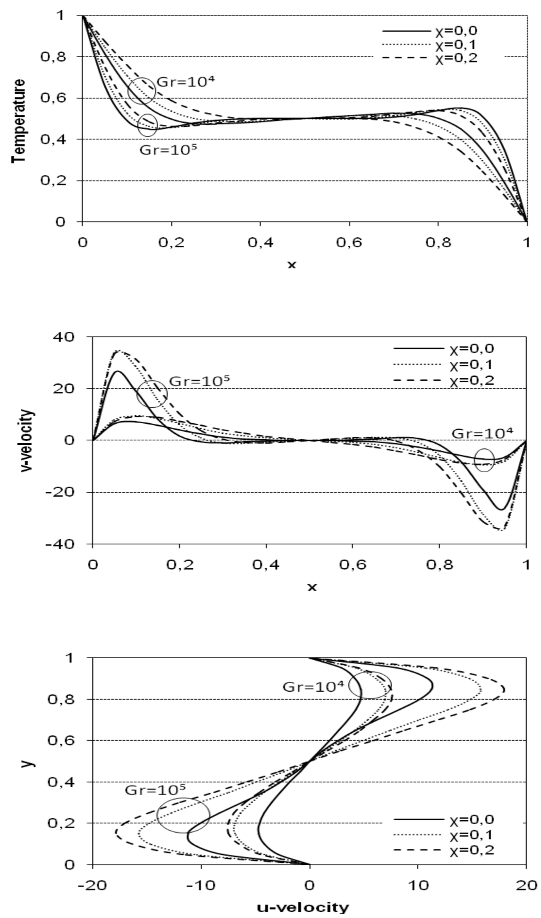

Figure 4: Comparison of the temperature and velocity profiles between nanofluid and pure fluid for various Grashof numbers ($Pr = 6.2$, $\chi = 10\%$ and 20%).

The effect of the volume fraction on the streamlines and isotherms of nanofluid for various Grashof numbers is shown in figs.5 and 6. They show that the intensity of the streamlines increase with an increase in the volume fraction as a result of high-energy transport through the flow associated with the irregular motion of the ultrafine particles.

The isotherms show that the vertical stratification of isotherms breaks down with an increase in the volume fraction for higher Grashof numbers. This is due to a number of effects such as gravity, Brownian motion.

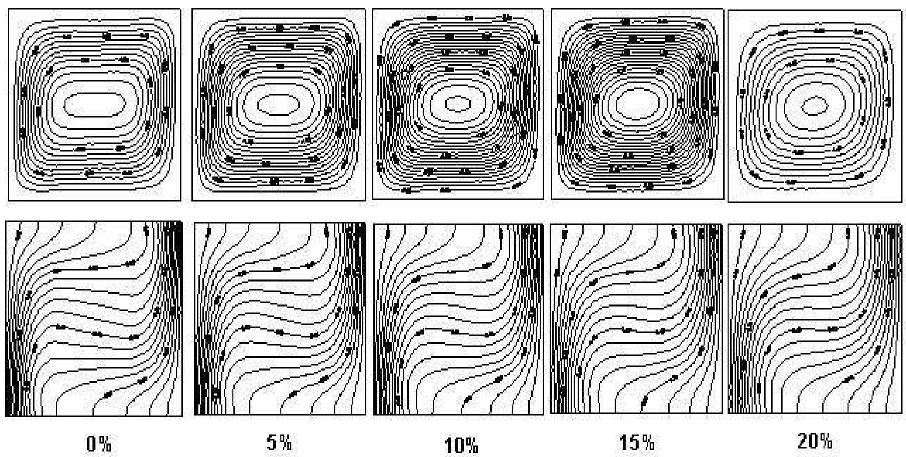

Figure 5: Streamlines contours and isotherms at various volume fractions ($Gr = 10^4$ and $Pr = 6.2$).

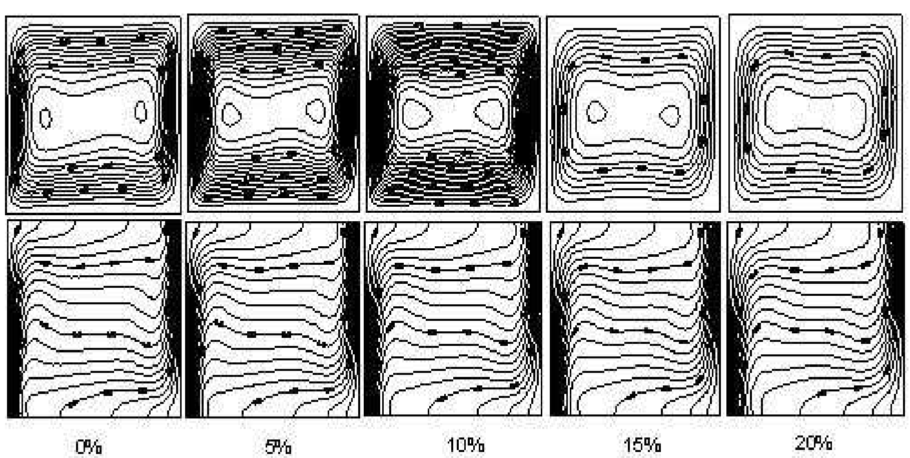

Figure 6: Streamlines contours and isotherms at various volume fractions ($Gr = 10^5$ and $Pr = 6.2$).

5 Conclusion

Heat transfer enhancement in a two-dimensional enclosure is studied numerically for a range of Grashof numbers and volume fractions. The present results illustrate that the suspended nanoparticles substantially increase the heat transfer rate with an increase in the nanoparticles volume fraction and at any given Grashof number.
This study shows that the results obtained by ADI numerical method have a good agreement with those obtained experimentally and numerically (finite volume method).

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Optimal harvesting policies for a fish population model

M. EL HOURMI[1], B. BOUAYAD[2], and E.H. EL MAZOUDI[3]
Laboratoire de Recherche en Economie de l'Energie, Environnement et Ressources
University Cadi Ayyad Marrakech, Morocco
e-mail[1]: elhourmi@yahoo.fr
e-mail[2]: bouayadb@yahoo.fr
e-mail[3]: h_mazoudi@yahoo.fr

Abstract

The present paper deals with the problem of the optimal harvesting of a renewable resource that can be continuously exploited by three actors using different fishing engines (trawler, coastal and artisanal vessels). The main purpose of the model is to analyze the interaction between optimal equilibrium stock and the distribution of fishing quotas in the steady state among the different agents exploiting the fishery.

Keywords: *Bioeconomics, optimal management, quota system, octopus.*

1 Introduction

We will analyze the problem of optimal management of fisheries resources under the quota system. We consider that the fishery is operated by three agents using different fishing engines (trawler, coastal and artisanal vessels). The main purpose of the model is to analyze the interaction between optimal equilibrium stock and the distribution of fishing quotas in the steady state among the different agents exploiting the fishery.
The issue will be treated under two hypotheses: first time, the resource price is constant and a second time, the price depends on the amount captured.

2 Model Formulation and determination of optimal solutions

The natural growth function of resource can be described by the following equation:

$$F(x) = rx[1 - \frac{x}{k}]. \quad (1)$$

Where $F(x)$ denotes the logistic equation, x represents the biomass at time t, K is called the carrying capacity of environment. The positive constant r is referred to as the intrinsic growth rate.
When fishery is exploited, the equation (1) is altered to:

$$\frac{dx}{dt} = F(x) - h(t). \quad (2)$$

with

$$h(t) = \sum_{i=1}^{3} h_i(t).$$

Where $i = 1, 2$ and 3 represent respectively the trawler costal and artisanal fleet's that are exploiting the resource, $h(t)$ is the total harvest rate, is the harvest rate of fleet i at time t, expressed as :

$$h_i(t) = G(E_i, x) = q_i E_i^s x^v. \quad (3)$$

Where E_i is the fishing effort of fleet i at time t.
$G(E_i, x)$ is the harvest function that links inputs (E_i and x) to the catch rate, $h_i(t)$, for the fleet i. s takes the values a, b and d depend on the fleet i and represent respectively the elasticities of harvest compared with fishing effort E_i, and as for v takes the values g, m and n dependent on the fleet i and represent respectively the elasticities of harvest compared with biomass size. q_i is the catchability coefficient which, for simplicity, is supposed to be constant for each fleet.
The total effort cost is expressed as:

$$C_i(E_i) = c_i E_i \quad (4)$$

Where c_i is the fleet i's unit cost of fishing effort.
From equation (3), we can write the cost function as follows:

$$C_i(h_i, x) = c_i E_i = C_i(x) h_i^{\frac{1}{s}} = C_i(x) \alpha_i^{\frac{1}{s}} h^{\frac{1}{s}} \quad (5)$$

with

$$C_i(x) = c_i(q_i x^g)^{-\frac{1}{S}}$$

Where $C_i(x)$ denote the unit cost of harvesting for each fleet related to the fleet i when the stock level is x. Let $\alpha_1 = \frac{h_1}{h}$, $\alpha_2 = \frac{h_2}{h}$, and $\alpha_3 = \frac{h_3}{h}$, respectively, shares of catching fleets trawler, coastal and artisanal, respectively . The equation (5) can be written as follows:

$$C_1(h_1, x) = C_1(x)h_1^{\frac{1}{b}} = C_1(x)(\alpha_1 h)^{\frac{1}{b}} = C_1(x)\beta_1 h^{\frac{1}{b}}$$

$$C_2(h_2, x) = C_2(x)h_2^{\frac{1}{d}} = C_2(x)(\alpha_2 h)^{\frac{1}{d}} = C_2(x)\beta_2 h^{\frac{1}{d}}$$

$$C_3(h_3, x) = C_3(x)h_3^{\frac{1}{n}} = C_3(x)(\alpha_3 h)^{\frac{1}{n}} = C_3(x)\beta_3 h^{\frac{1}{n}}$$

With $\beta_1 = \alpha_1^{\frac{1}{b}}$, $\beta_2 = \alpha_2^{\frac{1}{d}}$, and $\beta_3 = \alpha_3^{\frac{1}{n}}$
The discounted net cash flow from the fishery represents the objective function for all fleets, which can be expressed as:

$$\begin{aligned} VP &= \int_0^\infty e^{-\delta t}[(p\alpha_1 h - C_1(x)\beta_1 h^{\frac{1}{b}}) + (p\alpha_2 h - C_2(x)\beta_2 h^{\frac{1}{d}}) + (p\alpha_3 h - C_3(x)\beta_3 h^{\frac{1}{n}})]dt \\ &= \int_0^\infty e^{-\delta t}[ph - C_1(x)\beta_1 h^{\frac{1}{b}} - C_2(x)\beta_2 h^{\frac{1}{d}} - C_3(x)\beta_3 h^{\frac{1}{n}}]dt \end{aligned} \tag{6}$$

subject to:

$$\begin{aligned} \frac{dx}{dt} &= F(x) - h(t) \\ 0 &\leq h(t) \leq h_{max} \\ x(0) &= x_0 \end{aligned}$$

The total income for each fleet can be expressed as:

$$R_i(h) = p\alpha_i h(t) \tag{7}$$

Where p is the price of the harvest resource at time t (is a fixed constant), δ is the discount rate, h_{max} is maximum harvest rate and $x(0)$ is the initial population, assumed to be known.
Le Hamiltonian of our problem is:

$$\begin{aligned} H(x(t), h(t), \lambda(t)) &= e^{-\delta t}[ph - C_1(x)\beta_1 h^{\frac{1}{b}} - C_2(x)\beta_2 h^{\frac{1}{d}} - C_3(x)\beta_3 h^{\frac{1}{n}}] \\ &+ \lambda(t)[F(x) - h] + \rho(t)h - \phi(t)[h - h_{max}] \end{aligned} \tag{8}$$

where $\lambda(t)$, $\rho(t)$ and $\phi(t)$ represent the adjoints or costate variables.
The first order conditions of the problem are:

$$\frac{\partial H(.)}{\partial h(t)} = e^{-\delta t}[p - \frac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1} - \frac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1} - \frac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}] - \lambda(t) + \rho(t) - \phi(t) = 0 \tag{9}$$

$$-\frac{\partial H(.)}{\partial x(t)} = e^{-\delta t}[C_1'(x)\beta_1 h^{\frac{1}{b}} + C_2'(x)\beta_2 h^{\frac{1}{d}} + C_3'(x)\beta_3 h^{\frac{1}{n}}] - \lambda(t)F'(x) \tag{10}$$

$$\frac{\partial H(.)}{\partial \lambda(t)} = F(x) - h(t) \tag{11}$$

From equation (9), if:

$e^{-\delta t}[p - \frac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1} - \frac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1} - \frac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}] - \lambda(t) < 0$ with $\rho > 0$ and $\phi = 0$, then the harvest rate is zero;

$e^{-\delta t}[p - \frac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1} - \frac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1} - \frac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}] - \lambda(t) > 0$ with $\rho = 0$ and $\phi > 0$, then the harvest rate is maximal, $h(t) = h_{max}$;

$e^{-\delta t}[p - \frac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1} - \frac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1} - \frac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}] - \lambda(t) = 0$ with $\rho = \phi = 0$, then the harvest rate is positive, and should between 0 and h_{max} .

The general solution is the combination bang-bang and singular controls.
In developments that follow, we take the case of interior solution. After transformation (see Appendix 1):

$$\delta = F'(x^*) - \frac{C_1'(x^*)\beta_1 F(x^*)^{\frac{1}{b}} + C_2'(x^*)\beta_2 F(x^*)^{\frac{1}{d}} + C_3'(x^*)\beta_3 F(x^*)^{\frac{1}{n}}}{p - \frac{\beta_1}{b}C_1(x^*)F(x^*)^{\frac{1}{b}-1} - \frac{\beta_2}{d}C_2(x^*)F(x^*)^{\frac{1}{d}-1} - \frac{\beta_3}{n}C_3(x^*)F(x^*)^{\frac{1}{n}-1}} \quad (12)$$

Representing the equation that determines implicitly the resource stock.
This is the modified golden rule by the marginal effect of population, the discount rate and the relative variation of marginal profit.
If the elasticities of output with respect to fishing effort and resource are the unit for the three fleets (therefore $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, and $\beta_3 = \alpha_3$), then the equation (12) can be rewritten :

$$\delta = F'(x^*) - \frac{[C_1'(x^*)\alpha_1 + C_2'(x^*)\alpha_2 + C_3'(x^*)\alpha_3]F(x^*)}{p - \alpha_1 C_1(x^*) - \alpha_2 C_2(x^*) - \alpha_3 C_3(x^*)} \quad (13)$$

After transformation and consideration of explicit functional forms, we obtain:

$$\delta = r - \frac{2x^* r}{K} + \frac{[\frac{c_1\alpha_1}{q_1 x^{*2}} + \frac{c_2\alpha_2}{q_2 x^{*2}} + \frac{c_3\alpha_3}{q_3 x^{*2}}]rx^*(1 - \frac{x^*}{K})}{p - [\frac{c_1\alpha_1}{q_1 x^*} + \frac{c_2\alpha_2}{q_2 x^*} + \frac{c_3\alpha_3}{q_3 x^*}]} \quad (14)$$

Resulting that (See appendix 2):

$$\frac{2}{K}x^{*2} + x^*[\frac{\delta}{r} - 1 - \frac{\theta}{Kp}] - \frac{\delta\theta}{rp} = 0 \quad (15)$$

With

$$\theta = \frac{c_1\alpha_1 q_2 q_3 + c_2\alpha_2 q_1 q_3 + c_3\alpha_3 q_2 q_1}{q_1 q_2 q_3}$$

Solving the equation (15) admits a unique positive solution. As a result, the stock equilibrium is given by:

$$x^* = \frac{K}{4}[1 - \frac{\delta}{r} + \frac{\theta}{Kp} + \sqrt{(1 - \frac{\delta}{r} + \frac{\theta}{Kp})^2 + 8\frac{\delta\theta}{Krp}}] \quad (16)$$

Equation (16) shows that the share of optimal harvest affects the optimal biomass, x^*, through θ. This means that now the effects of quota sharing on the optimal stock will depend on their unit harvesting cost.
For the resource stock, given the initial period, the optimal policy capture can be described by:

$$\begin{array}{rcccl} h^*(t) & = & h_{max} & whenever & x > x^* \\ & = & F(x*) & whenever & x = x^* \\ & = & 0 & whenever & x > x^* \end{array} \quad (17)$$

If the country has monopoly power in the market, we can write the planner's problem as given by:

$$\begin{array}{rcl} VP & = & \int_0^\infty e^{-\delta t}[(p(h)\alpha_1 h - C_1(x)\beta_1 h^{\frac{1}{b}}) + (p(h)\alpha_2 h - C_2(x)\beta_2 h^{\frac{1}{d}}) \\ & + & (p(h)\alpha_3 h - C_3(x)\beta_3 h^{\frac{1}{n}})]dt \\ & = & \int_0^\infty e^{-\delta t}[p(h)h - C_1(x)\beta_1 h^{\frac{1}{b}} - C_2(x)\beta_2 h^{\frac{1}{d}} - C_3(x)\beta_3 h^{\frac{1}{n}}]dt \end{array} \quad (18)$$

subject to:

$$\begin{array}{rcl} \frac{dx}{dt} & = & F(x) - h(t) \\ 0 & \leq & h(t) \leq h_{max} \\ x(0) & = & x_0 \end{array}$$

Where $p(h)$ is the inverse demand function.
The present value Hamiltonian of this problem is given by:

$$\begin{array}{rcl} H(x(t), h(t), \lambda(t)) & = & e^{-\delta t}[p(h)h - C_1(x)\beta_1 h^{\frac{1}{b}} - C_2(x)\beta_2 h^{\frac{1}{d}} - C_3(x)\beta_3 h^{\frac{1}{n}}]dt \\ & + & \lambda(t)[F(x) - h] + \rho(t)h - \phi(t)[h - h_{max}] \end{array} \quad (19)$$

The optimality conditions are given by:

$$\begin{array}{rcl} \frac{\partial H(.)}{\partial h(t)} & = & e^{-\delta t}[p(h) + p'(h)h - \frac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1} - \frac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1} - \frac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}] \\ & - & \lambda(t) + \rho(t) - \phi(t) = 0 \end{array} \quad (20)$$

$$-\frac{\partial H(.)}{\partial x(t)} = e^{-\delta t}[C_1'(x)\beta_1 h^{\frac{1}{b}} + C_2'(x)\beta_2 h^{\frac{1}{d}} + C_3'(x)\beta_3 h^{\frac{1}{n}}] - \lambda(t)F'(x) \quad (21)$$

$$\frac{\partial H(.)}{\partial \lambda(t)} = F(x) - h(t) \quad (22)$$

In the case of a singular control, of the equation (20):

$$\lambda(t) = e^{-\delta t}[p(h) + p'(h)h - \frac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1} - \frac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1} - \frac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}] \quad (23)$$

After transformation (See Appendix 3), we obtain the following solution:

$$\delta = F'(x^*) - \frac{C_1'(x^*)\beta_1 h^{\frac{1}{b}} + C_2'(x^*)\beta_2 h^{\frac{1}{d}} + C_3'(x^*)\beta_3 h^{\frac{1}{n}}}{p(h) + p'(h)h - \frac{\beta_1}{b}C_1(x^*)h^{\frac{1}{b}-1} - \frac{\beta_2}{d}C_2(x^*)h^{\frac{1}{d}-1} - \frac{\beta_3}{n}C_3(x^*)h^{\frac{1}{n}-1}} \tag{24}$$

If the elasticities of output with respect to fishing effort and resource are the unit for the three fleets, then equation (24) can be written as follows:

$$\delta = F'(x^*) - \frac{[C_1'(x^*)\alpha_1 + C_2'(x^*)\alpha_2 + C_3'(x^*)\alpha_3]F(x^*)}{p(h) + p'(h)h - \alpha_1 C_1(x^*) - \alpha_2 C_2(x^*) - \alpha_3 C_3(x^*)} \tag{25}$$

With $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, and $\beta_3 = \alpha_3$. And can write that:

$$\delta = r - \frac{2x^*r}{K} - \frac{[\frac{c_1\alpha_1}{q_1x^{*2}} + \frac{c_2\alpha_2}{q_2x^{*2}} + \frac{c_3\alpha_3}{q_3x^{*2}}]rx^*(1-\frac{x^*}{K})}{p(h) + p'(h)h - [\frac{c_1\alpha_1}{q_1x^*} + \frac{c_2\alpha_2}{q_2x^*} + \frac{c_3\alpha_3}{q_3x^*}]} \tag{26}$$

After transformation(See appendix 4):

$$x^{*4}(\frac{4r\mu}{K^2}) + x^{*3}(\frac{2\delta\mu - 6r\mu}{K}) + x^{*2}(\frac{2\psi - 2K\delta\mu + 2rK\mu}{K}) + x^*(\frac{K\psi\delta - rK\psi - r\theta}{rK}) - \frac{\delta\theta}{r} = 0 \tag{27}$$

With $\theta = \frac{c_1\alpha_1q_2q_3 + c_2\alpha_2q_1q_3 + c_3\alpha_3q_2q_1}{q_1q_2q_3}$ and $p(h) = \psi - \mu h(t)$
Where ψ and μ are parameters of the inverse demand function. The equation (27) admits a numerical solution x^*.

3 Results

Before analyzing situations optimal exploitation of the resource octopus, we must submit technical data, biological and economic fisheries of cephalopods. According to the Fisheries Research Institute, the intrinsic growth rate r of octopus is estimated at 20% and the carrying capacity of environment of 1249373 tons.
The stock assessment of octopus on the model of logistic growth and utilization of the production function of Schaefer to determine the catchability coefficients $q_1 = 0.000184$, $q_2 = 0.0000817$, and $q_3 = 0.000236$ respectively fleets offshore, inshore and artisanal.
On the basis of annual catches by the three fleets between 1998 and 2000, the shares allocated annual quotas were established by public authorities as follows:

- The offshore fleet: $\alpha_1 = 51\%$;
- The coastal fleet: $\alpha_2 = 11\%$;

- The artisanal fleet: $\alpha_3 = 38\%$;

Based on the previously estimated parameters, solving the equation (16) determines the optimal biomass $x^* = 383960$ tons and therefore the optimal catch 53192 tons with an average landed price of 6 dollars per kilogram.
Starting with equal shares for each operator ($\alpha_1 = \alpha_2 = \alpha_3$), the equilibrium biomass is 381420 tons and the optimal harvest is 52995 tons. When the catch quota for the offshore fleet increases compared to that of the artisanal fleet (from $\alpha_1 = 33\%$ to $\alpha_1 = 61\%$), we find that the optimal biomass and capture the feel of balance by 1.1% and 0.6% . If we abandon the assumption of constant prices and we keep the same settings used in solving equation (16), the optimal stock is given by equation (27). The admissible solution is 1244300 tons.
The possibility of increasing the share of offshore fishery in relation to artisanal fishery will be beneficial to the recovery of the resource. This measure may be supported by reducing the artisanal fleet through conversion to other fisheries and / or cessation of activity against compensation.

4 Conclusion

In this paper we have examined the optimal management of a fishery under assumptions that the price is constant or depends on the amount captured, the natural growth function for the fishing resource depends on the own biomass and the sea's environmental conditions (considered to be stable and constant) and the differences in the harvesting cost are due to differences in the unit cost of the fishing effort.
We have demonstrated that the share of harvest affects the optimal biomass as long as the cost harvesting differs between agents.

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Appendix 1

Equation (9) is then reduced to:

$$\lambda(t) = e^{-\delta t}[p - \tfrac{\beta_1}{b} C_1(x) h^{\frac{1}{b}-1} - \tfrac{\beta_2}{d} C_2(x) h^{\frac{1}{d}-1} - \tfrac{\beta_3}{n} C_3(x) h^{\frac{1}{n}-1}] \qquad (A1)$$

Differentiating equation $(A1)$, with respect to time, we get:

$$\begin{array}{rcl} \dot{\lambda} & = & -\delta e^{-\delta t}[p - \tfrac{\beta_1}{b} C_1(x) h^{\frac{1}{b}-1} - \tfrac{\beta_2}{d} C_2(x) h^{\frac{1}{d}-1} - \tfrac{\beta_3}{n} C_3(x) h^{\frac{1}{n}-1}] \\ & + & e^{-\delta t}[-\tfrac{\beta_1}{b} C_1'(x) h^{\frac{1}{b}-1}\dot{x} - \tfrac{\beta_1(1-b)}{b^2} C_2(x) h^{\frac{1}{b}-2}\dot{h}] \\ & + & e^{-\delta t}[-\tfrac{\beta_2}{d} C_2'(x) h^{\frac{1}{d}-1}\dot{x} - \tfrac{\beta_2(1-d)}{d^2} C_2(x) h^{\frac{1}{d}-2}\dot{h} \\ & - & \tfrac{\beta_3}{n} C_3'(x) h^{\frac{1}{n}-1}\dot{x} - \tfrac{\beta_3(1-n)}{n^2} C_3(x) h^{\frac{1}{n}-2}\dot{h}] \end{array} \qquad \text{(A2)}$$

Substituting equation $(A1)$ in (10), we get:

$$\begin{array}{rcl} \dot{\lambda} & = & e^{-\delta t}[C_1'(x)\beta_1 h^{\frac{1}{b}} + C_2'(x)\beta_2 h^{\frac{1}{d}} + C_3'(x)\beta_3 h^{\frac{1}{n}}] \\ & - & e^{-\delta t}[p - \tfrac{\beta_1}{b} C_1(x) h^{\frac{1}{b}-1} - \tfrac{\beta_2}{d} C_2(x) h^{\frac{1}{d}-1} - \tfrac{\beta_3}{n} C_3(x) h^{\frac{1}{n}-1}]F'(x) \end{array} \qquad \text{(A3)}$$

By equating $(A2)$ and $(A3)$:

$$\begin{array}{rl} & (F'(x) - \delta)[p - \tfrac{\beta_1}{b} C_1(x) h^{\frac{1}{b}-1} - \tfrac{\beta_2}{d} C_2(x) h^{\frac{1}{d}-1} - \tfrac{\beta_3}{n} C_3(x) h^{\frac{1}{n}-1}] \\ - & [C_1'(x)\beta_1 h^{\frac{1}{b}} + C_2'(x)\beta_2 h^{\frac{1}{d}} + C_3'(x)\beta_3 h^{\frac{1}{n}}] \\ + & [-\tfrac{\beta_1}{b} C_1'(x) h^{\frac{1}{b}-1}\dot{x} - \tfrac{\beta_1(1-b)}{b^2} C_2(x) h^{\frac{1}{b}-2}\dot{h}] \\ + & [-\tfrac{\beta_2}{d} C_2'(x) h^{\frac{1}{d}-1}\dot{x} - \tfrac{\beta_2(1-d)}{d^2} C_2(x) h^{\frac{1}{d}-2}\dot{h} - \tfrac{\beta_3}{n} C_3'(x) h^{\frac{1}{n}-1}\dot{x} - \tfrac{\beta_3(1-n)}{n^2} C_3(x) h^{\frac{1}{n}-2}\dot{h}] \end{array} \qquad \text{(A4)}$$

Starting from the steady state, $\dot{x} = \dot{h} = 0$, the equation $(A4)$ can be written:

$$\begin{array}{rl} & (F'(x^*)-\delta)[p-\frac{\beta_1}{b}C_1(x^*)F(x^*)^{\frac{1}{b}-1}-\frac{\beta_2}{d}C_2(x^*)F(x^*)^{\frac{1}{d}-1}-\frac{\beta_3}{n}C_3(x^*)F(x^*)^{\frac{1}{n}-1}] \\ - & [C_1'(x^*)\beta_1F(x^*)^{\frac{1}{b}}+C_2'(x^*)\beta_2F(x^*)^{\frac{1}{d}}+C_3'(x^*)\beta_3F(x^*)^{\frac{1}{n}}]=0 \quad \text{(A5)}\end{array}$$

Appendix 2

The equation (14) implies that:

$$[\delta-r+\frac{2rx^*}{K}]=\frac{r[\frac{c_1\alpha_1q_2q_3+c_2\alpha_2q_1q_3+c_3\alpha_3q_2q_1}{q_1q_2q_3x^*}](1-\frac{x^*}{K})}{p-[\frac{c_1\alpha_1q_2q_3+c_2\alpha_2q_1q_3+c_3\alpha_3q_2q_1}{q_1q_2q_3x^*}]} \qquad (A6)$$

Assume that $\theta=\frac{c_1\alpha_1q_2q_3+c_2\alpha_2q_1q_3+c_3\alpha_3q_2q_1}{q_1q_2q_3}$, the equation $(A6)$ can be rewritten as follows:

$$[\frac{\delta}{r}-1+\frac{2x^*}{K}]=\frac{[\frac{\theta}{x^*}](1-\frac{x^*}{K})}{p-[\frac{\theta}{x^*}]}$$

$$p[\frac{x^*\delta}{r}-x^*+\frac{2x^{*2}}{K}-\frac{\delta\theta}{rp}-\frac{\theta x^*}{Kp}]=0 \qquad \text{(A7)}$$

Since the price is positive and non-zero, $p>0$, then $(A7)$:

$$\frac{x^*\delta}{r}-x^*+\frac{2x^{*2}}{K}-\frac{\delta\theta}{rp}-\frac{\theta x^*}{Kp}=0$$

With

$$\theta=\frac{c_1\alpha_1q_2q_3+c_2\alpha_2q_1q_3+c_3\alpha_3q_2q_1}{q_1q_2q_3} \qquad \text{(A8)}$$

Appendix 3

Differentiating equation (23), with respect to time, we get:

$$\begin{array}{rcl}\dot\lambda & = & -\delta e^{-\delta t}[p(h)+p'(h)h-\frac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1}-\frac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1}-\frac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}] \\ & + & e^{-\delta t}[p'(h)\dot h+p'(h)\dot h+p''(h)h\dot h-\frac{\beta_1}{b}C_1'(x)h^{\frac{1}{b}-1}\dot x-\frac{\beta_1(1-b)}{b^2}C_1(x)h^{\frac{1}{b}-2}\dot h] \\ & + & e^{-\delta t}[-\frac{\beta_2}{d}C_2'(x)h^{\frac{1}{d}-1}\dot x-\frac{\beta_2(1-d)}{d^2}C_2(x)h^{\frac{1}{d}-2}\dot h-\frac{\beta_3}{n}C_3'(x)h^{\frac{1}{n}-1}\dot x \\ & - & \frac{\beta_3(1-n)}{n^2}C_3(x)h^{\frac{1}{n}-2}\dot h] \qquad \text{(A9)}\end{array}$$

The substitution of $\lambda(t)$, equation (23), in equation (21) allows to write:

$$\begin{array}{rcl}\dot\lambda & = & e^{-\delta t}[C_1'(x)\beta_1h^{\frac{1}{b}}+C_2'(x)\beta_2h^{\frac{1}{d}}+C_3'(x)\beta_3h^{\frac{1}{n}}] \\ & - & e^{-\delta t}[p(h)+p'(h)h-\frac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1}-\frac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1}-\frac{\beta_1}{n}C_3(x)h^{\frac{1}{n}-1}]F'(x) \quad \text{(A10)}\end{array}$$

By equating the two equations $(A9)$ and $(A10)$, we obtain:

$$(F'(x)-\delta)[p(h)+p'(h)h-\tfrac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1}-\tfrac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1}-\tfrac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}]$$
$$-[C_1'(x)\beta_1h^{\frac{1}{b}}+C_2'(x)\beta_2h^{\frac{1}{d}}+C_3'(x)\beta_3h^{\frac{1}{n}}]$$
$$+[2p'(h)\dot{h}+p''(h)h\dot{h}-\tfrac{\beta_1}{b}C_1'(x)h^{\frac{1}{b}-1}\dot{x}-\tfrac{\beta_1(1-b)}{b^2}C_1(x)h^{\frac{1}{b}-2}\dot{h}]$$
$$+[-\tfrac{\beta_2}{d}C_2'(x)h^{\frac{1}{d}-1}\dot{x}-\tfrac{\beta_2(1-d)}{d^2}C_2(x)h^{\frac{1}{d}-2}\dot{h}$$
$$-\tfrac{\beta_3}{n}C_3'(x)h^{\frac{1}{n}-1}\dot{x}-\tfrac{\beta_3(1-n)}{n^2}C_3(x)h^{\frac{1}{n}-2}\dot{h}] \qquad \text{(A11)}$$

At steady state, $\dot{x}=\dot{h}=0$, the equation $(A11)$ becomes:

$$(F'(x)-\delta)[p(h)+p'(h)h-\tfrac{\beta_1}{b}C_1(x)h^{\frac{1}{b}-1}-\tfrac{\beta_2}{d}C_2(x)h^{\frac{1}{d}-1}-\tfrac{\beta_3}{n}C_3(x)h^{\frac{1}{n}-1}]$$
$$-[C_1'(x)\beta_1h^{\frac{1}{b}}+C_2'(x)\beta_2h^{\frac{1}{d}}+C_3'(x)\beta_3h^{\frac{1}{n}}] \qquad \text{(A12)}$$

Appendix 4

Assume that $\theta=\frac{c_1\alpha_1q_2q_3+c_2\alpha_2q_1q_3+c_3\alpha_3q_2q_1}{q_1q_2q_3}$ and $p(h)=\psi-\mu h(t)$ is the inverse demand function, the equation (4.36) is rewritten as follows:

$$[1-\tfrac{\delta}{r}-\tfrac{2x}{K}] = \frac{[\frac{\theta}{x}][1-\frac{x}{k}]}{\psi-2h\mu-[\frac{\theta}{x}]}$$

$$[1-\tfrac{\delta}{r}-\tfrac{2x}{K}]=\frac{\theta(1-\frac{x}{K})}{x(\psi-2h\mu)-\theta} \qquad \text{(A13)}$$

Substituting $h(t)$ by $F(x)$, we obtain:

$$[x\psi-2rx^2\mu+\tfrac{2rx^3\mu}{K}-\theta](\tfrac{\delta}{r}-1+\tfrac{2x}{K})=\theta(1-\tfrac{x}{K}) \qquad \text{(A14)}$$

$$\tfrac{4rx^4\mu}{K^2}-\tfrac{6rx^3\mu}{K}+\tfrac{2x^3\delta\mu}{K}+\tfrac{2x^2\psi}{K}-2x^2\delta\mu+2rx^2\mu+\tfrac{x\psi\delta}{r}-x\psi-\tfrac{r\theta}{k}-\tfrac{\delta\theta}{r}=0 \qquad \text{(A15)}$$

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Existence of Bounded Solutions for a Nonlinear Parabolic System with Nonlinear Gradient Term.

Hamid El Ouardi
Equipe Architectures des Systèmes
Université Hassan II Ain Chock
ENSEM, BP. 9118, Oasis, Casablanca, Morocco
e-mail: h.elouardi@ensem.ac.ma

Abstract

In this note we show the existence of bounded solutions of the nonlinear parabolic system

$$(u_1)_t - \Delta_{p_1} u_1 = a_1(x)\,|\nabla u_1|^{p_1} + f_1(x, u_2)$$
$$(u_2)_t - \Delta_{p_2} u_2 = a_2(x)\,|\nabla u_2|^{p_2} + f_2(x, u_1)$$

where $\Delta_s z = \operatorname{div}(|\nabla z|^{s-2}\nabla z)$***,*** $s > 1$ ***is the s-Laplacian operator and*** a_i***,*** f_i ***are given functions,*** $i = 1, 2$***.***

Keywords: *Nonlinear parabolic systems; nonlinear gradients terms; p-Laplacian; existence and bounded solutions.*

1 Introduction

Let Ω be an open and bounded subset in $\mathbb{R}^N$ with smooth boundary Γ and let T be a positive real number. In the cylinder $Q_T = \Omega \times]0, T[$, with lateral boundary $S_T = \Gamma \times]0, T[$, we consider the nonlinear system ($\boldsymbol{S}$)

$$\frac{\partial u_1}{\partial t} - \Delta_{p_1} u_1 = a_1(x)\,|\nabla u_1|^{p_1} + f_1(x, u_2) \quad (x, t) \in Q_T, \tag{1}$$
$$\frac{\partial u_2}{\partial t} - \Delta_{p_2} u_2 = a_2(x)\,|\nabla u_2|^{p_2} + f_2(x, u_1) \quad (x, t) \in Q_T, \tag{2}$$
$$u_1(x, t) = u_2(x, t) = 0 \quad (x, t) \in S_T, \tag{3}$$
$$(u_1(x, 0), u_2(x, 0)) = (u_{10}(x), u_{20}(x)) \quad x \in \Omega, \tag{4}$$

where $\Delta_{p_i} z = \text{div}(|\nabla z|^{p_i - 2} \nabla z)$, $p_i > 1, i = 1, 2$. Precise conditions on a_i, f_i and u_{i0} will be given later.

The prototype systems $(\boldsymbol{S})$ is only weakly coupled in the reaction terme f_i' s and turns up in many mathematical settings as non-Newtonian fluids, nonlinear filtration, population evolution, reaction diffusion problems, porus media and so forth. Therefore, it is important to obtain information about the existence of solutions for this problem.

When $a_i \equiv 0$, much attention has been given to the existence and the regularity of solutions of systems $(\boldsymbol{S})$, by using different approaches (see, for exemple, [12, 15] and references therein.

The case of a single equation of the type $(\mathbf{S})$ is studied in [10, 11]. The purpose of this paper is the natural extension to system $(\mathbf{S})$ of the result by [5], which concerns the single equation $\frac{\partial u}{\partial t} - \Delta_p u = d\,|\nabla u|^p + f(x,t)$.

Notation

We represent the Sobolev space of order m in Ω by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \forall |\alpha| \le m\},$$

with the norm

$$\|u\|_{m,p} = \Big(\sum_{|\alpha| \le m} |D^\alpha u|^p_{L^p(\Omega)} \Big)^{1/p}, u \in W^{m,p}(\Omega), 1 \le p < \infty.$$

Let $\mathcal{D}(\Omega)$ be the space of test functions in Ω and by $W_0^{m,p}(\Omega)$ we represent the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$. The dual space of $W_0^{m,p}(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$ with p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. We use the symbols $(\cdot,\cdot)$ and $|\cdot|$, to indicate the inner product and the norm in $L^2(\Omega)$. We use $\langle \cdot,\cdot \rangle_{W^{-1,p}(\Omega), W_0^{1,p}(\Omega)}$ to indicate the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $\|\cdot\|_0$ to indicate the norm $W_0^{1,p}(\Omega)$. The p_i - Laplacian operator $\mathcal{A}_i$ is such that for $i = 1, 2$

$$\begin{array}{rccc} \mathcal{A}_i : & W_0^{1,p_i}(\Omega) & \to & W^{-1,p_i'}(\Omega) \\ & u & \mapsto & -\Delta_{p_i} u \end{array}$$

and it satisfies the following properties:

- $\mathcal{A}_i$ is monotonic, that is, $\langle \mathcal{A}_i u - \mathcal{A}_i v, u - v \rangle \ge 0, \forall u, v \in W_0^{1,p}(\Omega)$;
- $\mathcal{A}_i$ is hemicontinuous, that is, for each $u, v, w \in W_0^{1,p}(\Omega)$ the function $\lambda \mapsto \langle \mathcal{A}_i(u + \lambda v), w \rangle$ is continuous in $\mathbb{R}$;
- $\langle \mathcal{A}_i u(t), u(t) \rangle_{W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)} = \|u\|_0^p$;
- $\langle \mathcal{A}_i u(t), u'(t) \rangle_{W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)} = \frac{1}{p} \frac{d}{dt} \|u\|_0^p$;
- $\|\mathcal{A}_i u(t)\|_{W^{-1,p'}(\Omega)} \le C \|u\|_0^{p-1}$, where C is a constant.

2 Assumptions and main results

First we specify our notion of weak solution.

Definition 2.1 *A pair* $(u_1,\ u_2)$ *is said to be a weak solution of* $(\boldsymbol{S})$ *if for* $i = 1, 2$

$$u_i \in C(0,T; L^2(\Omega));$$
$$\frac{\partial u_i}{\partial t} \in L^{p_i'}(0,T; W^{-1,p_i'}(\Omega)) + L^1(Q_T);$$
$$\int_\Omega u_i(x,\tau) w_i(x,\tau)dx - \int_\Omega u_i(x,0)w_i(x,0)dx - \int_0^\tau \langle v_{i1}, u_i\rangle dt - \int_0^\tau \int_\Omega v_{i2} u_i dxdt +$$
$$\int_0^\tau \int_\Omega |\nabla u_i|^{p_i-2} \nabla u_i \nabla w_i dxdt = \int_0^\tau \int_\Omega a_i(x) |\nabla u_i|^{p_i} w_i dxdt + \int_0^\tau \int_\Omega f_i(x,u) w_i dxdt.$$
$$\forall \tau \in [0,T], \forall w_i \in L^\infty(\Omega \times (0,\tau)) \cap L^{p_i}(0,\tau; W_0^{1,p_i}(\Omega))$$
$$\text{and} \quad \frac{\partial w_i}{\partial t} = v_{i1} + v_{i2} \in L^{p'}(0,\tau; W^{-1,p'}(\Omega)) + L^1(\Omega \times (0,\tau)).$$

We consider the following assumptions on the data:

(H1) $p_i \in [2, N[\ , (i = 1,2)$.

(H2) $u_{i0} \in L^{+\infty}(\Omega), (i = 1,2)$.

(H3) $a_i \in L^\infty(\Omega), (i = 1,2)$.

(H4) $f_i \in C^1\Omega \times (R), (i = 1,2)$.

The next lemma plays a central role in the proof of the existence theorem. Its proof can be found in [10].

Lemma 2.2 *For every* $\beta, f \in L^\infty(\Omega)$, $0 \le \beta(s) \le M, \forall s \in \mathbb{R}$ *and* $\alpha \in [2, N]$, *the problem*

$$u_t - \Delta_\alpha u = \beta(u) |\nabla u|^\alpha + f, \quad u = 0 \text{ on } \partial\Omega,$$

possesses a solution u *such that* $u \in L^\infty(Q_T) \cap L^\infty(0,T; L^2(\Omega)) \cap L^\alpha(0,T; W_0^{1,\alpha}(\Omega))$.

3 Existence of weak bounded solutions

Our main result is the following:

Theorem 3.1 *Let (H1) to (H4) be satisfied. Then there exists at least one weak bounded solution* (u_1, u_2) *of problem (S) such that for* $i = 1, 2$ *, we have* $u_i \in L^{p_i}(0,T; W_0^{1,p_i}(\Omega)) \cap C(0,T; L^{q_i}(\Omega)) \cap L^\infty(Q_T)$, *for all* $q_i \in [1, +\infty)$.

Denote $f_i(x,u_1,u_2)=\begin{cases} f_1(x,u_2) & \text{if } i=1 \\ f_2(x,u_1) & \text{if } i=2 \end{cases}$

We will prove Theorem 1 by means of nonlinear parabolic regularization.

Starting from a suitable initial iteration $(u_1^0,u_2^0)=(u_{10},u_{20})$, we construct a sequence $\{(u_1^n,u_2^n)\}_{n=1}^{\infty}$ from the iteration process

$$\frac{\partial u_i^n}{\partial t}+\mathcal{A}_i u_i^n = a_i(x)\min\{|\nabla u_i^n|^{p_i},n\}+f_i(x,u_1^{n-1},u_2^{n-1}) \quad (x,t)\in Q_T, \tag{5}$$
$$u_i^n(x,t)=0 \quad (x,t)\in S_T, \tag{6}$$
$$u_i^n(x,0)=u_{i0}(x) \quad x\in\Omega \tag{7}$$

where $i=1,2$. It is clear that for each $n=1,2,...,$ the above systems consists of two uncoupled initial boundary-value problems. By classical results, the existence of weak solution $u_i^n\in C(0,T;L^2(\Omega))\cap L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$ follows from [17]. By Lemma 1 we assert that

$$u_i^n\in L^\infty(Q_T), i=1,2. \tag{8}$$

To find a limit function $(u_1(x,t),u_2(x,t))$ of $(u_1^n(x,t),u_2^n(x,t))$ we will divide our proof in the following four lemmas.

Lemma 3.2 *There exist a constant c independent of n such that for $\tau\in[0,T]$*

$$\sup_{0\le\tau\le T}\int_\Omega |u_i^n(x,\tau)|^2\,dx+\int_0^T\int_\Omega |\nabla u_i^n|^{p_i}\,dxdt\le c. \tag{9}$$

Proof Since $u_i^n\in L^\infty(Q_T)\cap L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$, $\sinh(\lambda u_i^n)\in L^\infty(Q_T)\cap L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$ $(\lambda=\max(\|a_1\|_\infty,\|a_2\|_\infty))$ is a testing function for (5). For each $\tau\in[0,T]$, we derive

$$\int_\Omega\int_0^{u_i^n(x,\tau)}\sinh(\lambda s)dsdx+\lambda\int_0^\tau\int_\Omega\cosh(\lambda u_i^n)\,|\nabla u_i^n|^{p_i}\,dxdt+\lambda\int_0^\tau\int_\Omega|\sinh(\lambda u_i^n)|\,|\nabla u_i^n|^{p_i}\,dxdt$$

$$\int_0^\tau\int_\Omega|\sinh(\lambda u_i^n)|\left|f_i(x,u_1^{n-1},u_2^{n-1})\right|dxdt+\int_\Omega\int_0^{u_{i0}(x)}\sinh(\lambda s)dsdx. \tag{10}$$

It is not difficult to cheek that

$$\int_0^w\sinh(\lambda s)ds=\frac{1}{\lambda}\left[\cosh(\lambda w)-1\right]\ge\frac{\lambda}{2}(w)^2, \tag{11}$$
$$\cosh(s)\ge|\sinh(s)|,\cosh(s)\ge1. \tag{12}$$

By (8), (H2), (H4), (11), (12), we obtain

$$\int_\Omega |u_i^n|^2 (x,\tau)dx + \int_0^T \int_\Omega |\nabla u_i^n|^{p_i} dxdt \leq c_i(T). \tag{13}$$

Taking the supremum for $\tau \in [0,T]$, we have, for $\forall n \in \mathbb{N}$

$$\sup_{0\leq\tau\leq T} \int_\Omega |u_i^n|^2 (x,\tau)dx + \int_0^T \int_\Omega |\nabla u_i^n|^{p_i} dxdt \leq c_i(T). \tag{14}$$

This proves (9).

By lemma 2, there exit a subsequence $\{u_i^n\}, i = 1,2$ (denoted again by $\{u_i^n\}$) and a function $u_i(x,t) \in L^{p_i}(0,T;W_0^{1,p_i}(\Omega)) \cap L^\infty(Q_T)$ such that as $n \to +\infty$,

$$u_i^n \rightharpoonup u_i \text{ weakly in } L^{p_i}(0,T;W_0^{1,p_i}(\Omega)); \tag{15}$$
$$u_i^n \rightharpoonup u_i \text{ weakly * in } L^\infty(Q_T), \tag{16}$$

and u_i satisfies (8) and (9) by the weak lower semicontinuity. ■

Lemma 3.3

$$u_i^n \rightharpoonup u_i \textit{ strongly in } L^{p_i}(Q_T); \tag{17}$$
$$u_i^n \rightharpoonup u_i \textit{ a.e. in } Q_T. \tag{18}$$

Proof We have $\frac{\partial u_i^n}{\partial t} = -\mathcal{A}_i u_i^n + \left[a_i(x)\min\{|\nabla u_i^n|^{p_i}, n\} + f_i(x, u_1^{n-1}, u_2^{n-1})\right]$. By (8) and (14) we derive

$$\|\mathcal{A}_i u_i^n\|_{L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega))} \leq c\,; \tag{19}$$
$$\left\|a_i(x)\min\{|\nabla u_i^n|^{p_i}, n\} + f_i(x, u_1^{n-1}, u_2^{n-1})\right\|_{L^1(Q_T)} \leq c. \tag{20}$$

By virtue of lemma 4.2 in [5] we obtain

$$u_i^n \rightharpoonup u_i \text{ strongly in } L^{p_i}(Q_T). \tag{21}$$

Taking a subsequence of $\{u_i^n\}, i = 1,2$ (denoted again by $\{u_i^n\}$) further, we have

$$u_i^n \rightharpoonup u_i \text{ a.e. in } Q_T. \tag{22}$$

By Vitali's theorem, we have, for any $r \in (1,+\infty)$,

$$f_i(., u_1^n, u_2^n) \rightharpoonup f_i(., u_1, u_2) \text{ strongly in } L^r(Q_T). \quad ■ \tag{23}$$

Lemma 3.4 $\nabla u_i^n \rightharpoonup \nabla u_i$ *a.e. in* Q_T.

Proof Fix $\mu, \varepsilon > 0$. Due to (22), Egoroff's theorem implies that there exists a measurable set $A_\varepsilon \subset Q_T$ such that $L^{N+1}(Q_T \backslash A_\varepsilon) \leq \varepsilon$ and $u_i^n \rightharpoonup u_i$ uniformly on A_ε, which follows that

$$|u_i^n \rightharpoonup u_i^m| < \mu \text{ on } A_\varepsilon \tag{24}$$

if $n, m > M$. Let ξ be a cutoff function such that $\xi \equiv 1$ on A_ε, $spt\xi \subset Q_T$.

By subtracting $(5)_n$ and $(5)_m$ we have

$$\frac{\partial u_i^n}{\partial t} - \frac{\partial u_i^m}{\partial t} + (\mathcal{A}_i u_i^n - \mathcal{A}_i u_i^m) =$$

$$a_i(x)\left(\min\left\{|\nabla u_i^n|^{p_i}, n\right\} - \min\left\{|\nabla u_i^m|^{p_i}, m\right\}\right)$$

$$+ f_i(x, u_1^{n-1}, u_2^{n-1}) - f_i(x, u_1^{m-1}, u_2^{m-1}). \tag{25}$$

Choosing a testing function $\xi T_\varepsilon(u_i^n - u_i^m) = \xi \max\left\{-\varepsilon, \min\left\{(u_i^n - u_i^m), \varepsilon\right\}\right\}$ for (25) and noting that T_ε is an odd function satisfying $|T_\varepsilon| \leq \varepsilon$, we conclude that

$$\begin{aligned}
&\int_{Q_T} \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n - \nabla u_i^m)\xi T_\varepsilon'(u_i^n - u_i^m)\, dz \\
&\leq \int_{Q_T} (u_i^n - u_i^m) T_\varepsilon(u_i^n - u_i^m)\xi_t dz \\
&+ \int_{Q_T} \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).\nabla\left(\xi T_\varepsilon(u_i^n - u_i^m)\right)\, dz \\
&+ \int_{Q_T} a_i(x)\left(\min\left\{|u_i^n|^{p_i}, n\right\} - \min\left\{|u_i^m|^{p_i}, m\right\}\right)\xi T_\varepsilon(u_i^n - u_i^m)\, dz \\
&+ \int_{Q_T} \left(f_i(x, u_1^{n-1}, u_2^{n-1}) - f_i(x, u_1^{m-1}, u_2^{m-1})\right)\xi T_\varepsilon(u_i^n - u_i^m)\, dz \\
&\leq c_i(\mu)\varepsilon.
\end{aligned}$$

By virtue of (8) and (9). It follows that from (23) that

$$\lim_{n,m\to+\infty} \sup \int_{A_\varepsilon} \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n - \nabla u_i^m) dz \leq C_i(\mu)\varepsilon.$$

By the arbitrariness of ε it results that

$$\lim_{n,m\to+\infty} \sup \int_{A_\varepsilon} \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m\right).(\nabla u_i^n - \nabla u_i^m) dz = 0. \tag{26}$$

Since $p_i > 1$, we obtain

$$\int_{A_\varepsilon} |\nabla u_i^n - \nabla u_i^m|^{p_i}\, dz \le c \int_{A_\varepsilon} \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m \right).(\nabla u_i^n - \nabla u_i^m) dz.$$

Therefore it follows from (26) that

$$\lim_{n,m\to+\infty} \sup \int_{A_\varepsilon} |\nabla u_i^n - \nabla u_i^m|^{p_i}\, dz = 0. \tag{27}$$

We deduce that

$$\nabla u_i^n \rightharpoonup \nabla u_i \ \ a.e. \text{ in } \ A_\varepsilon.$$

This is true for each $\varepsilon > 0$ and so

$$\nabla u_i^n \rightharpoonup \nabla u_i \ \ a.e. \text{ in } \ Q_T. \tag{28}$$

By (9) and Vitali's theorem, we have, for any $r_i \in (1, p_i)\,, i = 1, 2$

$$\nabla u_i^n \rightharpoonup \nabla u_i \ \ strongly\ in \ \ L^{r_i}(Q_T), i = 1, 2. \ \ \blacksquare \tag{29}$$

Lemma 3.5 $\nabla u_i^n \rightharpoonup \nabla u_i$ *strongly in* $L^{p_i}(Q_T), i = 1, 2.$

Taking a testing function $sinh(\lambda(u_i^n - u_i^m)) \in L^\infty(Q_T) \cap L^{p_i}(0, T; W_0^{1,p_i}(\Omega))$ for (25) ($\lambda = \max\left(\|a_1\|_\infty, \|a_2\|_\infty\right) + 1$), we deduce that

$$\lambda \int_{Q_T} \cosh(\lambda(u_i^n - u_i^m)) \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m \right).(\nabla u_i^n - \nabla u_i^m) dz$$

$$\le A \int_{Q_T} \sinh(\lambda(u_i^n - u_i^m)) \left(|\nabla u_i^n|^{p_i} + |\nabla u_i^m|^{p_i} \right) dz +$$

$$\int_{Q_T} |\sinh(\lambda(u_i^n - u_i^m))| \left| \left(f_i(x, u_1^{n-1}, u_2^{n-1}) - f_i(x, u_1^{m-1}, u_2^{m-1}) \right) \right| dz, \tag{30}$$

where $A = \max\left(\|a_1\|_\infty, \|a_2\|_\infty\right).$

Since $\sinh(\lambda s)$ is an odd function. The above inequality becomes

$$\begin{aligned} &\int_{Q_T} \lambda \cosh(\lambda(u_i^n - u_i^m)) dz \\ &- \int_{Q_T} A\sinh(\lambda(u_i^n - u_i^m)) \nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i^m|^{p_i-2}\nabla u_i^m.(\nabla u_i^n - \nabla u_i^m) dz \\ &\le A \int_{Q_T} |\sinh(\lambda(u_i^n - u_i^m))| \left(|\nabla u_i^n|^{p_i-2}\, \nabla u_i^n.\nabla u_i^m + |\nabla u_i^m|^{p_i-2}\, \nabla u_i^m.\nabla u_i^n \right) dz \\ &\quad + \int_{Q_T} |\sinh(\lambda(u_i^n - u_i^m))| \left| \left(f_i(x, u_1^{n-1}, u_2^{n-1}) - f_i(x, u_1^{m-1}, u_2^{m-1}) \right) \right| dz. \end{aligned} \tag{31}$$

Recalling (28) and let $m \to +\infty$, by Fatou's lemma we deduce that

$$\int_{Q_T} \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i|^{p_i-2}\nabla u_i\right).(\nabla u_i^n - \nabla u_i)dz$$

$$\leq A\int_{Q_T} |\sinh(\lambda(u_i^n - u_i))| \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n.\nabla u_i + |\nabla u_i^m|^{p_i-2}\nabla u_i.\nabla u_i^n\right)dz$$

$$+\int_{Q_T} |\sinh(\lambda(u_i^n - u_i))| \left|\left(f_i(x,u_1^{n-1},u_2^{n-1}) - f_i(x,u_1,u_2)\right)\right| dz = J_1 + J_2 \quad (32)$$

And we use (8) and (9) to estimate J_1 and J_2 as below. For J_1, by Hölder inequality we have

$$\begin{aligned} J_1 \;\leq\; & A\left(\int_{Q_T} |\sinh(\lambda(u_i^n - u_i))|^{p_i} |\nabla u_i|^{p_i} dz\right)^{\frac{1}{p_i}} \left(\int_{Q_T} |\nabla u_i^n|^{p_i})dz\right)^{\frac{p_i-1}{p_i}} \\ & +A\left(\int_{Q_T} |\sinh(\lambda(u_i^n - u_i))|^{\frac{p_i-1}{p_i}} |\nabla u_i|^{p_i} dz\right)^{\frac{p_i-1}{p_i}} \left(\int_{Q_T} |\nabla u_i^n|^{p_i})dz\right)^{\frac{1}{p_i}} \\ \leq\; & C\left(\int_{Q_T} |\sinh(\lambda(u_i^n - u_i))|^{p_i} |\nabla u_i|^{p_i} dz\right)^{\frac{1}{p_i}} + \\ & +C\left(\int_{Q_T} |\sinh(\lambda(u_i^n - u_i))|^{p_i'} |\nabla u_i|^{p_i} dz\right)^{\frac{1}{p_i'}}. \end{aligned} \quad (33)$$

Since $|\sinh(\lambda(u_i^n - u_i))|$ is uniformly bounded for $\forall n \in \mathbb{N}$, in view of (18) and (23) we assert that $J_1 + J_2$ tends to zero when $n \to \infty$ by Lebesgue dominated convergence theorem. Then

$$\lim_{n\to+\infty}\int_{Q_T} \left(|\nabla u_i^n|^{p_i-2}\nabla u_i^n - |\nabla u_i|^{p_i-2}\nabla u_i\right).(\nabla u_i^n - \nabla u_i)\, dz = 0. \quad (34)$$

With the similar process to (27), it follows that

$$\lim_{n\to+\infty}\int_{Q_T} \left(|\nabla u_i^n|^{p_i} - |\nabla u_i|^{p_i}\right)\, dz = 0, \quad (35)$$

which implies that

$$\begin{gathered} \mathcal{A}_i u_i^n \to \mathcal{A}_i u_i \text{ strongly in in } L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega)); \\ f_i(.,u_1^n,u_2^n) \to f_i(.,u_1^n,u_2^n) \text{ strongly in in } L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega)); \\ a_i(x)\min\{|\nabla u_i^n|^{p_i}, n\} \to a_i(x)\,|\nabla u_i|^{p_i} \text{ strongly in } \quad \text{in } L^1(Q_T). \end{gathered}$$

Thus

$$\frac{\partial u_i^n}{\partial t} \to \frac{\partial u_i}{\partial t} \quad \text{strongly in in } L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega)) + L^1(Q_T).$$

Therefore

$$\begin{aligned} \frac{\partial u_i}{\partial t} &\in L^{p_i'}(0,T_0;W_0^{-1,p_i'}(\Omega)) + L^1(Q_T); \\ \frac{\partial u_i}{\partial t} + \mathcal{A}_i u_i &= a_i(x)\,|\nabla u_i|^{p_i} + f_i(x,u_1,u_2). \end{aligned}$$

As $L^{p_i'}(0,T;W_0^{-1,p_i'}(\Omega)) + L^1(Q_T) \subset L^1(0,T;H^{-s}(\Omega))$ for large enough, then u_i^n converges strongly to u_i in $C(0,T;H^{-s}(\Omega))$ and

$$u_i^n(x,0) \to u_i(x,0) \text{ strongly in } H^{-s}(\Omega)$$

implies $u_i(x,0) = u_{i0}(x)$.

The proof of Theorem 1 is completed.

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Stabilized finite element method for the Navier-Stokes problem

A. Elakkad [(1)], A. Elkhalfi [(1)], and N. Guessous [(2)]

[(1)] Laboratoire Génie Mécanique - Faculté des Sciences et Techniques
B.P. 2202 - Route d'Imouzzer - Fès, Morocco.
e-mail: elakkadabdeslam@yahoo.fr
[(2)] Département de mathématiques et informatique, Ecole normale
Supérieure de Fès, B.P. 5206, Bensouda, Fès, Morocco.
e-mail: ngessous@yahoo.fr

Abstract

A stabilized finite element method for the two-dimensional stationary incompressible Navier-Stokes equations is investigated in this work. A macroelement condition is introduced for constructing the local stabilized formulation of the stationary Navier-Stokes equations. By satisfying this condition, the stability of the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangular element is established. Moreover, the well-posedness and the optimal error estimate of the stabilized finite element method for the stationary Navier-Stokes equations are obtained. In order to evaluate the performance of the method, the numerical results are compared with some previously published works or with others coming from commercial code like Adina system.

Keywords: *Incompressible Navier-Stokes Equations, Stabilized finite element, A posteriori error estimates, Adina system.*

1 Introduction

A posteriori error analysis in problems related to fluid dynamics is a subject that has received a lot of attention during the last decades. In the conforming

case there are several ways to define error estimators by using the residual equation. In particular, for the Stokes problem, M. Ainsworth, J. Oden [2], C. Carstensen, S.A. Funken [6], D. Kay, D. Silvester [16] and R. Verfurth [22] introduced several error estimators and provided that they are equivalent to the energy norm of the errors. Other works for the stationary Navier-Stokes problem have been introduced in [4, 8, 13, 15, 19, 23, 24].
We were interested in the resolution of the incompressible Navier-Stokes equations in two dimensions on the fields where the numerical problem is well posed with boundary conditions and other aspects of the problem. A discretization by quadrangular finite elements is used. Two iterative methods are used to solve the not-symmetrical discrete system of the Navier-Stokes equations. The method BiConjugate Gradients Stabilized Method (BICGSTAB) and minimal residual generalized method (GMRES) are given in [8, 12]. The technique of preconditioning of the linear systems of big sizes is used to reduce the time of convergence of the iterative methods. This technique of preconditioning has allowed us to accelerate the convergence of the iterative methods independently of the Reynolds number and the number of meshes. Moreover, the methods of Picard or Newton are used to solve the non-linear algebraic systems resulting from the discretization.
Section 2 presents the model problem used in this paper. The Stabilized finite-element approximation described is in section 3. Section 4 shows the methods of a posteriori error bounds of the computed solution. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in section 5.

2 Incompressible Navier-Stokes equations

Let Ω be a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$ and to satisfy a further condition stated in (B1) below. We consider the steady-state Navier-Stokes equations for the flow of a Newtonian incompressible viscous fluid with constant viscosity:

$$\begin{cases} -\nu\Delta u + (u.\nabla)u + \nabla p = f, \ \ \nabla.u = 0 \ \ x \in \Omega; \\ u|\partial\Omega = 0, \end{cases} \tag{1}$$

where $\nu > 0$ is a given constant called the kinematic viscosity.
u is the fluid velocity, p is the pressure field, ∇ is the gradient and $\nabla.$ is the divergence operator.
This system is the basis for computational modeling of the flow of an incompressible Newtonian fluid such as air or water. The presence of the nonlinear convection term $u.\nabla u$ means that boundary value problems associated with the Navier-Stokes equations can have more than one solution.
We define the spaces:

$$X = H_0^1(\Omega)^2,\ V = L^2(\Omega)^2,$$
$$W = L_0^2(\Omega) = \{q \in L^2(\Omega) :\ \int_\Omega q\ dx = 0\},$$

The spaces $L^2(\Omega)^m$, m= 1, 2, 4 are endowed with the L^2-scalar product and $L^2 - norm$ denoted by $(.,\ .)$ and $|.|$. The spaces $H_0^1(\Omega)$ and X are equipped with the scalar product and norm $((u,v)) = (\nabla u, \nabla v),\ \|u\| = (\nabla u, \nabla u)^{\frac{1}{2}}$.
As mentioned above, we need a further assumption on Ω:
(B1) Assume that Ω is regular so that the unique solution $(v,q) \in (X,M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = l,\ \nabla.v = 0\ in\ \Omega,\ v|\partial\Omega = 0, \tag{2}$$

for a prescribed $l \in V$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \leq C_0|l|, \tag{3}$$

where $C_0 > 0$ is a constant depending on Ω and $\|.\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $H^i(\Omega)^2$ for i = 1, 2.
We introduce the following Laplace operator
$Au = -\Delta u,\ \forall u \in D(A) = H^2(\Omega)^2 \cap X,$
and the bilinear operator

$$B(u,v) = (u.\nabla)v + \frac{1}{2}(\nabla.u)v,\ \forall u, v \in X, \tag{4}$$

Moreover, we define the continuous bilinear forms a(., .) and d(., .) on $X \times X$ and $X \times W$, respectively, by: $a(u,v) = \nu((u,v)),\ \forall u, v \in X,$
$and\ d(v,q) = -(v, \nabla q) = (q, \nabla.v), \forall v \in X,\ q \in W,$
and a generalized bilinear form on $(X,M) \times (X,M)$ by
$B((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q),$
and a trilinear form on $X \times X \times X$ by
$b(u,v,w) = \langle B(u,v), w\rangle_{X' \times X} = ((u.\nabla)v, w) + \frac{1}{2}((\nabla.u)v, w)$
$= \frac{1}{2}((u.\nabla)v, w) - \frac{1}{2}((u.\nabla)w, v),\ \forall u, v, w \in X.$
We remark that the validity of assumption (B1) is known (see [13]) if $\partial\Omega$ is of C^2, or if Ω is a two-dimensional convex polygon. From assumption (B1), it is easily shown [13] that

$$|v| \leq \gamma_0\|v\|,\ \|v\| \leq \gamma_0 \mid PAv \mid,\ \|v\|_2 \leq \gamma_1 \mid PAv \mid, \tag{5}$$

where P is the L^2-orthonormal projection of V onto the space
$\{v \in L^2(\Omega)^2 :\ \nabla.v = 0\ in\ \Omega\ and\ v.n|_{\partial\Omega} = 0\}$, and $\gamma_0,\ \gamma_1,\ .\ .\ .$ are positive constants depending only on Ω.
It is easy to verify that B and b satisfy the following properties (see [10]):

$$\begin{cases} \nu\|u\|^2 = B((u,p);(u,p)), \\ |B((u,p);(v,q))| \leq \gamma_2(\|u\| + |p|)(\|v\| + |q|), \\ \alpha_0(\|u\| + |p|) \leq \sup_{(v,q)\in(X,W)} \frac{B((u,p);(v,q))}{\|v\|+|q|} \end{cases} \tag{6}$$

hold for all $(u,p),\ (v,q) \in (X,W)$ and constants $\gamma_2 > 0$ and $\alpha_0 > 0$,

$$b(u,v,w) = -b(u,w,v), \tag{7}$$

$$|b(u,v,w)| = \frac{1}{2}c_0|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}(\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}} + |v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|w\|), \tag{8}$$

for all $u, v, w \in V$ and

$$|b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)| \leq c_1\|u\||Av||w|, \tag{9}$$

for all $u \in X,\ v \in D(A),\ w \in Y$, where $c_0,\ c_1\ ,\ .\ .\ .\ .$, are positive constants depending on the domain Ω.
Under the above notations, the variational formulation of the problem (1) reads as follows: find $(u,p) \in (X,M)$ such that for all $(v,q) \in (X,M)$:

$$B((u,p);(v,q)) + b(u,u,v) = (f,v). \tag{10}$$

The following existence and uniqueness results are classical (see [10, Chapter IV]).

Theorem 2.1 *Assume that ν and $f \in V$ satisfy the following uniqueness condition:*

$$1 - \frac{c_0\gamma_0^2}{\nu^2}|f| > 0. \tag{11}$$

Then the problem (10) admits a unique solution $(u,p) \in (D(A) \cap X, H^1(\Omega) \cap W)$ such that

$$\|u\| \leq \frac{\gamma_0}{\nu}|f|,\ \ |Au| + \|p\|_1 \leq c_0|f|, \tag{12}$$

where γ_0 and c_0 are defined in (5) and (8), respectively.

3 Stabilized finite element approximation

In this section we apply the stabilized finite element method developed for the Stokes equations to consider the numerical solution of the two-dimensional stationary incompressible Navier-Stokes equations (1). Let $h > 0$ be a real positive parameter. The finite element subspace (X_h, M_h) of (X, M) is characterized by $\tau_h = \tau_h(\Omega)$, a partitioning of $\overline{\Omega}$ into triangles or quadrilaterals, assumed to be regular in the usual sense (see [10, 16]), i.e., for some σ and ω with $\sigma > 1$ and $0 < \omega < 1$,
$h_K \leq \sigma\rho_K\ \forall K \in \tau_h$,
$|\cos(\theta_{iK})| \leq \omega$, i=1, 2, 3, 4, $\forall K \in \tau_h$,

where h_K is the diameter of element K, ρ_K is the diameter of the inscribed circle of element K, and θ_{iK} are the angles of K in the case of a quadrilateral partitioning. The mesh parameter h is given by $h = max\{h_K\}$ and the set of all interelement boundaries will be denoted by Γ_h.
The finite element subspaces of interest in this paper are defined by setting

$$R_1(K) = \begin{cases} P_1(K) \ if \ K \ is \ triangular, \\ Q_1(K) \ if \ K \ is \ quadrilateral, \end{cases} \tag{13}$$

giving the continuous piecewise bilinear velocity subspace

$$X_h = \{v \in X : \ v_i|_K \in R_1(K), \ i = 1,2, \ \forall K \in \tau_h\}, \tag{14}$$

and the piecewise constant pressure subspace

$$M_h = \{q \in M : \ q|_K \in P_0(K), \ \forall K \in \tau_h\}. \tag{15}$$

Note that neither of these methods are stable in the standard Babuska-Brezzi sense; $P_1 - P_0$ triangle locks on regular grids (since there are more discrete incompressibility constraints than velocity degrees of freedom), and the $Q_1 - P_0$ quadrilateral is the most infamous example of an unstable mixed method.
With the above choices of the velocity-pressure finite element spaces $(X_h, M_h) \subset (X, M)$, a globally stabilized discrete formulation of the Navier-Stokes problem (10) can be defined as follows.

Definition 3.1 *Globally stabilized formulation: find* $(u_h, p_h) \in (X_h, M_h)$ *such that for all* $(v, q) \in (X_h, M_h)$:

$$B_h((u_h, p_h); (v, q)) + b(u_h, u_h, v) = (f, v), \tag{16}$$

where
$B_h((u,p);(v,q)) = B((u,p);(v,q)) + \beta C_h(p,q), \ \forall (u,p),(v,q) \in (X, W)$
$C_h(p,q) = \sum_{e \in \Gamma_h} h_e \int_e [p]_e [q]_e ds, \ \forall p, q \in W,$
and $[.]_e$ *is the jump operator across* $e \in \Gamma_h$*, and* $\beta > 0$ *is the global stabilization parameter [14].*

In order to define a locally stabilized formulation of the Navier-Stokes problem, we introduce a macroelement partitioning Λ_h as follows: Given any subdivision τ_h, a macroelement partitioning Λ_h may be defined such that each macroelement K is a connected set of adjoining elements from τ_h. Every element K must lie in exactly one macroelement, which implies that macroelements do not overlap. For each K, the set of interelement edges, which are strictly in the interior of K, will be denoted by Γ_K, and the length of an edge $e \in \Gamma_K$ is denoted by h_e. With these additional definitions a locally stabilized formulation of the Navier-Stokes problem (10) can be stated as follows.

Definition 3.2 *Locally stabilized formulation: find* $(u_h, p_h) \in (X_h, M_h)$, *such that for all* $(v, q) \in (X_h, M_h)$

$$B_h((u_h, p_h); (v, q)) + b(u_h, u_h, v) = (f, v), \tag{17}$$

where
$C_h(p, q) = \sum_{K\in\bigwedge_h} \sum_{e\in\Gamma_K} h_e \int_e [p]_e [q]_e ds,\ \forall p, q \in W,$
$[.]_e$ *is the jump operator across* $e \in \Gamma_K$ *and* $\beta > 0$ *is the local stabilization parameter.*

The following stability results of these mixed methods for the macroelement partitioning defined above were formally established by Kay and Silvester [16].

Theorem 3.3 *Given a stabilization parameter* $\beta \geq \beta_0 > 0$, *suppose that every macroelement* $K \in \tau_h$ *belongs to one of the equivalence classes* $\varepsilon_{\widehat{K}}$, *and that the following macroelement connectivity condition is valid: for any two neighboring macroelements* K_1 *and* K_2 *with* $\int_{K_1\cap K_2} ds \neq 0$ *there exists* $v \in X_h$ *such that*
supp $v \subset K_1 \cup K_2$ *and* $\int_{K_1\cap K_2} v.nds \neq 0$
Then,

$$\|B_h((u, p); (v, q))\| = \gamma_3(\|u\| + |p|)(\|v\| + |q|),\ \forall (u, p), (v, q) \in (X, M), \tag{18}$$

$$\alpha(\|u_h\| + |p_h|) = sup_{(v,q)\in(X_h,M_h)} \frac{B_h((u_h, p_h); (v, q))}{\|v\| + |q|},\ \forall (u_h, p_h) \in (X_h, M_h), \tag{19}$$

$$|C_h(p - J_h p, q_h)| \leq c_4 h \|p\|_1 |q_h|,\ C_h(p, q_h) = 0,\ \forall p \in H^1(\Omega) \cap M,\ q_h \in M_h, \tag{20}$$

where $\alpha > 0$, $\gamma_3 > 0$ *are two constants independent of* h *and* β, *and* β_0 *is any fixed positive constant and* n *is the outnormal vector.*
In the suite we shall assume that $\beta \geq \beta_0$.

Theorem 3.4 *Under the assumptions of Theorem 2.1 and Theorem 3.3, the problem (17) admits a unique solution* $(u_h, p_h) \in (X_h, M_h)$ *satisfying*

$$\|u_h\| \leq \frac{\gamma_0}{\nu}|f|,\ |p_h| \leq \alpha^{-1}(c_0\nu^{-2}\gamma_0^3|f|^2 + \gamma_0|f|). \tag{21}$$

4 Error estimates

In order to derive error estimates of the stabilized finite element solution (u_h, p_h), we also need the Galerkin projection $(R_h, Q_h) : (X, M) \longrightarrow (X_h, M_h)$ defined by

$$B_h((R_h(v, q) - v, Q_h(v, q) - q); (v_h, q_h)) = 0,\ \forall (v_h, q_h) \in (X_h, M_h), \tag{22}$$

for each $(v, q) \in (X, M)$. Note that, due to Theorem 3.3, (R_h, Q_h) is well defined. By using an exact similar argument to the one used by Layton, Tobiska [17], we may obtain the following approximation properties.

Lemma 4.1 *Under the assumptions of Theorem 3.3, the projection* (R_h, Q_h) *satisfies*

$$|v - R_h(v,q)| + h\|v - R_h(v,q)\| + h|q - Q_h(v,q)| \leq c_5 h(|Av| + |q|), \quad (23)$$

for all $(v, q) \in (X, M)$ *and*

$$|v - R_h(v,q)| + h\|v - R_h(v,q)\| + h|q - Q_h(v,q)| \leq c_5 h^2(|Av| + \|q\|_1), \quad (24)$$

for all $(v, q) \in (D(A), H^1(\Omega) \cap M)$.

Theorem 4.2 *Assume that the assumptions of Theorem 2.1 and Theorem 3.3 hold. Then the stabilized finite element solution* (u_h, p_h) *satisfies the error estimates:*

$$|u - u_h| + h(\|u - u_h\| + |p - p_h|) \leq ch^2, \quad (25)$$

where $c > 0$ *is a general constant depending on the data* $(\Omega, \nu, \beta_0, f)$.

Proof Since $C_h(p, q_h) = 0, \forall p \in H^1(\Omega) \cap M, q_h \in M_h$,
we derive from (10) and (17) that for all $(v, q) \in (X_h, M_h)$

$$B_h((e_h, \eta_h); (v, q)) + b(u - R_h(u,p) + e_h, u, v) + b(u_h, u - R_h(u,p) + e_h, v) = 0, \quad (26)$$

where $e_h = R_h(u,p) - u_h$ and $\eta_h = Q_h(u,p) - p_h$. Taking $(v, q) = (e_h, \eta_h)$ in (26) and using (7), we arrive at

$$\nu\|e_h\|^2 + \beta_0 C_h(\eta_h, \eta_h) + b(e_h, u, e_h) \leq |b(u - R_h(u,p), u, e_h)| + |b(u_h, u - R_h(u,p), e_h)|. \quad (27)$$

We find from (8), (12), (21) and (24) that

$$\nu\|e_h\|^2 - |b(e_h, u, e_h)| \geq \nu\|e_h\|^2 - c_0\gamma_0\|u\|\|e_h\|^2 \geq \nu(1 - c_0\gamma_0^2|f|\nu^{-2})\|e_h\|^2, \quad (28)$$

$$|b(u_h, u - R_h(u,p), e_h)| + |b(u - R_h(u,p), u, e_h)|$$
$$\leq c_0\gamma_0(\|u\| + \|u_h\|)\|e_h\|\|u - R_h(u,p)\| \leq ch\|e_h\|. \quad (29)$$

Combining (27) with (28-29) yields

$$\|e_h\| \leq ch. \quad (30)$$

Moreover, by using (8-9), (12), (24) and (30), we have

$$
\begin{aligned}
&|b(u_h, u - R_h(u,p), e_h)| + |b(u - R_h(u,p), u, e_h)| \\
&\le |b(u, u - R_h(u,p), e_h)| + |b(u - R_h(u,p), u, e_h)| \\
&+|b(u - R_h(u,p), u - R_h(u,p), e_h)| + |b(e_h, u - R_h(u,p), e_h)| \\
&\le c_1|Au|\ |u - R_h(u,p)|\ \|e_h\| \\
&c_0\gamma_0(\|u - R_h(u,p) + \|e_h\|)\|u - R_h(u,p)\|\ \|e_h\| \le ch^2\ \|e_h\|.
\end{aligned} \tag{31}
$$

Combining (27-28) with (31) gives

$$\|e_h\| \le ch^2, \tag{32}$$

Moreover, one finds from (24), (32) and (12) that

$$|u - u_h| \le |e_h| + |u - R_h(u,p)| \le \gamma_0\|e_h\| + c_5h^2(|Au| + \|p\|_1) \le ch^2 \tag{33}$$

$$\|u - u_h\| \le \|e_h\| + \|u - R_h(u,p)\| \le ch^2 + c_5h(|Au| + \|p\|_1) \le ch \tag{34}$$

Using again (19), (26), (12) and (21), we obtain

$$|\eta_h| \le \alpha^{-1}c(\|u\| + \|u_h\|)\|u - u_h\| \le c\|u - u_h\|. \tag{35}$$

It follows from (24), (34-35) and (12) that

$$|p - p_h| \le |p - Q_h(u,p)| + |\eta_h| \le ch(|Au| + \|p\|_1) + c\|u - u_h\| \le ch \tag{36}$$

Combining (34-35) with (36) yields (25). ■

5 Numerical simulations

In this section some numerical results of calculations with finite element Method and ADINA system will be presented. Using our solver, we run two traditional test problems (Channel domain [8] and Backward-facing step problem [17, 20]) with a number of different model parameters.

If points in Ω are denoted by $\xi = \frac{x}{L}$, then denotes points of a normalized domain. In addition, let the velocity u be defined so that $u = Uu_*$ where U is a reference value-for example, the maximum magnitude of velocity on the inflow. If the pressure is scaled so that $p(L\xi) = U^2 p_*(\xi)$ on the normalized domain. The flow Reynolds number is defined by $R = UL/v$.

Notice that taking the limit $R \mapsto \propto$ gives the reduced hyperbolic problem

$$\begin{cases} u^*.\nabla u^* + \nabla p_* = f_*, \\ \nabla.u^* = 0. \end{cases} \tag{37}$$

The relative velocity error is $e_h = \frac{\|u - u_h\|}{\|u\|}$ and pressure error is $\eta_h = \frac{|p - p_h|}{|p|}$.

Example 5.1 *Square domain $\Omega = (-1,1)^2$, parabolic inflow boundary condition, natural outflow boundary condition, analytic solution. The Poiseuille channel flow solution*

$$u_x = 1 - y^2;\ u_y = 0;\ p = -2\nu x; \tag{38}$$

is also an analytic solution of the Navier-Stokes equations, since the convection term $(u.\nabla)u$ is identically zero. It also satisfies the natural outflow condition

$$\begin{cases} \nu \frac{\partial u_x}{\partial x} - p = 0 \\ \frac{\partial u_y}{\partial x} = 0. \end{cases} \tag{39}$$

The pressure gradient is proportional to the viscosity parameter. This makes sense physically; if a fluid is not very viscous then only a small pressure difference is needed to maintain the flow. Notice also that in the extreme limit $\nu \longrightarrow 0$, the parabolic velocity solution specified in (38) satisfies the Euler equations (37) together with a constant pressure solution.

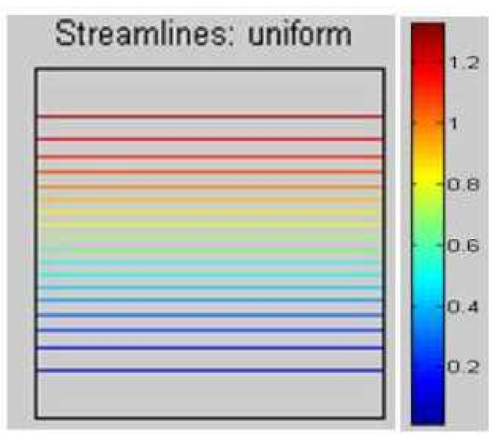

Fig. 1. Streamline plot associated with a 32×32 square grid, $Q_1 - P_0$ approximation, $\nu = \frac{1}{100}$ and $\beta = \frac{1}{4}$.

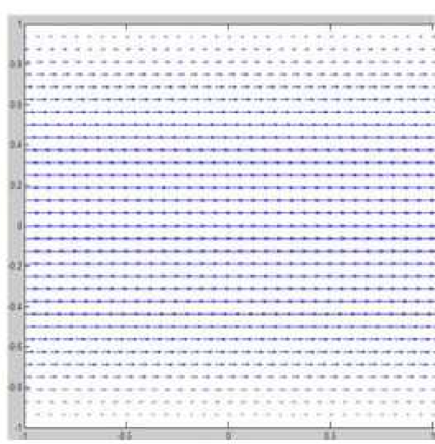

Fig. 2. Velocity vectors solution, associated with a 32×32 *square grid,* $Q_1 - P_0$ *approximation,* $\nu = \frac{1}{100}$ *and* $\beta = \frac{1}{4}$.

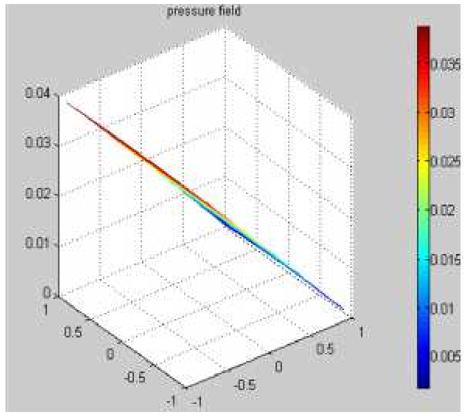

Fig. 3. Pressure plot for the flow with a 32×32 *square grid and* $\beta = \frac{1}{4}$.

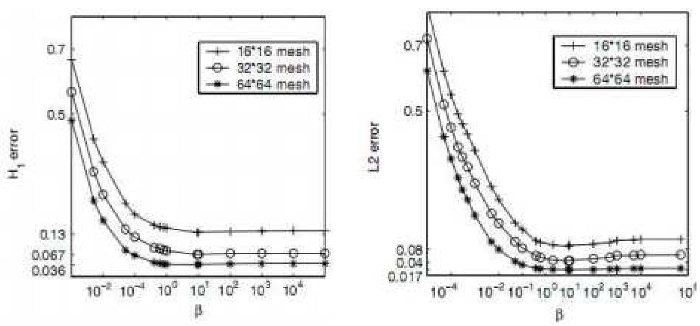

Fig. 4. The left plot shows the relative velocity error curve with respect to β, *and the right plot shows the relative pressure error curve with respect to* β.

h	$\frac{\Vert u-u_h \Vert}{\Vert u \Vert}$	$\frac{\vert u-u_h \vert}{\vert u \vert}$	$\frac{\vert p-p_h \vert}{\vert p \vert}$
1/16	0.1349	0.0528	0.0890
1/32	0.0674	0.0442	0.0442
1/64	0.0365	0.0286	0.0170

Table 1. Numerical results of the stabilized finite element method.

Example 5.2 *L-shaped domain* Ω, *parabolic inflow boundary condition, natural outflow boundary condition.*
This example represents flow in a rectangular duct with a sudden expansion; a Poiseuille flow profile is imposed on the inflow boundary (x=-1; $0 \leq y \leq 1$*), and a no-flow (zero velocity) condition is imposed on the walls.*
The Neumann condition (39) is applied at the outflow boundary (x=5; $-1 < y < 1$*) and automatically sets the mean outflow pressure to zero.*

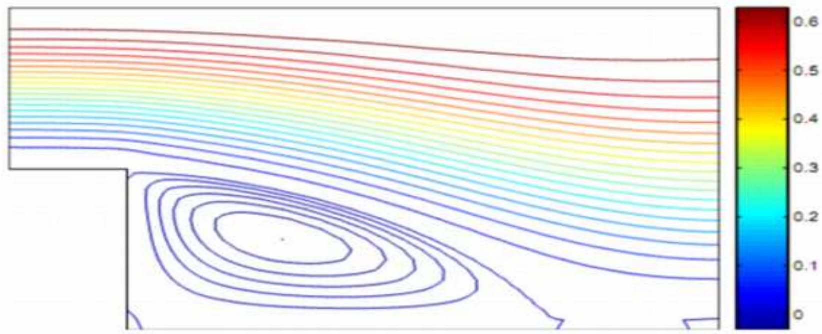

Fig. 5. Equally spaced streamline plot associated with a 32×96 *square grid,* $Q_1 - P_0$ *approximation,* $\nu = 1/50$ *and* $\beta = \frac{1}{4}$.

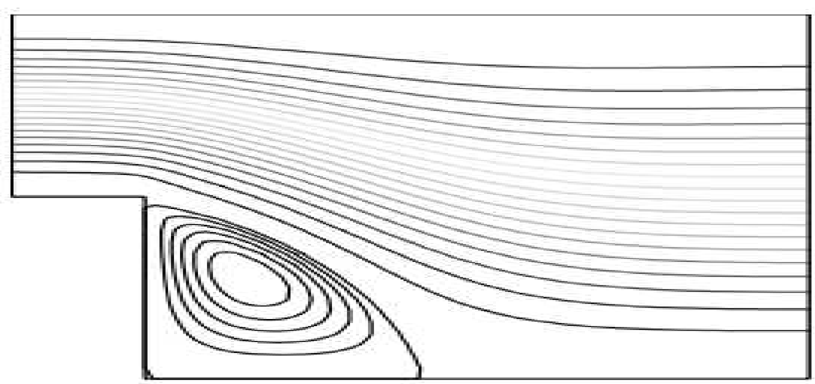

Fig. 6. Equally spaced streamline obtained by H.C. Elman and al [8], associated with a 32×96 *square grid,* $Q_1 - P_0$ *approximation,* $\nu = 1/50$ *and* $\beta = \frac{1}{4}$.

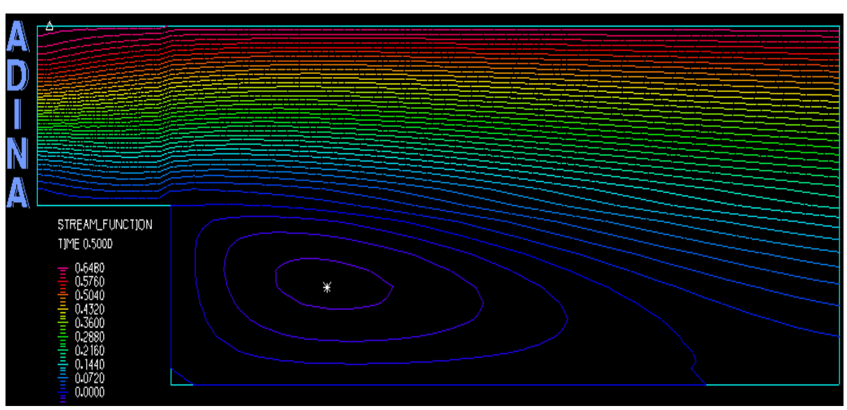

Fig. 7. The solution computed with ADINA system. The plots show the streamlines associated with a 32×96 *square grid,* $\nu = 1/50$.

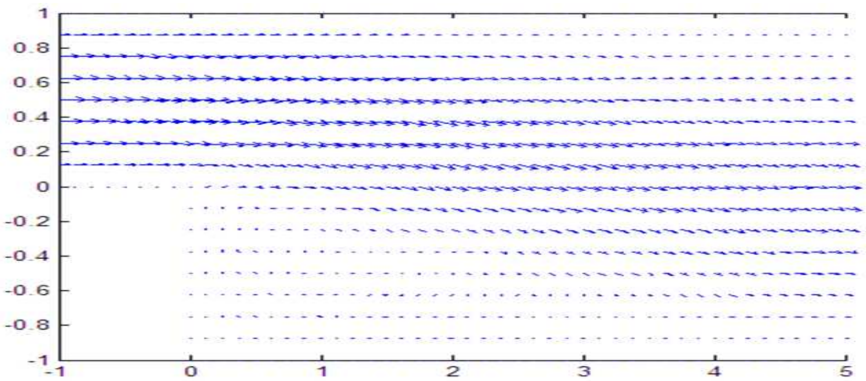

Fig. 8. Velocity vectors solution by MFE with a 32×96 *square grid,* $\nu = 1/50$ *and* $\beta = \frac{1}{4}$.

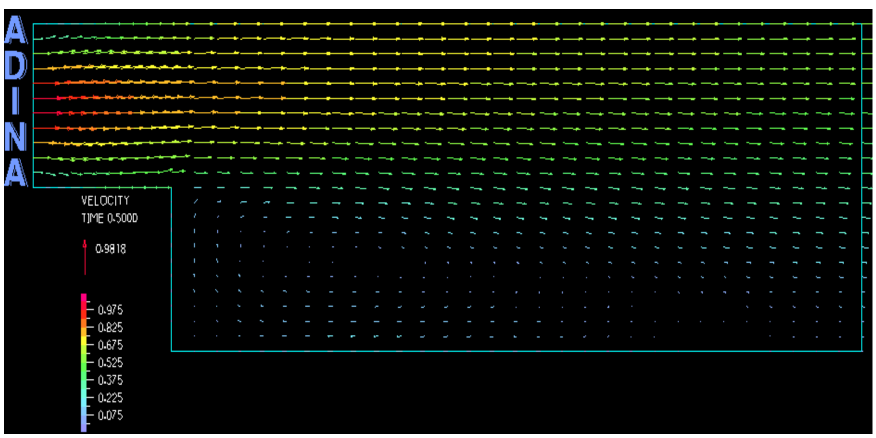

Fig. 9. The solution computed with ADINA System. The plots show the velocity vectors solution with a 32×96 *square grid,* $\nu = 1/50$.

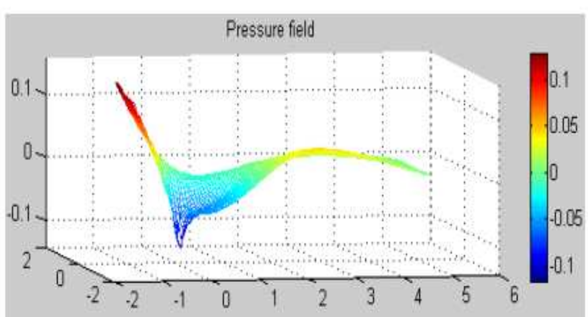

Fig. 10. Pressure plot for the flow with a 32×32 *square grid and* $\beta = \frac{1}{4}$.

The increased velocity caused by convection makes it harder for the fluid to flow around the corner, and a slow-moving component of the fluid becomes entrained behind the step. There are two sets of streamlines at equally spaced levels plotted in figure 1; one set is associated with positive stream function values and shows the path of particles introduced at the inflow. These pass over the step and exit at the outflow. The second set of streamlines is associated with negative values of the stream function. These streamlines show the path of particles in the recirculation region near the step; they are much closer in value, reflecting the fact that recirculating flow is relatively slow-moving.
If L is taken to be the height of the outflow region, then the flow pattern shown in figure 1 corresponds to a Reynolds number of 200. If the viscosity parameter were an order of magnitude smaller, then the steady flow would be unstable. The singularity at the origin is an important feature of the flow even in the convection-dominated case.

6 Conclusion

In this work, we were interested in the numerical solution of the partial differential equations by simulating the flow of an incompressible fluid. We have provided a theoretical analysis of the stabilized finite element method for the two-dimensional stationary Navier-Stokes equations. Also, we proposed methods of the estimation of error for the calculated solution.
Our results for Backward-facing step problem agree with H.C. Elman and al [8], and with ADINA system.
Numerical results are presented to see the performance of the method, and seem to be interesting by comparing them with other recent results.

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Modeling of the orthotropic bridge's impact response

R. Lassoued[1], M. Ouchenane[2], and M. Guenfoud[3]

[1]Laboratory of materials and durability of buildings,Department of Civil Engineering,
Faculty of Engineering science,University of Mentouri, Constantine, Algeria
e-mail: rachid_lassoued@yahoo.fr
[2]Laboratory of materials and durability of buildings,Department of Civil Engineering,
Faculty of Engineering science,University of Jijel, Jijel, Algeria
e-mail: ouchenane_ m@yahoo.fr
[3]Laboratory of civil Engineering and Hydraulic , Department of Civil Engineering,
University of 08 May 1945 of Guelma, Algeria
e-mail: gue2005@yahoo.fr

Abstract

The determination of the dynamic response is fundamental for analysis of a bridge structure. The bridge is modeled as an orthotropic rectangular plate with a pair of parallel edges simply supported under moving load. An orthotropic plate is defined as an element of structure having various properties in the two orthogonal directions. The study of the free vibration is based on the resolution of the differential equation depending on the mechanical properties of the plates. For the determination of natural frequencies, we develop a computer code using a bisection method with interpolation which precision reached 10- 12. We propose in this analysis the evolution of the response versus the rigidity structure ratio. This later is subjected to moving loads by using the modal superposition method and the integral convolution. The effect of the eccentricity of the loads, simulating real trajectories, is analyzed according to various speeds and intensities of loading.

Keywords: *Orthotropic plate, Method of bisection, Optimization, moving load.*

1 Introduction

The much diversified application of the orthotropic plates subjected to fixed or mobile loads "specific forces or masses" in various fields such as: "aeronautics, acoustics, mechanics, electronics, and the civil engineering," present a real interest of the researchers implied in this field. An orthotropic plate is defined as an element of structure having various properties in the two orthogonal directions. Most bridge decks or railway are orthotropic because of shape orthotropy. So, there is a particular need for access to highly accurate eigenvalues for plates and beams. For example, Wu and Dai [1] used the transfer matrix method to determine the natural frequencies and mode shapes of multi-span of a beams. They determine the dynamic performances of the considered beam subjected to moving loads. Moussu and Nivoiti [2] has determined an elastic constants of orthotropic plates by modal analysis. Later, D.J Gorman [3] use a computed method to determine eigenvalues for a completely free orthotropic plates by using a superposition method. He also [4], used the superposition method to obtain accurate analytical type solutions for the free in-plane vibration of rectangular plates with uniform, symmetrically distributed elastic edge supports acting normally to the boundaries. In addition, an excellent reference source concerning vibration of such plates may be found in the work of Leissa [5,6]. We can find exact characteristic equations for rectangular thin plates having two opposite sides simply supported. However, the analysis of thick plates has been presented by Lim and all [7]. According to all what has been stated before, the authors has determined initially the free frequencies in order to predict the dynamic behavior of the studied structures. Indeed, the dynamic response of bridge structures under moving loads at high speed is a problem of great concern in the design of high-speed railway bridges. In the literature, a large number of investigations have been carried out, with the bridge modeled as a beam and the vehicles as moving loads or moving masses. In this paper, this dynamic behavior is analyzed using the orthotropic plate theory and modal superposition. So, we present, firstly, an accurate method to calculate the free vibrations. This simple and fast method does not require a great place memory.On the other hand, it presents an excellent precision which reaches 10-12 . The strategy presented is based on the bisection method with interpolation to determine the eigenfrequencies. However, to determine the corresponding modes, the algorithm which we developed uses the Gauss method with a partial optimization of the "pivots" combined with an inverse power procedure. The dynamic response of an orthotropic bridge deck under moving load is studied. For this, we use the theory of the orthotropic plates and a modal superposition principle. We, also, analysis the effect of the loads characteristics on the bridge deck or on the railway.

2 Formulation of the problem : Case of free vibration

A Schematic of a two dimensional plate is shown in "Fig.1". It's a rectangular plate with its left and right edges simply supported and the other two opposite edges free. If, it's also solicited by an external load F, the governing equations of motion of this orthotropic plate can be written, according to Huffington and Hoppman [8] as follows:

$$D_x(\frac{\partial^4 w}{\partial x^4}) + 2D_{xy}(\frac{\partial^4 w}{\partial x^2 \partial y^2}) + D_y(\frac{\partial^4 w}{\partial y^4}) + C(\frac{\partial w}{\partial t}) + \rho h(\frac{\partial^2 w}{\partial t^2}) = F(x, y, t) \quad (1)$$

Where:

$$D_x = \frac{E_x h^3}{12(1 - \upsilon_{xy}\upsilon_{yx})},\ D_y = \frac{E_y h^3}{12(1 - \upsilon_{xy}\upsilon_{yx})},\ D_{xy} = (D_{xy}\upsilon_{xy} + 2D_k),\ D_k = \frac{G_{xy} h^3}{12} \quad (2)$$

D_x,D_y : flexural rigidities of the plate in the x, y direction
D_{xy}: torsional rigidities.
D_k: twisting rigidity of the plate.
G_{xy} : Shear modulus.
ρ: mass density of plate material.
h: thickness of the plate.
$W(x, y, t)$: displacement of plate in the z direction.

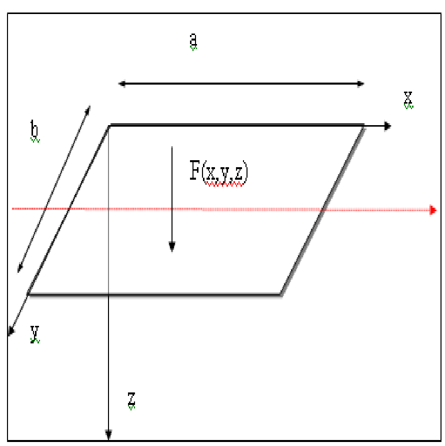

Figure 1: Considered plate

Let us note that the side effects (shearing and rotational inertia) are neglected. The resolution of the differential equation governing the movement is obtained by using the modal superposition method and the integral of convolution, by the separation of the temporal and space variables. Thus one expresses the dynamic response in the form of series. The free displacement at the point (x, y) of the plate and at the moment t is expressed in the form of series given by [9] :

$$W(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{m,n}(x,y)q_{m,n}(t) \tag{3}$$

Where : $U_{m,n}(x,y) = Y_{m,n}(y)\sin(\theta_m.x)$, $q_{m,n}(t) = \sin(\omega_{m,n}t)$ and $\theta_m = \frac{m\pi}{a}$

$U_{m,n}(x,y)$ is the mode shape, $\omega_{m,n}$ is the natural frequency which correspond to the m^{th} mode in the x direction and the n^{th} mode in the y one.

Substituting equation (03) in equation (01), we obtain:

$$D_y Y_{mn}^4(y) - 2D_{xy}(\frac{m\pi}{a})^2 Y_{m,n}^2(y) + [D_x(\frac{m\pi}{a})^4 - \rho h\omega_{mn}^2]Y_{mn}(y) = 0 \tag{4}$$

According to the properties of the plate, we can obtain:

$$Y_{mn}(y) = Ce^{\alpha y} \tag{5}$$

Then, the roots α, [Grace et Kennedy 1985][10] are as follows :

$$\alpha = ^{-}_{+} (A_1 {}^{-}_{+} \sqrt{A_1^2 - A_2 + \lambda_0^2})^{1/2} \tag{6}$$

with :

$$A_1 = \frac{D_{xy}}{D_y}(\frac{m\pi}{a})^2, A_2 = \frac{D_x}{D_y}(\frac{m\pi}{a})^4, \lambda_0^2 = \frac{\rho h\omega^2}{D_y} \tag{7}$$

Substituting the expressions A_1 , A_2 etλ_0 in the equation (06), we can express the roots of the considered equation. This later correspond to the resolution of the differential equation in term of the inflexion rigidity in the two directions of the orthotropic plate as well as the torsional rigidity. The analysis of the equation (06) will enable us to release three categories of orthotropic plates defined by the shape of the roots of the considered equation (the boundary conditions are also considered). This classification of the plates will be primarily based on the mechanical behavior of the structure, depend on the torsion rigidities and the inflection .

1. $Y_{mn}(y) = X_{1mn}\sin(r_{2mn}y) + X_{2mn}\cos(r_{2mn}y) + X_{3mn}\sinh(r_{1mn}y)$

$$+X_{4mn}\cosh(r_{1mn}y) \tag{8}$$

if $D_x < D_1$ where $D_1 = \rho h\omega_{mn}^2\theta_m^{-4}$

2. $Y_{mn}(y) = X_{1mn}\sin(r_{1mn}y) + X_{2mn}\cos(r_{1mn}y) + X_{3mn}\sinh(r_{3mn}y)$

$$+X_{4mn}\cosh(r_{3mn}y) \tag{9}$$

if $\frac{D_{xy}^2}{D_y} + D_1 > D_x > D_1$

3. $Y_{mn}(y) = \cosh(r_{4mn}y)(X_{1mn}\cos(r_{5mn}y)) + X_{2mn}\sin(r_{5mn}y)$

$$+\sinh(r_{4mn}y)(X_{3mn}\cos(r_{5mn}y)) + X_{4mn}\sin(r_{5mn}y) \quad (10)$$

if $D_x > \frac{D_{xy}^2}{D_y} + D_1$

The free boundary conditions (eq. 11) at $y = 0$ and $y = b$ allow to lead to a system of equation. Its resolution permits to determine the coefficients X_{1mn}, X_{2mn}, X_{3mn} and X_{4mn}.

$\frac{\partial^2 w}{\partial_y^2} + \nu_{xy}\frac{\partial^2 w}{\partial_y^2} = 0, -D_{xy}\frac{\partial^3 w}{\partial_x^2\partial_y} - D_y\frac{\partial^3 w}{\partial_y^3} = 0,$

$$2D_k\frac{\partial^3 w}{\partial_x^2\partial_y} = 0, -D_{xy}\frac{\partial^3 w}{\partial_x^2\partial_y} - D_y\frac{\partial^3 w}{\partial_y^3} - 2D_k\frac{\partial^3 w}{\partial_x^2\partial_y} = 0, \quad (11)$$

The parameters r_{imn} depend on the plate considered and the modes of vibration [10].

$r_{1mn} = \frac{m\pi}{a}\sqrt{\frac{D_{xy}+\sqrt{D_{xy}^2+D_y\rho h\omega_{mn}^2(\frac{a}{m\pi})^4-D_xD_y}}{D_y}},$

$r_{2mn} = \frac{m\pi}{a}\sqrt{\frac{D_{xy}+\sqrt{D_y^2+D_y\rho h\omega_{mn}^2(\frac{a}{m\pi})^4-D_xD_y}}{D_y}},$

$r_{3mn} = \frac{m\pi}{a}\sqrt{\frac{D_{xy}+\sqrt{D_{xy}^2+D_y\rho h\omega_{mn}^2(\frac{a}{m\pi})^4-D_xD_y}}{D_y}},$

$r_{4mn} = \frac{m\pi}{a}\sqrt{\frac{1}{2}(\frac{D_{xy}}{D_x} + \sqrt{\frac{D_x}{D_y} - \frac{\rho h\omega_{mn}^2}{D_y}}(\frac{a}{m\pi}{}^4))}$

$$r_{5mn} = \frac{m\pi}{a}\sqrt{\frac{1}{2}(\frac{D_{xy}}{D_x} + \sqrt{\frac{D_x}{D_y} - \frac{\rho h\omega_{mn}^2}{D_y}}(\frac{a}{m\pi}{}^4))} \quad (12)$$

The application of the boundary conditions (eq. 11) according to the various cases considered (eq.8-9-10) permits to lead to the system:

$$[M].[X] = 0 \quad (13)$$

M is a matrix which coefficients m_{ij} depend on boundary conditions and X is a vector with: $[X] = [X1_{mn},\ X2_{mn},\ X3_{4mn},\ X4_{mn}]^T$.

To obtain noncommonplace solutions, it is necessary that the determinant of the system will be null. Writing this determinant permits to lead to the frequencies equation. Knowing that the parameters r_{imn} are not independent variables but are function of the pulsation ω (eq. 12), the resolution of the frequencies equation is not easy and then requires an adequate data-processing treatment.

We seek to determine the pulsations ω checking this equation. For that, we develop a code which calculates the eigenvalues of the frequencies equation.

It is based on a bisection method with interpolation which precision reaches 10^{-12}[12].

This method permits to record the eigenvalues of the frequencies corresponding to the different mode of vibration "Fig.2". For each index m, we find an infinity of solutions $m = 1, \ldots, \infty$. Each solution is then located by a double index ω_{rs}.

The resolution of the system of equations is done by the inverse power method. This one is very similar to that of Gauss (triangularisation of M), but with a partial optimization of the pivots. Indeed, as for the method of Gauss, some problems of overshoot capacity and numerical errors appeared when the pivots of the matrix M that we triangularis are null or only very small. This method consists then, to replace a null pivot by a very small value (equal to the precision: in our case 10^{-15}), to avoid the capacity overshooting. At the end, we normalize the solution obtained.

On the other hand, the use of the inverse power procedure permit to have the fundamental modes which correspond to the lowest frequencies. This method [13] has much more importance than the traditional one because it permit to have the smallest eigenvalues which correspond to the lowest modes of vibration. Those are decisive for the structure stability.

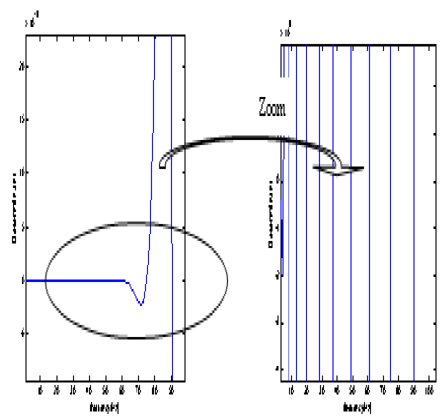

Figure 2: Evaluation of libres frequencies

3 Formulation of the problem : Case of forced vibration

When the orthotropic plate is under moving load, we can expressing the force $F(x, y, t)$ as a time step function and the equation (01) can be written as:

$$D_x\frac{\partial^4 w}{\partial x^4}+2D_{xy}\frac{\partial^4 w}{\partial x^2\partial x^2}+D_y\frac{\partial^4 w}{\partial y^4}+C\frac{\partial w}{\partial t}+\rho h\frac{\partial^2 w}{\partial t^2}=\sum_{l=1}^{L}p_l(t)\delta(x-\hat{x}_l(t))\delta(y-\hat{y}_l(t)) \tag{14}$$

where $p_l(t)$ is the moving load at a position $\hat{x}_l(t)$, $\hat{y}_l(t)$. We consider a convoy of loads spacing by a constant a_1.

The substitution of the equation (3) in the equation (9) permit to write:

$$q{\prime\prime}_{mn}(t)+2\alpha_{mn}\omega_{mn}q{\prime}_{mn}(t)+\omega_{mn}^2 q_{mn}(t)=\frac{2}{\rho ha\int_0^L Y_{mn}^2(y)dy}\sum_{l=1}^{L}p_l(t)U_{mn}(\hat{x}_l,\hat{y}_l) \tag{15}$$

with: $\alpha_{mn}(t)=\frac{C}{2\rho h\omega_{mn}}$

The solution of the equation (10) is obtained in the time domain by the following convolution integral:

$$q_{mn}(t)=\frac{1}{M_{mn}}\int_0^t H_{mn}(t-\tau)f_{mn}(\tau)d\tau \tag{16}$$

where:
$M_{mn}=\frac{\rho ha}{2}\int_0^b Y_{mn}^2(y)dy$
$H_{mn}=\frac{1}{\omega_{mn}^b}\sin(\omega_{mn}(t)), t\geq 0$
$f_{mn}(t)=\sum_{l=1}^{L}p_l(t)\frac{\rho ha}{2}Y_{mn}^2(y)dyU_{mn}(\hat{x}_l,\hat{y}_l)$

We started by evaluating M_{mn} by using the trapeze method, the result was very satisfactory compared to the Simpson one. The evaluation of $q_{mn}(t)$ was easier by separating the variables t and τ. Then, in the case of only one load, equation (11) can be written as follows :

$q(t)=\frac{1}{M_{mn}\omega_{mn}}\sin(\omega_{mn}t)\int_0^t f_{mn}(\tau)\cos(\omega_{mn}\tau)d\tau$

$$-\frac{1}{M_{mn}\omega_{mn}}\frac{1}{M_{mn}\omega_{mn}}\cos(\omega_{mn}t)\int_0^t f_{mn}(\tau)\sin(\omega_{mn}\tau)d\tau \tag{17}$$

The calculation of $q(t)$, which represents the Duhamel integral, requires the evaluation numerically of both the two integrals present in the equation (12). We also, choose the trapeze method. After evaluation of $q(t)$, total displacement can be evaluated according to (eq. 03).

4 Results and discussions

4.1 Simply supported beam slab type bridge deck

The bridge considered simply supported on the two side and the other two opposite edges free.

Is show "Fig.3" .The physical parameters are : length $a = 25.7m$, width $b = 11.0m$, Young modulus $E = 2.1e^9 N/m^2$, parameters of beam I: cross section of beam $A = 0.7465m^2$,$I = 0.213m^4$,$e_1 = 0.18m$,total height of the beam $h_1 = 1.50m$, between axle $m = 1.80m$,Poisson ratio $\mu = 0.33$,$L = 0.75m$, $E_c = 0.22m$, $E_j = 0.03m$.

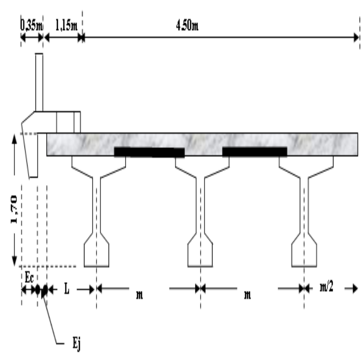

Figure 3: Bridge simply supported on the two sides and the other two opposite edges free

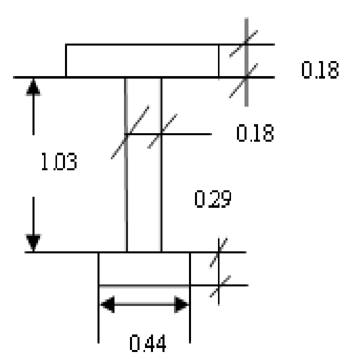

Figure 4: Section of homogenized beam

In according with the theory of Guyon-Massonet [14], we can simulate the bridge deck as an orthotropic plate whose mechanical characteristics are as follows:

The rigidities in the x and y directions of the orthotropic bridge deck can be calculated as:

$D_x = \frac{Eh^3}{12(1-\mu^2)} + \frac{EI}{m}$

$D_y = \frac{Eh^3}{12(1-\mu^2)}$

$D_{xy} = \mu D_y + \frac{Gh^3}{12} + \frac{Ge_1^3 h_1 \alpha_1}{m}$

with : $\alpha_1 = 0.152$ coefficient on the equivalent torsional moment of inertia of the I section. G: shear modulus Parameter of torsion:

$\alpha_1 = \frac{\gamma_P - \gamma_D}{2\sqrt{\rho_P \rho_D}} = 0.152$

γ_P:The flexural rigidity (longitudinal and transversal) corresponding to per unit length.

γ_D:Torsional rigidity of beams and diaphragms per unit length.

Parameter of diaphragms

$\theta = \frac{b}{L}\sqrt[4]{\frac{\rho_P}{\rho_D}} = 0.8146$

4.2 Analyze of effect eccentricity of moving load on the dynamic response

Under the influence two various speed " 10 m/s and 20 m/s", we can note a reduction in the dynamic amplitude response when the eccentricity of moving load increases for the points analyzed, the non charged side and the plate center. For the charged side, one observes an increase in the dynamic response.This is due to the rigidity D_{xy} of the bridge which is more significant than rigidity D_y "Fig.5".

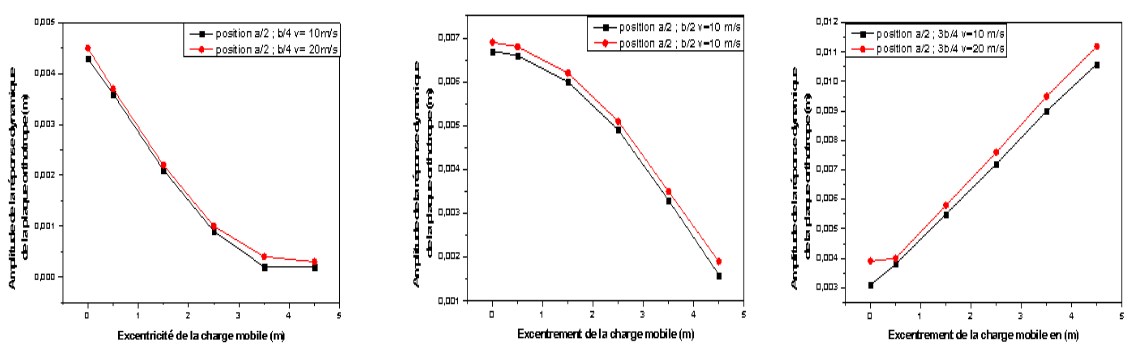

Figure 5: Evolution of the dynamic amplitude response function of the eccentricity of moving load

In the y direction. The eccentric moving load have less effect on the central is displacements.Concerning the influence speed, one similar observes an increase in the response for two speeds considered.

4.3 Analyze under the effect of the moving load convoy

We consider the effect of the moving load convoy on the dynamic response. This convoy is composed of two moving loads. The intensity of each force is 150000 N, they are spaced of 4m, and the speeds considered are 10m /s and

20m /s. The dynamic response due to an equivalent moving load in intensity. We note that the convoy of load with spacing is not the most unfavorable case "reduction in the dynamic response" for the two cases speed considered. Thus we can conclude that the spacing of the moving loads influences the dynamic response "Fig.5".

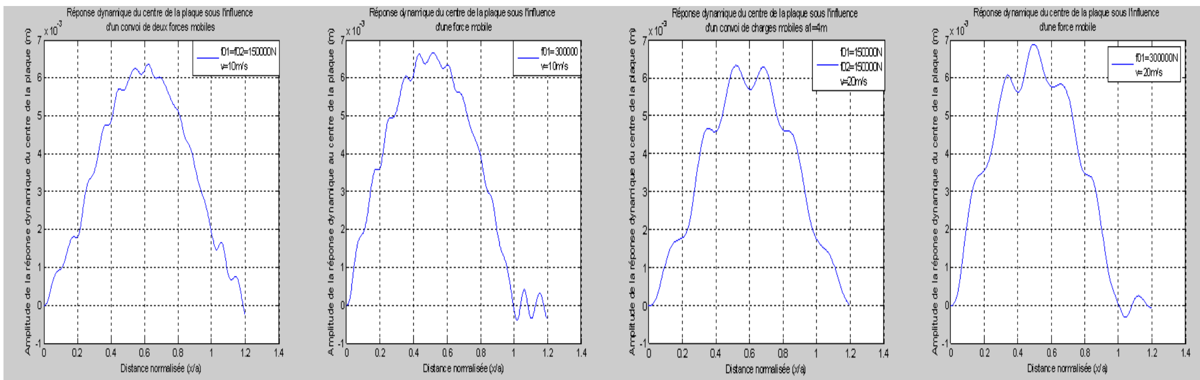

Figure 6: Dynamic response under the effect of the moving load convoy function a speed of load

5 Conclusion

A method is proposed to analyze the dynamic behavior of the orthotropic bridge.

Smaller torsion rigidity would lead to a greater torsion response at the side of the plate.

An equivalent beam model of the plate simulating the bridge, could give an evaluation of the amplification factor dynamic along the central line of the plate, but it would underestimate the responses dynamic at the side of the structure.

The principal beams are usually rigidified between them by diaphragms in the bridges, which the structure is composed by beam and plate. The existence of the diaphragms creates point of inflection between the beams, which increases the torsional rigidity of the bridge, which reduces alternatively the dynamic response and the amplitude of the torsional modes.

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GLOBAL CONVERGENCE OF THE QUASI NEWTON BFGS ALGORITHM WITH NEW NONMONOTONE LINE SEARCH TECHNIQUE

IVAN SUBHI LATIF

Dept. of Mathematics
College of Scientific Education
University of Salahaddin, Erbil-Iraq
e-mail: ivansubhi2001@yahoo.com

Abstract

The BFGS method is the most effective of the quasi-Newton algorithm for solving unconstrained optimization problem. In this work we develop a new nonmonotone line search of quasi-Newton algorithm for minimizing function having Lipschitz continuous partial derivatives. The nonmonotone line search can guarantee the global convergence of the original quasi-Newton BFGS algorithm. Numerical experiments on sixteenth wellknown test functions with various dimensions generally encouraging results show that the new algorithm line search is available and efficient in practical computation by comparing with other same algorithm in many situations.

Keywords : *Unconstrained optimization, BFGS update, Descent condition, Nonmonotone line searches.*

1 Introduction

Consider the unconstrained optimization problem

$$\min_{x \in \mathcal{R}} f(x), \qquad x \in \mathcal{R}^n \tag{1}$$

where $f : \mathcal{R}^n \to \mathcal{R}^1$ is a continuously differentiable function in $\mathcal{R}^n$ and $\mathcal{R}^n$ is the n-dimensional Euclidean space. We consider the case where the methods are implemented without regular restarts. The iterative formula is given by

$$x_{k+1} = x_k + \lambda_k d_k, \tag{2}$$

where λ_k is a steplength obtained by a line search, and d_k is the search direction defined by

$$d_k = \left\{ \begin{array}{ccc} -H_k g_k & \text{for} & k = 1 \\ -H_k g_k + \beta_k d_{k-1} & \text{for} & k \geq 2 \end{array} \right\}, \tag{3}$$

where λ_k denotes $\nabla f(x_k)$, and β_k is a scalar.

Quasi-Newton methods for solving (1) often need to update the iterate matrix H_k see [5]. Traditionally, $\{H_k\}$ satisfies the following quasi-Newton equation:

$$H_{k+1} V_k = Y_k, \tag{4}$$

where

$$V_k = x_{k+1} - x_k, \quad Y_k = g_{k+1} - g_k. \tag{5}$$

The famous update H_k is the BFGS formula

$$H_{k+1} = H_k - \frac{H_k V_k V_k^T H_k}{V_k^T H_k V_k} + V_k V_k^T / V_k^T Y_k. \tag{6}$$

It has shown that the BFGS method is the most effective quasi-Newton methods see [5],[6] from the computation point of view. The convergence properties of the BFGS method for convex minimization have been studied by many researchers for example [4], [6], [8], [12]. It is now Known that the BFGS method may fail for non convex functions [4]. Hence great efforts have been made to find new line search that not only possesses global convergence but also is superior the BFGS from the numerical performance.

Nonmonotone line search methods have been presented during recent decades [1], [8], [12], [13] the nonmonotone procedure is mainly to choose a large step size for line search methods and a void the iterates trapped in a narrow curved valley of objective functions. Many researchers used the non monotone technique methods [3], [4], [7], [9], [10], [11]. In this paper, we first propose a new nonmonotone type line search then apply it for BFGS method. In the next section, we present this concreter algorithm and establish some global convergent properties also we report some numerical result.

2 The New Nonmonotone Line Search With BFGS Algorithm (New)

Monotone method for solving (1) require that $f(x_k) \leq f(x_{k+1})$ hold at each iteration. But this does not necessarily hold at some iterations for nonmonotone methods. In this paper, we consider the following nonmonotone line search and then apply it for BFGS quasi-Newton method [10], [11], [12], [13].

2.1 New Nonmonotone Line Search

Given constant $\delta_1, \sigma \in (0,1), \delta_2 > 0$. Compute step α_k is the largest one in $\{s_k, s_k\beta, s_k\beta^2, \cdots\}$ such that

$$f(x_k + \beta^M d_k) \leq \max_{0\leq j\leq M} f(x_{k-j}) + \delta_\beta^m g_k^T d_k - \delta_2 \|\beta^M Z_k\|^2 \tag{7}$$

where

$$Z_k = y_k + C\|d_k^T g_k\|^r v_k + \max\left\{0, \frac{-y_k^T v_k}{\|v_k\}}\right\} v_k \tag{8}$$

were $C, r > 0$ are given constants.

2.2 Outlines of the New Non-Monotone Line Search for BFGS Algorithm (NEW)

Step 1 : Let $x_0 \in \mathcal{R}^n$ be initial point, $H_0 = I$, compute g_0; if $g_0 =$ and x_0 is a stationary point of (1) stop; else let $\delta_1, \rho \in (0,1), \delta_2 > 0$ and nonnegative integer M and ϵ is a small positive value, let $k = 0, M = \min(K, M), C = .1, r = 3$.
Step 2 : If $\|g_k\| < \epsilon$ then stop! Else go to step3.
Step 3 : Compute direction search d_k by (3) using QN formula for β_k.
Step 4 : $x_{k+1} = x_k + \alpha_k d_k$ the step size α_k is chosen by New nonmonotone line search rule (7), (8). Proposed update H_k to get H_{k+1} by formula (6).
Step 5 : Compute g_{k+1}; if $\|g_{k+1}\| = 0$ and x_{k+1} is a stationary point of (1) stop; else let $k = k+1$, go to step 6.
Step 6 : If the available storage is exceeded, then employ a restart option either with $k = n$ or $g_{k+1}^T g_{k+1} > g_{k+1}^T g_k$.
Step 7 : Set $k = k + 1$ and go to step 2.

2.3 Some Theoretical Properties of the (New) Algorithm

The New line search rule was implemented by considering the following assumptions.
(H1) The objective function $f(x)$ has a lower bounder on $\mathcal{R}^n$.
(H2) The gradient $g(x)$ of $f(x)$ is Lipschitz continuous in an open convex set B that contains the level set $L_0 = \{x \in \mathcal{R}^n, f(x) \leq f(x_0)\}$ with x_0 given i.e, there is a constant $L > 0$ such that

$$\|g(x) - g(y)\| \leq L\|x - y\|, \quad \forall` \; x, y \in B. \tag{9}$$

Some time require that $f(x)$ is twice continuously differentiable. In what follows, we first describe the non-monotone line search.

(H3) The Matrices B_k are uniformly positive definite, i.e. there exists $0 \leq m \leq M$ such that for any $k > 0$

$$m\|d\|^2 \leq d^T B_k d \leq M\|d\|^2. \tag{10}$$

Lemma 2.1 *Assume that (H3) holds and let $H_k = B_k^{-1}$. Then*

$$\frac{1}{M}\|d\|^2 \leq d^T H_k d \leq \frac{1}{m}\|d\|^2, \quad \forall \ k. \tag{11}$$

For the proof see [9].

Property 2.3.1 : The quasi Newton formula has an attractive property that for each, k it always holds that

$$z_k^T v_k \geq C\|g_k^T d_k\|\|v_k\|^2 > 0. \tag{12}$$

This property is independent of the convexity of f as well as the line search used. Thus the search direction defined by (3) is always a descent direction of the objective function, namely $g_k^T d_k < 0$. The following result shows that the nonmonotone line search (7) is well defined.

Theorem 2.2 *The nonmonotone line search with BFGS algorithm is well defined.*

Proof Infect, we only need to prove that step length λ_k can be obtained in finite steps. If it is not true, then for all sufficiently large positive integer m, we have

$$\begin{aligned} f(x_k + \beta^M d_k) &\geq \max_{0 \leq j \leq M} f(x_{k-j}) + \delta_1 \beta^M g_k^T d_k - \delta_2 \|\beta^M Z_k\|^2 \\ &\geq f(x_k) + \delta_1 \beta^M g_k^T d_k - \delta_2 \|\beta^M Z_k\|^2 \end{aligned} \tag{13}$$

where $\rho_1 \in (0,1), \sigma_2 > 0, M = \min(k, m)$ where M nonnegative integer, $k = 0, 1, 2, \cdots$

$$Z_k = y_k + C\|d_k^T g_k\|^r v_k + \max\left\{0, \frac{-y_k^T v_k}{\|v_k\|}\right\} v_k. \tag{14}$$

Let $m \to \infty$ in (13) then

$$g_k^T d_k \geq \sigma_1 g_k^T d_k \tag{15}$$

which implies that $g_k^T d_k \geq 0$. Since $\sigma_1 \in (0,1)$. This yields a contradiction so algorithm (2.2) is well defined. ■

Theorem 2.3 *Let Assumption (H1), (H2) hold, if steplength $\alpha_k > 0$ is computed by the nonmonotone line search (7) then we have*

$$\lim_{k\to\infty} \alpha_k d_k = 0. \quad \lim_{k\to\infty} \alpha_k g_k^T d_k = 0. \tag{16}$$

Proof Let $k-j$ be an integer satisfying $k-M \leq k-j \leq k$,

$$f(xk-1) = \max_{0\leq j\leq m} f(x_{k-1}). \tag{17}$$

It follows from (7) that the sequence $\{f(x_{k-j})\}$ is decreasing in fact, note that $g_k^T d_k < 0$, we have from (7) that

$$\begin{aligned} f(x_{k+1-j}) &= \max_{0\leq j\leq M} f(x_{k+1-j}) \\ &= \max\left(\max_{0\leq j\leq M} f(x_{k-j}), f(x_{k+1}\right) \\ &\leq \max\left(\max_{0\leq j\leq M-1} f(x_{k-j}), f(x_{k-M}), f(x_{k+1})\right) \\ &= \max\left(\max_{0\leq j\leq M} f(x_{k-j}), f(x_{k+1})\right) \\ &= f(x_{k-j}). \end{aligned} \tag{18}$$

We have from the last inequality (7) and (17) that

$$\begin{aligned} f(x_{k-j}) &= f(x_{k-j-1}) + \alpha_{k-j-1} d_{k-j-1} \\ &\leq \max_{0\leq j\leq M} f(x_{k-j-1}) + \delta_1 \alpha_{k-j-1} g_{k-j-1}^T d_{k-j-1} - \delta_2 \|\alpha_{k-j-1} z_k\|^2 \\ &= f(x_{k-j-1}) + \delta_1 \alpha_{k--1} g_{k-j-1}^T d_{k-j-1} - \delta_2 \|\alpha_{k-j-1} z_k\|^2. \end{aligned} \tag{19}$$

Since $\{f(x_{k-j})\}$ is decreasing and bounded from assumption (H1), let $k\to\infty$ in the above inequality, we have

$$\lim_{k\to\infty} \alpha_{k-j-1} d_{k-j-1} = 0. \tag{20}$$

Let $p = k+M+2$. Now by induction, we prove that for any $j \geq 1$ the following two formula hold

$$\lim_{k\to\infty} \alpha_{k-2j} d_{k-2j} = 0 \tag{21}$$

$$\lim_{k\to\infty} f(x_{k-2j}) = \lim_{k\to\infty} f(x_{k-j}). \tag{22}$$

For $j = 1$ since $\{p\} \subset \{k-j\}$, it follow from (20) that (21) hold, which shows that $\|x_{p-j} - x_{p-j-1}\| \to 0$ as $f(x)$ is uniformly continuous in the level set, (22) holds for $j = 1$.

Now suppose that (21) and (22) holds for given j. It follows from (7) that

$$f(x_{p-2j}) \leq f(x_{p-2j-1}) + \delta_1 \alpha_{p-2j-1} g_{p-2j-1}^T d_{p-2j-1} - \delta \|\alpha_{p-2j-1} Z_{p-2j-1}\|^2 \quad (23)$$

where

$$\begin{aligned} \|\alpha_k Z_k\| &= \left\| \alpha_k y_k + \alpha_k C g_k^T d_k + \alpha_k \max\left\{0, -\frac{y_k^T v_k}{\|v_k\|}\right\} \right\| \\ &< \|\alpha_k y_k\| + \|\alpha_k C g_k^T d_k\| + \left\| \alpha_k \max\left\{0, -\frac{y_k^T v_k}{\|v_k\|}\right\} \right\| \\ &< \|\alpha_k C g_k^T d_k\| \\ &\leq \|\alpha_k g_k^T d_k\|. \end{aligned} \quad (24)$$

Let $k \to \infty$ we get from (22) $\lim_{k\to\infty} \|x_{p-2j} - x_{p-2j-1}\| \to 0$. Since is uniformly continuous in the level set.

$$\lim_{k\to\infty} f(x_{p-2j-1}) = \lim_{k\to\infty} f(x_{p-2j}) = \lim_{k\to\infty} f(x_{p-j}). \quad (25)$$

Thus (21), (22) hold for any $j \geq 1$. Now for any k, it hold that

$$x_{k+1} = x_{k-j} - \sum_{j=1}^{p-j-k-1} \alpha_{p-2j} Z_{p-2j}. \quad (26)$$

Since $p-k-1 = k+M+2-k-l-1 < k+M+2-k-1 = M+1$, we have from (21) and (26) that

$$\lim_{k\to\infty} \|x_{k+1} - x_{p-j}\| = 0. \quad (27)$$

We get the uniform continuity of $f(x)$ that

$$\lim_{k\to\infty} f(x) = \lim_{k\to\infty} f(x_{p-j}). \quad (28)$$

It follows from (7), (24) that

$$f(x_{k+1}) \leq f(x_{p-j}) + \delta_1 \alpha_k g_k^T d_k - \delta_2 \|\alpha_k Z_k\|^2. \quad (29)$$

Let $k \to \infty$ we have

$$\lim_{k\to\infty} \alpha_k d_k = 0, \qquad \lim_{k\to\infty} \alpha_k g_k^T d_k = 0. \quad \blacksquare$$

3 Numerical Results

In this section we present the computational performance of a newly-programmed FORTRAN implementation of the new algorithm on as set of 16 unconstrained optimization test problem (Appendix 1). The test problem in the CUTE [1] library, along with other inexact line search optimization problem in extended or generalized form. Each problem is tested 4 time for a gradually increasing number of variable; $n = 5, 10, 100, 1000$ at the same time we present comparisons with standard BFGS with Armijo line search algorithms, including the performance profiles of new nonmonotone line search. The same stopping criterion employed was $\|g_k\| < 1 \times 10^{-6}$ for all two algorithms report some numerical results obtained by newly-programmed FORTRAN procedure with double precision.

In comparison of algorithms the function evaluation is normally assumed to be the most costly factor in each iteration and the number of iterations. We solve each of these test function by the

1. BFGS with Inexact Line Search (BFGSAR) Algorithm:
2. The New Nonmonotone line Search for BFGS (NEW) Algorithm.

All the numerical results are summarized in table (1) and (3). They present the number of iterations (NOI) versus the number of function evaluations (NOF) while table (2) give the percentage performance of the new algorithm based on both (NOI) and (NOF) against the original algorithm.

The important thing is that the new algorithm solves each particular problem measured by (NOI) and (NOF) respectively. Moreover, the new proposed algorithm always performs more stably and efficiently.

Namely there are about (42-53)% improvements of NOI for all dimensions Also there are (35 -41)% improvements of NOF for all test functions.

N of OF TEST	TEST FUNCTION	BFGS NOF (NOI)				NEW NOF (NOI)			
		5	10	100	1000	5	10	100	1000
1	Gen-Cubic	207	207	209	209	42	43	46	49
		125	125	126	126	31	32	34	36
2	Liarwhd	24	51	98	1116	27	37	33	70
		15	41	49	388	20	29	24	64
3	Shanno	32	49	39	25	24	35	2‘	25
		20	34	19	17	18	28	14	17
4	Ex-Beale	32	28	29	29	19	19	19	26
		20	20	21	21	15	15	15	20
5	Gen-Wood	295	268	285	299	244	231	190	236
		260	240	254	268	222	213	167	208
6	dqrie	28	28	24	18	24	26	20	17
		18	18	16	11	18	20	15	12
7	Gen-Helical	85	85	87	87	56	57	53	74
		59	59	60	60	44	44	39	54
8	Gen-Beale	61	61	62	62	15	15	15	24
		34	34	35	35	12	12	12	19
9	Gen-Recip	19	20	20	22	14	14	14	14
		14	15	15	16	11	11	11	11
10	Gen-Edger	21	21	22	22	11	11	11	12
		12	12	13	13	8	8	8	9
11	Gen-	155	122	173	168	52	97	74	81
	Non-Digonal	92	69	103	95	32	75	58	65
12	APQ	15	17	84	372	11	19	73	354
		9	13	74	354	8	15	68	345
13	TPQ	19	21	81	370	16	19	75	343
		13	15	72	352	12	15	70	334
14	Gen-Shallow	119	103	113	120	15	15	15	15
		114	100	108	115	11	11	11	11
15	Gen-Powell	128	132	157	177	60	60	121	69
		112	116	141	161	52	52	111	60
16	Arwwhed	31	31	32	32	10	13	21	35
		15	15	16	16	6	8	13	24
General	Total of	1271	1244	1515	3128	640	711	801	1444
functions	16	932	911	1122	2048	520	588	670	1289

Table (1) : Comparison between the standard BFGS with Inexact Line Search (BFGS) Algorithm and New proposed algorithms using different value of $5 < N < 1000$ for group of test function.

N	Costs	NEW
5	NOF	49.64
	NOI	41.21
10	NOF	42.84
	NOI	35.46
100	NOF	47.13
	NOI	40.29
1000	NOF	53.84
	NOI	37.06

Table 2 : Percentage performance of the standard BFGS with Inexact Line Search (BFGS) Algorithm and New algorithm for 100% in both NOI and NOF.

4 Conclusions

In this Paper, a new form of nonmonotone line search has been proposed for guaranteeing the global convergence of Quasi-Newton method for minimizing un constrained optimization problem. Numerical experiments show that the new nonmonotone line search is more efficient and available for BFGS method.

Appendix 1

All the test functions used in Table 1 for this paper are from general literature [2]:

1. Generalized Cubic function:

$$f(x) = \sum_{i=1}^{n/2}[100(x_{2i} - x_{2i-1}^3)^2 + (1 - x_{2i-1})^2], x_0 = [-1.2, 1, \cdots, -1.2, 1.].$$

2. Liarwhd Function (cute):

$$f(x) = \sum_{i=1}^{n} 4(-x_1 + x_i^2)^2 + \sum_{i=1}^{n}(x_i - 1)^2, x_0 = [4., 4., \cdots, 4.].$$

3. Nondia (Shanno-78) Function (Cute):

$$f(x) = (x_i - 1)^2 + \sum_{i=2}^{n} 100(x_1 - x_{i-1}^2)^2, \quad x_0 = [-1., -1., \cdots, -1.].$$

4. Extended Beale Function:

$$\begin{aligned} f(x) &= \sum_{i=1}^{n/2}[1.5 - x_{2i-1} + (1 - x_{2i})]^2 + [2.25 - x_{2i-1}(1 - x_{2i}^2)]^2 \\ &\quad + [2.625 - x_{2i-1}(1 - x_{2i}^3)]^2, \\ x_0 &= [1., 0.8, \cdots, 1., 0.8]. \end{aligned}$$

5. Generalized wood function:

$$\begin{aligned} f(x) &= \sum_{i=1}^{n/2} 100(x_{4i-2} - x_{4i-2}^2)^2 + (1 - x_{4i-3})^2 + 90(x_{4i} - x_{4i-1}^2)^2 + (1 - x_{4i-1})^2 \\ &\quad +10.1(x_{4i-2} - 1)^2 + (x_{4i} - 1)^2 + 19.8(x_{4i-2} - 1)(x_{4i-2} - 1)(x_{4i-2} - 1) \\ x_0 &= [-3., -1., -3., -1., \cdots, -3., -1., -3., -1.]. \end{aligned}$$

6. Dqudrtic Function (CUTE):

$$f(x) = \sum_{i=1}^{n-2} (x_i^2 + cx_{i+1}^2 + dx_{i+2}^2), x_0 = [3., 3., \cdots, 3., 3.], c = 100, d = 100.$$

7. General Helical Function:

$$f(x) = \sum_{i=1}^{n/3} (1 - -x_{3i} - 10^* H_i)^2 + 100(R_i - 1)^2 + x_{3i}^2,$$

$$R_i = \sqrt{(x_{3i-2}^2 + x_{3i-1}^2)}, H_1 = \frac{\tan^{-1} \frac{x_{3i-1}}{x_{3i-2}}}{2.PI}$$

where

$$x_0 = [-1., 0., 0., \cdots, -1., 0.], 0.$$

8. Generalized Beale Function:

$$\begin{aligned} f(x) &= \sum_{i=1}^{n/2} [1.5 - x_{2i} + (1 - x_{2i})]^2 + [2.25 - x_{2i-1}(1 - x_{2i}^2)]^2 \\ &\quad +[2.625 - x_{2i-1}(1 - x_{2i}^2]^2, \\ x &= [-1., -1., \cdots, -1., -1.]. \end{aligned}$$

9. Generalized Recip Function:

$$f(x) = \sum_{i=1}^{n/3} \left[(x_{3i-1} - 5)^2 + x_{9i-1}^2 + \frac{x_{3i}^2}{(x_{3i-1} - x_{3i} - 2)^2} \right], x_0 = [2., 5., 1., \cdots, 2., 5., 1.].$$

10. Generalized Edger Function:

$$f(x) = \sum_{i=1}^{n/2} (x_{2i-1} - 2)^4 + (x_{2i-1} - 2)^2 x_{2i}^2 + (x_{2i} + 1)^2, x_0 = [1., 0., \cdots, 1., 0.].$$

11. Generalized Non diagonal function:

$$f(x) = \sum_{i=2}^{n} [100(x_1 - x_i^2)^2 + (1 - x_i)^2, \quad x_0 = [-1., \cdots, -1.].$$

12. Almost Perturbed Quadratic Function:

$$f(x) = \sum_{i=1}^{n} i x_i^2 + \frac{1}{100}(x_1 + x_n)^2, \quad x_0 = [0.5, 0.5, \cdots, 0.5, 0.5].$$

13. Tridiagonal Perturbed Quadratic Function:

$$f(x) = x_i^2 + \sum_{i=2}^{n-1} i x_i^2 + (x_{i-1} + x_i + x_{i+1})^2, \text{‘} \ x_0 = [0.5, 0.5, \cdots, 0.5, 0.5].$$

14. Generalized Shallow Function:

$$f(x) = \sum_{i=1}^{n/2} (x_{2i-1}^2 - x_{2i})^2 + (1 - x_{2i-1})^2, \quad x_0 = [-2., -2., \cdots, -2., -2.].$$

15. Generalized Powell function:

$$\begin{aligned} f(x) &= \sum_{i=1}^{n/3} \left\{ 3 - \left[\frac{1}{1 + (x_i - 2x_i)^2} \right] - \sin\left(\frac{\pi x_{2i} x_{3i}}{2} \right) - \exp\left[- \left(\frac{x_i + x_{3i}}{x_{2i}} - 2 \right)^2 \right] \right\}, \\ x_0 &= [0., 1., 2., \cdots, 0., 1., 2.]. \end{aligned}$$

16. Arwhead Function (CUTE):

$$f(x) = \sum_{i=1}^{n-1} (4x_i + 3) + \sum_{i=1}^{n-1} (x_i^2 + x_n^2)^2, \quad x_0 = [1., 1., \cdots, 1., 1.].$$

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Optimization of an intense magnetic environment effect on the electrical characteristics of a MOS transistor

S. LATRECHE[1], S. LABIOD[2], C. GONTRAND[3]

[1]Laboratoire (LHS), Dept. Electronique, Univ. Mentouri. Constantine. Algeria
e-mail:latreche.saida@gmail.com
[2]Laboratoire (LHS), Dept. Electronique, Univ. Mentouri. Constantine. Algeria
e-mail: samir.labiod@gmail.com
[3]Institut de Nanotechnologie de Lyon (INL). INSA-Lyon. Univ.lyon. France

Abstract

The development of electronics components has generated ineluctable reduction of all devise dimensions, especially, channel width and oxide thickness. This miniaturization induces many electrical instabilities in the components, which disturb their behavior. Among these components, we were interested to a MOS (Metal-Oxide-Semiconductor) transistor. In order to optimize the performances of this last one, we present in this paper a study concerning the influence of a magnetic field on its electrical characteristics. The different simulations carried out with (ISE-TCAD 8.0) software showed that the presence of a constant magnetic field influences the behavior of MOS transistor. In fact, the magnetic field leads to a displacement of the operating point, an appearance of magneto-resistance effect, and a reduction of the threshold potential with the considered magnetic induction. It was shown in addition that the transistor is more sensitive to a perpendicular magnetic field (along Z axis) than to a parallel one).

Keywords: *Drift-Diffusion Model, Lorentz Force, Magneto-Resistance, MOS Transistor.*

1 Introduction

The miniaturization of transistor's dimensions, the reduction of their voltage levels and the increase of their speed make their behavior more complex. The influence of those parameters is then harmful and generates consequently many perturbations such as, capacitive coupling, noise, and an electromagnetic influence. We quote for example that in some applications which are based on the nuclear magnetic induction, a CMOS circuitry is placed and must work under an intense magnetic field. In this paper, the influence of magnetic field on electrical characteristics of a MOS transistor is studied. The sensitivity of the device in the presence of the magnetic field is also studied.

2 Simulated Structure

The 2-D structure used in the simulation of the active device is shown in Fig 1. It is an N-channel MOS with a substrate doping $N_A = 5.10^{16} cm^{-3}$, an oxide thickness $T_{OX} = 4nm$. The device with is $1\mu m$. The separation between the drain and source is $L = 0.15\mu m$, with gate length $L_G = 0.18\mu m$

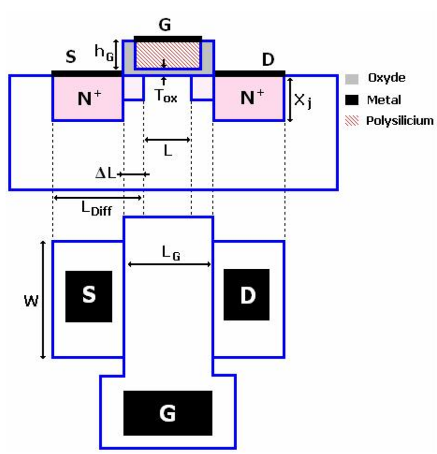

Figure 1: A view of the simulated MOS

As shown in Fig.2. The channel is doped Boron following Gaussian function. This allow to control threshold voltage of the device [1]. The drain and source are doped Arsenic using Gaussian function. The surface concentration is $5.10^{21} cm^{-3}$.

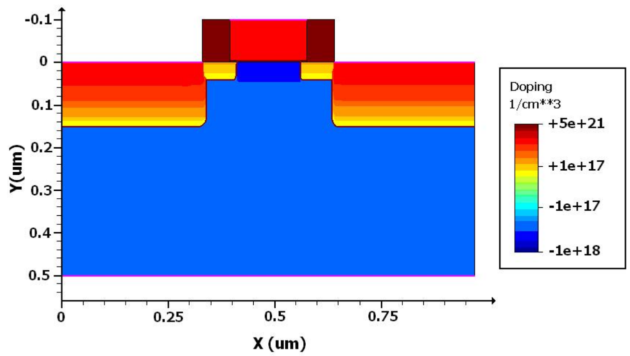

Figure 2: The Doping Profile

On Fig.3, we show the variation of the output characteristics for different gate ($V_{gs} = 1v, V_{gs} = 1.5v$) voltage values. The transfert characteristic is represented for $V_{ds} = 1.5v$

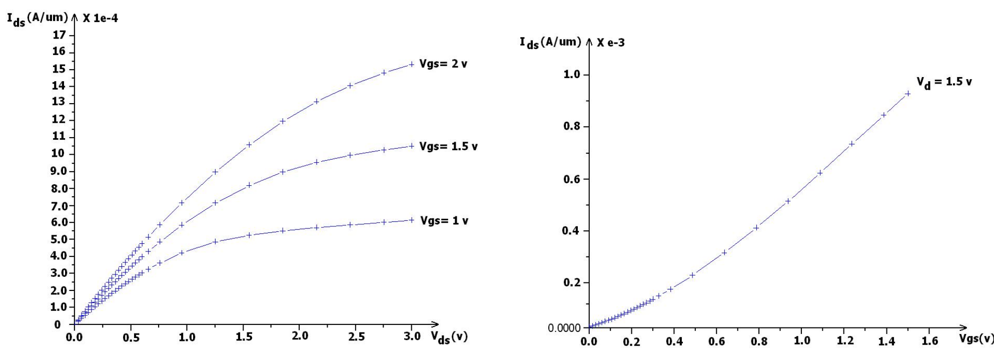

Figure 3: The current-voltage I-V characteristics

3 Mathematical model

For the analysis of magnetic field effects in MOS transistors, one has to set up and solve the transport equations governing the flow of electrons (n) and holes (p) in the device. To this end we must introduce, to the commonly used drift-diffusion model,the magnetic-field dependent term[2]. These laters are added to the action of the LORENTZ force on the motion of the carriers [3]. While the continuity and Poisson's equation remain the same as zero fields, the current density has to be modified, to account for the LORENTZ force [4]:

$$\vec{j}^{B}_{n,p} = -\sigma_{n,p}\vec{\nabla}\varphi_{n,p} - \sigma_{n,p}\frac{1}{1+(\mu_h B)^2}[\mu_h\vec{B}\wedge\vec{\nabla}\varphi_{n,p} + \mu_h\vec{B}\wedge(\mu_h\vec{B}\wedge\vec{\nabla}\varphi_{n,p})] \quad (1)$$

Where $\sigma_{n,p}$represent the electric conductivity of the electrons or holes,$\nabla\varphi_{n,p}$ the gradient of the electron quasi-Fermi potential,μ_h the hall mobility related to the normal mobility as $\mu_h = r_{n,p}\mu_{n,p}$ with $r_{n,p}$ the Hall scattering factor, and $\vec{B}$ the magnetic field applied with the Poisson and continuity equations[5].A device under magnetic field can be properly simulated under the drifft-difussion approximation. The vector product between the current density and the magnetic field is computed by considering a local coordinate system and the neighboring points [6,7]. The two-dimensional conductivity of the electron under a magnetic field, applied perpendiculary to the current flow plane is described by

$$\begin{cases} j_x &= \sigma_{xx}E_x + \sigma_{xy}E_y \\ j_y &= -\sigma_{xy}E_x + \sigma_{xx}E_y \end{cases} \tag{2}$$

Where E_x and E_y are the components of the electric field in the (x, y) plane E_x and σ_{xx} and σ_{xy} are the components of the conductivity tensor.

$$\begin{cases} \sigma_{xx} &= \frac{\sigma_0}{1+(\mu_h B)^2} \\ \sigma_{xx} &= \frac{\sigma_0 \mu_h B}{1+(\mu_h B)^2} \end{cases} \tag{3}$$

In the case of the HALL bar geometry $L \gg W$, we have $j_y = 0$. This leads to $E_x = \frac{j_x}{\sigma_0}$, which is independent on B. In the case of the long and narrow devices $W \gg L$, then $E_y = 0$ and $j_x = \sigma_{xx}E_x$ [8,9].

Then,

$$E_x = j_x \frac{1 + (\mu_h B)^2}{\sigma_0} \tag{4}$$

so we can determine the new resistance between drain and source

$$R_{ds} = R_0(1 + (\mu_h B)^2) \tag{5}$$

4 Results and discussions

This study concerns the effect of a transverse magnetic field applied in the plane of the MOS transistor and perpendicular to the current flow.

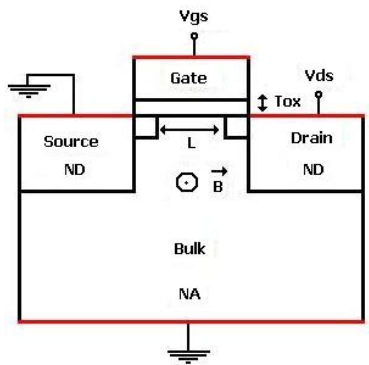

Figure 4: Cross-section view of a MOS transistor

4.1 Sensitivity

A variation of sensitivity versus a gate to source and a drain to source biases with an external applied magnetic field B is studied. The ranges of V_{gs}, V_{ds} and $\vec{B}$ fields are $0v \leq V_{gs} \leq 1.6v$, $0v \leq V_{ds} \leq 3.2v$ and $B = 0.6Tesla$

The device sensitivity is defined as [10]:

$S(B) = \frac{I_{ds}(B\neq 0-I_{ds}(B=0)}{BI_{ds}(B=0)}$

The study of the sensitivity variations with drain and gate voltage is effectuated for a magnetic induction B=0.6 Tesla. The results are shown in Fig.5. We note that in the case of a parallel magnetic field, the sensitivity device is negligible. The perpendicular magnetic field however has a great influence on the device sensitivity

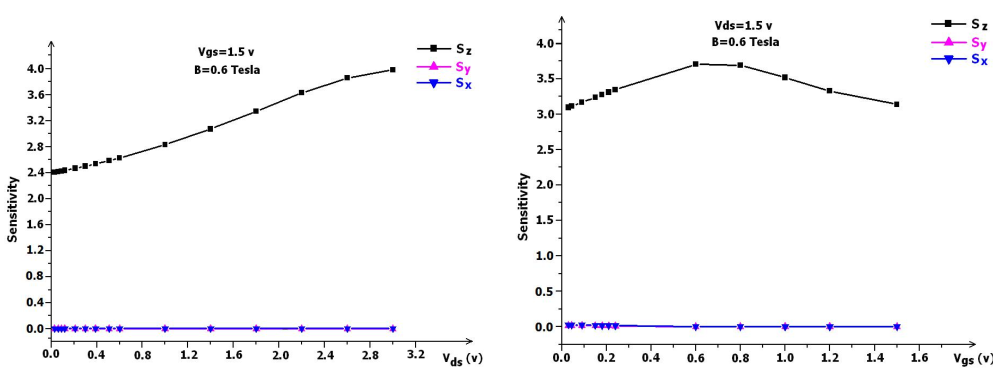

Figure 5: Sensitivity variation versus V_{gs} and V_{ds}

A variation of sensitivity versus a gate to source V_{gs} and drain to source V_{ds} biases and an external applied magnetic fields B is studied.

4.2 Magneto-resistance in inversion layer

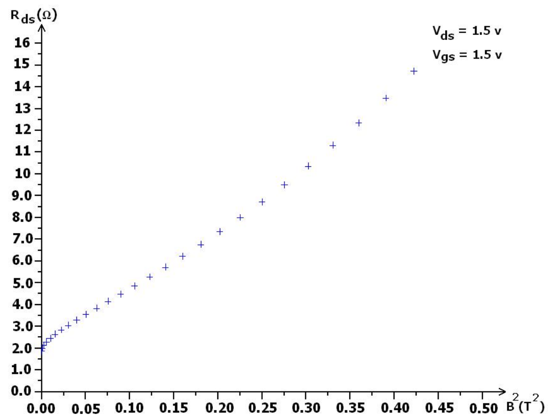

Figure 6: Variations of threshold voltage versus magnetic filed

The variations of the channel resistance R_{ds} with the magnetic field are shown in Fig.6 for a gate voltage $V_{gs} = 1.5v$ at $V_{ds} = 1.5v$. The quasi linearity of R_{ds} versus B^2 confirms the validity of the magneto-resistance analysis carried out in equation (5).

4.3 Threshold voltage

The application of a magnetic field displaces the inversion layer. The force is offset by the Hall voltage developed vertically in the channel which effectively adds another term to the threshold voltage V_T:

$$V_T = 2\varphi_{FI} + \frac{1}{C_{OX}}(4qN_A\varepsilon_s\varphi_{FI})^{\frac{1}{2}} + V_H$$

Where the third term is due to the developed Hall voltage due to the magnetic field. The Fig.7 show the variation of the threshold voltage V_T versus a magnetic field applied on the considered MOS transistor.

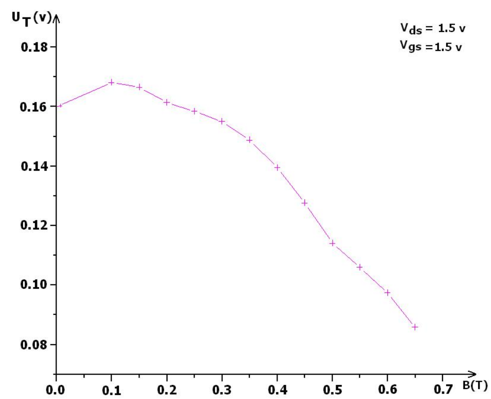

Figure 7: Variations of threshold voltage versus magnetic field

5 Conclusion

A magnetic field is applied perpendicular to the carrier flow in order to generate a vertical Hall voltage. this later induce a shift of the threshold voltage of the considered device. The orientations of the magnetic field according to x and y axes, have a negligible effect on the device sensitivity. This later increases with drain voltage, and reach its maximum values for gate voltage $V_{gs} = 0.6v$. The electrons deviations caused by the Lorentz force effect reduce the electron's number in the inversion layer. This induce an augmentation of the the inversion layer resistance with B^2 and a displacement of the operating point, as shown in Fig.6. The relation between R_{ds} and B^2 is almost linear .

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Optimizing the Time Spent by Diffusion Processes in Intervals

Mario Lefebvre

École Polytechnique de Montréal, Canada
e-mail: mlefebvre@polymtl.ca

Abstract

The problem of optimally controlling a one-dimensional diffusion process $X(t)$ ***is considered. The aim is either to minimize or to maximize the time spent by*** $X(t)$ ***in an interval. It is shown that it is sometimes possible to obtain the optimal control by considering the corresponding uncontrolled process. The problem formulation generalizes that in Whittle (1982). The same type of optimization problem is also treated for a two-dimensional degenerate diffusion process*** $(X(t), Y(t))$ ***for which the derivative of*** $X(t)$ ***is a deterministic function of*** $X(t)$ ***and*** $Y(t)$***. This problem has applications in reliability theory.***

Mathematics Subject Classification: 93E20.

Keywords: *Exit time, LQG homing, mean time to failure, reliability, survival optimization, Wiener process.*

1 Introduction

We consider the one-dimensional controlled diffusion process $\{X(t), t \geq 0\}$ defined by the stochastic differential equation

$$dX(t) = a[X(t)]\,dt + b[X(t)]\,L[u(t)]\,dt + \{N[X(t)]\}^{1/2}\,dB(t), \qquad (1)$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion, $u(t)$ is the control variable, $a(\cdot)$ is a real function, $b(\cdot) \neq 0$ and $N(\cdot) > 0$. Moreover, $L(\cdot)$ is a differentiable function such that $L'(\cdot) \neq 0$.

Assume that $X(t_0) = x \in I := (d_1, d_2)$, and define

$$T(x, t_0) = \inf\{t > t_0 : X(t) \notin I \mid X(t_0) = x\}.$$

That is, $T(x, t_0)$ is the first time the controlled process $X(t)$ leaves the interval I. Notice that, by continuity, we can write that $X[T(x, t_0)] = d_1$ or d_2.

Our aim is to find the control u^* that minimizes the expected value of the cost criterion

$$J(x, t_0) := \int_{t_0}^{T(x,t_0)} \tfrac{1}{2} q[X(t)]\, C[u(t)]\, dt + K[T(x, t_0)],$$

where $q(\cdot) > 0$ and $C(\cdot) \geq 0$. The case where $K[T(x, t_0)] = \lambda T(x, t_0)$ has already been treated. When the parameter λ is positive, the optimizer seeks to minimize the time spent by $\{X(t), t \geq 0\}$ in I, taking the control costs into account. This type of problem has been termed *LQG homing* by Whittle (1982, p. 289) (see also Whittle (1990, p. 222) for a risk-sensitive formulation). If λ is negative, the objective is to maximize the survival time in I (again, taking the control costs into account).

The author considered LQG homing problems in a number of papers (see, for instance, Lefebvre (2004)). Kuhn (1985) and, very recently, Makasu (2009) treated risk-sensitive homing problems.

If $L[u(t)] = u(t)$ and $C[u(t)] = u^2(t)$, and if the relation

$$\alpha N(x) = \frac{b^2(x)}{q(x)} \tag{2}$$

holds for a positive constant α, then using a theorem due to Whittle (1982) we can express the optimal control u^* $(= u^*(t_0))$ as follows:

$$u^* = \frac{N(x)}{b(x)} \frac{G_x(x, t_0)}{G(x, t_0)},$$

where

$$G(x, t_0; \alpha) := E\left[\exp\{-\alpha K[\tau(x, t_0)]\}\right]. \tag{3}$$

In the above equation, $\tau(x, t_0)$ is the same as the random variable $T(x, t_0)$, but for the *uncontrolled* process $\{\xi(t), t \geq 0\}$ defined by

$$d\xi(t) = a[\xi(t)]\, dt + \{N[\xi(t)]\}^{1/2}\, dB(t).$$

That is, $\{\xi(t), t \geq 0\}$ is the diffusion process obtained by setting $L[u(t)] = 0$ in (1), and

$$\tau(x, t_0) = \inf\{t > t_0 : \xi(t) \notin I \mid \xi(t_0) = x\}.$$

Remarks. (i) Notice that the constant α in (2) always exists if N, b and q are constants.

(ii) In the present paper, we will choose the termination cost

$$K[T(x,t_0)] = -\ln[T(x,t_0)+1]/\alpha.$$

Then G is simply the mathematical expectation of $\tau(x,t_0)+1$:

$$G(x,t_0;\alpha) = E\left[\exp\{-\alpha(-\ln[\tau(x,t_0)+1]/\alpha)\}\right] = 1 + E\left[\tau(x,t_0)\right].$$

In Section 2, we will show that we can obtain the optimal control from the mathematical expectation in (3) when we choose $C[u(t)] = L^2[u(t)]$, thus generalizing the theorem in Whittle (1982). Next, in Section 3 the same type of problem will be considered for a degenerate two-dimensional controlled diffusion process $(X(t), Y(t))$ for which the derivative of $X(t)$ is a deterministic function $\rho[X(t), Y(t)]$. The results obtained in that section are useful in reliability theory, when one wants to maximize the lifetime of a device (see Lefebvre (2009)). In both sections, examples will be presented. Finally, we will give some concluding remarks in Section 4.

2 Optimal Control in One Dimension

Let $F(x,t_0)$ be the value function defined by

$$F(x,t_0) = \inf_u E\left[J(x,t_0) \mid X(t_0) = x\right].$$

Assuming that the function F exists and is twice differentiable, we can show that it satisfies the dynamic programming equation

$$\begin{aligned} 0 \;=\; & \inf_u \{F_{t_0}(x,t_0) + a(x)\,F_x(x,t_0) + b(x)\,L(u)\,F_x(x,t_0) \\ & \quad + \tfrac{1}{2}q(x)\,C(u) + \tfrac{1}{2}N(x)\,F_{xx}(x,t_0)\}. \end{aligned}$$

It follows that the optimal control u^* is such that

$$b(x)\,L'(u^*)\,F_x(x,t_0) + \tfrac{1}{2}q(x)\,C'(u^*) = 0. \tag{4}$$

We make the following assumption:

H_1: Eq. (4) can be solved explicitly for u^*.

Once we have found u^* in terms of $F_x(x,t_0)$, we must solve the (generally non-linear) partial differential equation

$$\begin{aligned} 0 \;=\; & F_{t_0}(x,t_0) + a(x)\,F_x(x,t_0) + b(x)\,L(u^*)\,F_x(x,t_0) \\ & + \tfrac{1}{2}q(x)\,C(u^*) + \tfrac{1}{2}N(x)\,F_{xx}(x,t_0). \end{aligned} \tag{5}$$

This equation is valid for $x \in (d_1, d_2)$. The boundary conditions are

$$F(x, t_0) = K(t_0) \quad \text{if } x = d_1 \text{ or } d_2. \tag{6}$$

Consider now the particular case where $C(u) = L^2(u)$. Then Eq. (4) becomes

$$b(x)\,L'(u^*)\,F_x(x, t_0) + \tfrac{1}{2}q(x)\,2\,L(u^*)\,L'(u^*) = 0.$$

Since we have assumed that $L'(\cdot) \neq 0$, we can write that

$$b(x)\,F_x(x, t_0) + q(x)\,L(u^*) = 0. \tag{7}$$

Hence, in this particular case, the assumption H_1 is equivalent to
H_1^*: the function L^{-1} exists.
The optimal control can thus be expressed as follows:

$$u^* = L^{-1}\left(-\frac{b(x)\,F_x(x, t_0)}{q(x)}\right).$$

Therefore, to obtain the value of u^*, we must find the derivative of the value function $F(x, t_0)$ with respect to x, and determine the inverse function L^{-1}.

Remark. If the inverse function L^{-1} does not exist, then the optimal control might not be unique. For example, if $L(u) = u^2$, then there are two values of u that minimize the expected value of the cost criterion $J(x, t_0)$.

Next, since (from Eq. (7)) $L(u^*) = -b(x)\,F_x(x, t_0)/q(x)$, substituting into Eq. (5) we find that $F(x, t_0)$ satisfies the partial differential equation

$$0 = F_{t_0}(x, t_0) + a(x)\,F_x(x, t_0) - \frac{b^2(x)}{2\,q(x)}\,[F_x(x, t_0)]^2 + \tfrac{1}{2}\,N(x)\,F_{xx}(x, t_0). \tag{8}$$

In the case where $F_{t_0}(x, t_0) = 0$, this equation is a Riccati equation for $F_x(x, t_0)$.

It is sometimes possible to solve explicitly the second order non-linear differential equation (8). However, we obtain a very interesting probabilistic interpretation by making the same transformation as in Whittle (1982): we assume that the relation in (2) holds, and we define

$$H(x, t_0; \alpha) = e^{-\alpha F(x, t_0)}.$$

We then find that Eq. (8) is transformed into the linear equation

$$0 = H_{t_0}(x, t_0; \alpha) + a(x)\,H_x(x, t_0; \alpha) + \tfrac{1}{2}\,N(x)\,H_{xx}(x, t_0; \alpha). \tag{9}$$

Furthermore, the boundary conditions become

$$H(x, t_0; \alpha) = e^{-\alpha K(t_0)} \quad \text{if } x = d_1 \text{ or } d_2. \tag{10}$$

Now, Eq. (9) is the differential equation satisfied by the function $G(x, t_0; \alpha)$ defined in (3), and the conditions in (10) are the appropriate boundary conditions. Hence, we can write that $H(x, t_0; \alpha) \equiv G(x, t_0; \alpha)$, and we obtain the following proposition.

Proposition 2.1 *Under the above hypotheses, if $C(u) = L^2(u)$, then the optimal control u^* can be obtained from the mathematical expectation $G(x, t_0; \alpha)$ for the uncontrolled process that corresponds to $\{X(t), t \geq 0\}$.*

Remark. We also assume that $P[\tau(x, t_0) < \infty] = 1$, so that the solution of (9) and (10) is unique.

Particular case. Let us consider the case where $\{X(t), t \geq 0\}$ is a controlled standard Brownian motion, so that $a[X(t)] \equiv 0$ and $N[X(t)] \equiv 1$. Furthermore, let us choose $b[X(t)] \equiv b_0$ ($\neq 0$) and $q[X(t)] \equiv q_0$ (> 0). Then the positive constant α in (2) is given by b_0^2/q_0. Finally, assume that $K[T(x, t_0)] = -\ln[T(x, t_0)+1]/\alpha$. Then, as mentioned in Section 1, $G(x, t_0; \alpha)$ becomes $1 + E[\tau(x, t_0)]$.

Now, as is well known (see, for instance, Lefebvre (2007, p. 220)), in this case the function $m(x, t_0) := E[\tau(x, t_0)]$ satisfies the second-order ordinary differential equation

$$\tfrac{1}{2} m_{xx}(x, t_0) = -1,$$

subject to the boundary conditions

$$m(x, t_0) = t_0 \quad \text{if } x = d_1 \text{ or } d_2.$$

We easily find that

$$m(x, t_0) = -x^2 + (d_1 + d_2)x + t_0 - d_1 d_2.$$

It follows that the function $G(x, t_0; \alpha)$ is given by

$$G(x, t_0; \alpha) = -x^2 + (d_1 + d_2)x + t_0 - d_1 d_2 + 1,$$

so that $G_x(x, t_0; \alpha) = -2x + (d_1 + d_2)$, and the optimal control is

$$u^* = L^{-1}\left\{\frac{-2x + d_1 + d_2}{b_0[-x^2 + (d_1 + d_2)x + t_0 - d_1 d_2 + 1]}\right\}.$$

In the special case where $d_1 = -d$, $d_2 = d$, $t_0 = 0$, and $L(u) = u^3$, we obtain that

$$u^* = \left\{\frac{-2x}{b_0(d^2 - x^2 + 1)}\right\}^{1/3}.$$

Remarks. (i) Because the constant α is positive, the termination cost function $K[T(x, t_0)] = -\ln[T(x, t_0) + 1]/\alpha$ is negative. That is, a reward is given for survival in the continuation region $I := (d_1, d_2)$. Hence, the optimizer wants to maximize the time spent by the controlled process in I.

(ii) The function $F(x, t_0)$ is given by

$$F(x, t_0) = -\ln[G(x, t_0; \alpha)]/\alpha = -\frac{q_0}{b_0^2} \ln\left[-x^2 + (d_1 + d_2)x + t_0 - d_1 d_2 + 1\right].$$

We can check that this function satisfies both the partial differential equation (8) and the boundary conditions (6) (with $K(t_0) = -\ln(t_0 + 1)/\alpha$).

In the next section, we will consider an optimal control problem in two dimensions.

3 Optimal Control in Two Dimensions

Let $(X(t), Y(t))$ be the two-dimensional degenerate diffusion process defined by the system of stochastic differential equations

$$\begin{aligned} dX(t) &= -\frac{X^{-d}(t)}{d_0[Y(t) - c]} dt, \\ dY(t) &= a[X(t), Y(t)]\, dt + b[X(t), Y(t)]\, L[u(t)]\, dt + \{N[X(t), Y(t)]\}^{1/2} dB(t), \end{aligned}$$

where $d_0 > 0$ and $d > -1$. This system is a particular case of the controlled stochastic processes proposed by Rishel (1991) to model the wear of machines.

Define the first passage time $T(x, y, t_0)$ as follows:

$$T(x, y, t_0) = \inf\{t > t_0 : \{X(t) = 0\} \cup \{Y(t) = c\} \mid X(t_0) = x, Y(t_0) = y\},$$

where $x > 0$ and $y > c$. The random variable $X(t)$ denotes the remaining lifetime of the machine, whereas $Y(t)$ is a random variable that is closely correlated with the lifetime of this machine. We assume that production must be stopped (and the machine repaired) if $Y(t)$ decreases to c before $X(t)$ reaches the origin.

We want to find the control u^* that minimizes the expected value of

$$J(x, y, t_0) := \int_{t_0}^{T(x,y,t_0)} \tfrac{1}{2} q[X(t), Y(t)]\, C[u(t)]\, dt + K[T(x, y, t_0)],$$

where $q(\cdot, \cdot) > 0$ and $C(\cdot) \geq 0$.

Next, we define the value function $F(x, y, t_0)$ by

$$F(x, y, t_0) = \inf_u E\left[J(x, y, t_0) \mid X(t_0) = x, Y(t_0) = y\right].$$

Proceeding as in the previous section (assuming in particular that the inverse function L^{-1} exists), we find that the optimal control can be expressed as follows:

$$u^* = L^{-1}\left(-\frac{b(x, y)\, F_y(x, y, t_0)}{q(x, y)}\right).$$

Moreover, if the relation

$$\alpha N(x,y) = \frac{b^2(x,y)}{q(x,y)} \tag{11}$$

is valid for a positive constant α, then the function $F(x, y, t_0)$ can be obtained from the formula

$$e^{-\alpha F(x,y,t_0)} = E\left[\exp\{-\alpha K[\tau(x,y,t_0)]\}\right],$$

where $\tau(x, y, t_0)$ is the random variable that corresponds to $T(x, y, t_0)$ for the uncontrolled process $(\xi(t), \eta(t))$ defined by

$$\begin{aligned} d\xi(t) &= -\frac{\xi^{-d}(t)}{d_0\,[\eta(t)-c]}\,dt, \\ d\eta(t) &= a[\xi(t),\eta(t)]\,dt + \{N[\xi(t),\eta(t)]\}^{1/2}\,dB(t). \end{aligned}$$

Furthermore, we assume that $P[\tau(x, y, t_0) < \infty] = 1$.

Particular case. We consider the case where $a[X(t), Y(t)] \equiv 0$, $N[X(t), Y(t)] \equiv 1$, $b[X(t), Y(t)] \equiv b_0$ $(\neq 0)$ and $q[X(t), Y(t)] \equiv q_0$ (> 0). It follows that the constant α defined in (11) is again given by b_0^2/q_0. Moreover, we choose

$$K[T(x,y,t_0)] = -\ln[T(x,y,t_0)+1]/\alpha,$$

so that

$$E\left[\exp\{-\alpha K[\tau(x,y,t_0)]\}\right] = 1 + E\left[\tau(x,y,t_0)\right].$$

Thus, the stochastic optimal control problem is reduced to the calculation of the expected value of the first passage time $\tau(x, y, t_0)$.

Now, the value of $m(x, y, t_0 = 0) := E[\tau(x, y, t_0 = 0)]$ has been computed explicitly by Lefebvre and Ait Aoudia (2010), by making use of the method of similarity solutions to solve the appropriate differential equation. They found that

$$m(x,y,t_0=0) = \frac{d_0\,x^{d+1}}{d+1}\,(y-c).$$

This result can be generalized to

$$m(x,y,t_0) = \frac{d_0\,x^{d+1}}{d+1}\,(y-c) + t_0.$$

Therefore, we can write that

$$e^{-\alpha F(x,y,t_0)} = \frac{d_0\,x^{d+1}}{d+1}\,(y-c) + t_0 + 1.$$

Hence, the optimal control is given by

$$u^* = L^{-1}\left(\frac{1}{b_0}\,\frac{\frac{d_0\,x^{d+1}}{d+1}}{\frac{d_0\,x^{d+1}}{d+1}\,(y-c)+t_0+1}\right).$$

With $c = d = t_0 = 0$ and $d_0 = b_0 = 1$, the optimal control reduces to

$$u^* = L^{-1}\left(\frac{x}{xy+1}\right).$$

This control enables the optimizer to maximize the expected remaining lifetime of the machine, or the expected time until the machine must be repaired (taking the control costs into account).

4 Conclusion

We have generalized a theorem due to Whittle which enables us, under certain assumptions, to optimally control a diffusion process by considering the corresponding uncontrolled process. Here, we have replaced the linear control in the plant equation by a function $L(u)$, and the quadratic control in the cost criterion by $C(u)$. We have obtained a result similar to that of Whittle for the case where $C(u) = L^2(u)$. We could now consider other cases, for example the one where $C(u) = L^{1/2}(u)$ (assuming that $L(u) \geq 0$).

In the particular cases presented in Sections 2 and 3, we have chosen a termination cost function that leads to the computation of the expected value of the first passage time for the uncontrolled process. This case is important for the applications. Indeed, as we have seen in Section 3, it is a natural problem to try to maximize the (expected) lifetime of a device.

Finally, it would be interesting to generalize even further Whittle's theorem by obtaining an expression for the optimal control in terms of a mathematical expectation for an uncontrolled process, but in the case where the relation in (2) or in (11) is not necessarily satisfied.

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Delta-Nabla Isoperimetric Problems

Agnieszka B. Malinowska[1], **Delfim F. M. Torres**[2]

[1]Białystok University of Technology, 15-351 Białystok, Poland
e-mail: abmalinowska@ua.pt
[2]University of Aveiro, 3810-193 Aveiro, Portugal
e-mail: delfim@ua.pt

Abstract

We prove general necessary optimality conditions for delta-nabla isoperimetric problems of the calculus of variations.

Keywords: *calculus of variations, delta and nabla derivatives and integrals, isoperimetric problems, necessary optimality conditions, time scales.*

2010 Mathematics Subject Classification: *49K05, 39A12, 34N05.*

1 Introduction

Isoperimetric problems consist in maximizing or minimizing a cost functional subject to integral constraints. They have found a broad class of important applications throughout the centuries. Areas of application include astronomy, geometry, algebra, and analysis [4]. The study of isoperimetric problems is nowadays done, in an elegant and rigorously way, by means of the theory of the calculus of variations [18], and concrete isoperimetric problems in engineering have been investigated by a number of authors [9]. For recent developments on isoperimetric problems we refer the reader to [2, 1, 11] and references therein.

A new delta-nabla calculus of variations has recently been introduced by the authors in [14]. The new calculus of variations allow us to unify and extend the two standard approaches of the calculus of variations on time scales [10, 16, 17], and is motivated by applications in economics [8].

The delta-nabla variational theory is still in the very beginning, and much remains to be done. In this note we develop further the theory by introducing the isoperimetric problem in the delta-nabla setting and proving respective necessary optimality conditions. Section 2 reviews the Euler-Lagrange equations of the delta-nabla calculus of variations [14] and recalls the results of the literature needed in the sequel. Our contribution is given in Section 3, where the delta-nabla isoperimetric problem is formulated and necessary optimality conditions for both normal and abnormal extremizers are proved (see Theorems 3.3 and 3.5). We proceed with Section 4, illustrating the applicability of our results with an example. Finally, we present the conclusion (Section 5) and some open problems (Section 6).

2 Preliminaries

We assume the reader to be familiar with the theory of time scales. For an introduction to the calculus on time scales we refer to the books [6, 7, 13].

Let $\mathbb{T}$ be a given time scale with jump operators σ and ρ, and differential operators Δ and ∇. Let $a, b \in \mathbb{T}$, $a < b$, and $(\mathbb{T} \setminus \{a, b\}) \cap [a, b] \neq \emptyset$; and $L_\Delta(\cdot,\cdot,\cdot)$ and $L_\nabla(\cdot,\cdot,\cdot)$ be two given smooth functions from $\mathbb{T} \times \mathbb{R}^2$ to $\mathbb{R}$. The results here discussed are trivially generalized for admissible functions $y : \mathbb{T} \to \mathbb{R}^n$ but for simplicity of presentation we restrict ourselves to the scalar case $n = 1$. Throughout the text we use the operators $[y]$ and $\{y\}$ defined by

$$[y](t) := \left(t, y^\sigma(t), y^\Delta(t)\right) , \quad \{y\}(t) := \left(t, y^\rho(t), y^\nabla(t)\right) .$$

In [14] the problem of extremizing a delta-nabla variational functional subject to given boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ is posed and studied:

$$\begin{gathered} \mathcal{J}(y) = \left(\int_a^b L_\Delta[y](t)\Delta t\right)\left(\int_a^b L_\nabla\{y\}(t)\nabla t\right) \longrightarrow \text{extr} \\ y \in C^1_\diamond\left([a,b],\mathbb{R}\right) \\ y(a) = \alpha\,, \quad y(b) = \beta\,, \end{gathered} \tag{1}$$

where $C^1_\diamond\left([a,b],\mathbb{R}\right)$ denote the class of functions $y : [a,b] \to \mathbb{R}$ with y^Δ continuous on $[a,b]^\kappa$ and y^∇ continuous on $[a,b]_\kappa$.

Definition 2.1 *We say that* $\hat{y} \in C^1_\diamond([a,b],\mathbb{R})$ *is a weak local minimizer (respectively weak local maximizer) for problem* (1) *if there exists* $\delta > 0$ *such that* $\mathcal{J}(\hat{y}) \leq \mathcal{J}(y)$ *(respectively* $\mathcal{J}(\hat{y}) \geq \mathcal{J}(y)$*) for all* $y \in C^1_\diamond([a,b],\mathbb{R})$ *satisfying the boundary conditions* $y(a) = \alpha$ *and* $y(b) = \beta$*, and* $||y - \hat{y}||_{1,\infty} < \delta$*, where* $||y||_{1,\infty} := ||y^\sigma||_\infty + ||y^\rho||_\infty + ||y^\Delta||_\infty + ||y^\nabla||_\infty$ *and* $||y||_\infty := \sup_{t\in[a,b]^\kappa_\kappa} |y(t)|$.

The main result of [14] gives two different forms for the Euler–Lagrange equation on time scales associated with the variational problem (1).

Theorem 2.2 (The general Euler-Lagrange equations on time scales [14]) *If $\hat{y} \in C^1_\diamond$ is a weak local extremizer of problem (1), then $\hat{y}$ satisfies the following delta-nabla integral equations:*

$$\begin{aligned}\mathcal{J}_\nabla(\hat{y}) &\left(\partial_3 L_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right)\\ &+ \mathcal{J}_\Delta(\hat{y}) \left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in [a,b]_\kappa\,; \end{aligned} \tag{2}$$

$$\begin{aligned}\mathcal{J}_\nabla(\hat{y}) &\left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^{t} \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right)\\ &+ \mathcal{J}_\Delta(\hat{y}) \left(\partial_3 L_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in [a,b]^\kappa\,. \end{aligned} \tag{3}$$

Remark 2.3 *In the classical context (i.e., when $\mathbb{T} = \mathbb{R}$) the necessary conditions (2) and (3) coincide with the Euler–Lagrange equations recently obtained in [8].*

Our main goal is to generalize Theorem 2.2 by covering variational problems subject to isoperimetric constraints. In order to do it (cf. proof of Theorem 3.3) we use some relationships of [3] between the delta and nabla derivatives, and some relationships of [12] between the delta and nabla integrals.

Proposition 2.1 (Theorems 2.5 and 2.6 of [3]) *(i) If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$ and f^Δ is continuous on $\mathbb{T}^\kappa$, then f is nabla differentiable on $\mathbb{T}_\kappa$ and*

$$f^\nabla(t) = \left(f^\Delta\right)^\rho(t) \quad \text{for all } t \in \mathbb{T}_\kappa\,. \tag{4}$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}_\kappa$ and f^∇ is continuous on $\mathbb{T}_\kappa$, then f is delta differentiable on $\mathbb{T}^\kappa$ and

$$f^\Delta(t) = \left(f^\nabla\right)^\sigma(t) \quad \text{for all } t \in \mathbb{T}^\kappa\,. \tag{5}$$

Proposition 2.2 (Proposition 7 of [12]) *If function $f : \mathbb{T} \to \mathbb{R}$ is continuous, then for all $a, b \in \mathbb{T}$ with $a < b$ we have*

$$\int_a^b f(t)\Delta t = \int_a^b f^\rho(t)\nabla t\,, \tag{6}$$

$$\int_a^b f(t)\nabla t = \int_a^b f^\sigma(t)\Delta t\,. \tag{7}$$

We also use the nabla Dubois–Reymond lemma of [16].

Lemma 2.4 (Lemma 14 of [16]) *Let* $f \in C_{ld}([a,b],\mathbb{R})$*. If*

$$\int_a^b f(t)\eta^{\nabla}(t)\nabla t = 0 \quad \text{for all } \eta \in C_{ld}^1([a,b],\mathbb{R}) \text{ with } \eta(a) = \eta(b) = 0\,,$$

then $f(t) = c$ *on* $t \in [a,b]_\kappa$ *for some constant* c*.*

3 Main Results

We consider delta-nabla isoperimetric problems on time scales. The problem consists of extremizing

$$\mathcal{L}(y) = \left(\int_a^b L_\Delta[y](t)\Delta t\right)\left(\int_a^b L_\nabla\{y\}(t)\nabla t\right) \longrightarrow \text{extr} \tag{8}$$

in the class of functions $y \in C_\diamond^1([a,b],\mathbb{R})$ satisfying the boundary conditions

$$y(a) = \alpha\,, \quad y(b) = \beta\,, \tag{9}$$

and the constraint

$$\mathcal{K}(y) = \left(\int_a^b K_\Delta[y](t)\Delta t\right)\left(\int_a^b K_\nabla\{y\}(t)\nabla t\right) = k, \tag{10}$$

where α, β, k are given real numbers.

Definition 3.1 *We say that* $\hat{y} \in C_\diamond^1([a,b],\mathbb{R})$ *is a weak local minimizer (respectively weak local maximizer) for* (8)–(10) *if there exists* $\delta > 0$ *such that*

$$\mathcal{L}(\hat{y}) \leq \mathcal{L}(y) \quad (\textit{respectively} \;\; \mathcal{L}(\hat{y}) \geq \mathcal{L}(y))$$

for all $y \in C_\diamond^1([a,b],\mathbb{R})$ *satisfying the boundary conditions* (9)*, the isoperimetric constraint* (10)*, and* $||y - \hat{y}||_{1,\infty} < \delta$*.*

Definition 3.2 *We say that* $\hat{y} \in C_\diamond^1$ *is an extremal for* $\mathcal{K}$ *if* $\hat{y}$ *satisfies the delta-nabla integral equations* (2) *and* (3) *for* $\mathcal{K}$*, i.e.,*

$$\begin{aligned}\mathcal{K}_\nabla(\hat{y})&\left(\partial_3 K_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\\ &+ \mathcal{K}_\Delta(\hat{y})\left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in [a,b]_\kappa\,;\end{aligned} \tag{11}$$

$$\mathcal{K}_{\nabla}(\hat{y})\left(\partial_3 K_{\Delta}[\hat{y}](t)-\int_a^t \partial_2 K_{\Delta}[\hat{y}](\tau)\Delta\tau\right)$$
$$+\,\mathcal{K}_{\Delta}(\hat{y})\left(\partial_3 K_{\nabla}\{\hat{y}\}(\sigma(t))-\int_a^{\sigma(t)} \partial_2 K_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)=const \quad \forall t\in[a,b]^{\kappa}\,. \tag{12}$$

An extremizer (i.e., a weak local minimizer or a weak local maximizer) for the problem (8)–(10) *that is not an extremal for* $\mathcal{K}$ *is said to be a normal extremizer; otherwise (i.e., if it is an extremal for* $\mathcal{K}$*), the extremizer is said to be abnormal.*

Theorem 3.3 *If* $\hat{y}\in C^1_{\diamond}([a,b],\mathbb{R})$ *is a normal extremizer for the isoperimetric problem* (8)–(10)*, then there exists* $\lambda\in\mathbb{R}$ *such that* $\hat{y}$ *satisfies the following delta-nabla integral equations:*

$$\mathcal{L}_{\nabla}(\hat{y})\left(\partial_3 L_{\Delta}[\hat{y}](\rho(t))-\int_a^{\rho(t)} \partial_2 L_{\Delta}[\hat{y}](\tau)\Delta\tau\right)$$
$$+\,\mathcal{L}_{\Delta}(\hat{y})\left(\partial_3 L_{\nabla}\{\hat{y}\}(t)-\int_a^{t} \partial_2 L_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)$$
$$-\,\lambda\left\{\mathcal{K}_{\nabla}(\hat{y})\left(\partial_3 K_{\Delta}[\hat{y}](\rho(t))-\int_a^{\rho(t)} \partial_2 K_{\Delta}[\hat{y}](\tau)\Delta\tau\right)\right.$$
$$\left.+\mathcal{K}_{\Delta}(\hat{y})\left(\partial_3 K_{\nabla}\{\hat{y}\}(t)-\int_a^{t} \partial_2 K_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)\right\}=const \quad \forall t\in[a,b]_{\kappa}\,; \tag{13}$$

$$\mathcal{L}_{\nabla}(\hat{y})\left(\partial_3 L_{\Delta}[\hat{y}](t)-\int_a^{t} \partial_2 L_{\Delta}[\hat{y}](\tau)\Delta\tau\right)$$
$$+\,\mathcal{L}_{\Delta}(\hat{y})\left(\partial_3 L_{\nabla}\{\hat{y}\}(\sigma(t))-\int_a^{\sigma(t)} \partial_2 L_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)$$
$$-\,\lambda\left\{\mathcal{K}_{\nabla}(\hat{y})\left(\partial_3 K_{\Delta}[\hat{y}](t)-\int_a^{t} \partial_2 K_{\Delta}[\hat{y}](\tau)\Delta\tau\right)\right.$$
$$\left.+\,\mathcal{K}_{\Delta}(\hat{y})\left(\partial_3 K_{\nabla}\{\hat{y}\}(\sigma(t))-\int_a^{\sigma(t)} \partial_2 K_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)\right\}=const \quad \forall t\in[a,b]^{\kappa}\,. \tag{14}$$

Proof Consider a variation of $\hat{y}$, say $\bar{y}=\hat{y}+\varepsilon_1\eta_1+\varepsilon_2\eta_2$, where for each $i\in\{1,2\}$, $\eta_i\in C^1_{\diamond}([a,b],\mathbb{R})$ and $\eta_i(a)=\eta_i(b)=0$, and ε_i is a sufficiently small parameter (ε_1 and ε_2 must be such that $||\bar{y}-\hat{y}||_{1,\infty}<\delta$ for some $\delta>0$). Here,

η_1 is an arbitrary fixed function and η_2 is a fixed function that will be chosen later. Define the real function

$$\bar{K}(\varepsilon_1, \varepsilon_2) = \mathcal{K}(\bar{y}) = \left(\int_a^b K_\Delta[\bar{y}](t)\Delta t\right)\left(\int_a^b K_\nabla\{\bar{y}\}(t)\nabla t\right) - k.$$

We have

$$\begin{aligned}\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} &= \mathcal{K}_\nabla(\hat{y})\int_a^b \left(\partial_2 K_\Delta[\hat{y}](t)\eta_2^\sigma(t) + \partial_3 K_\Delta[\hat{y}](t)\eta_2^\Delta(t)\right)\Delta t \\ &\quad + \mathcal{K}_\Delta(\hat{y})\int_a^b \left(\partial_2 K_\nabla\{\hat{y}\}(t)\eta_2^\rho(t) + \partial_3 K_\nabla\{\hat{y}\}(t)\eta_2^\nabla(t)\right)\nabla t = 0\,.\end{aligned}$$

We now make use of the following formulas of integration by parts [6]: if functions $f, g : \mathbb{T} \to \mathbb{R}$ are delta and nabla differentiable with continuous derivatives, then

$$\begin{aligned}\int_a^b f^\sigma(t)g^\Delta(t)\Delta t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t)\Delta t\,,\\ \int_a^b f^\rho(t)g^\nabla(t)\nabla t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\nabla(t)g(t)\nabla t\,.\end{aligned}$$

Having in mind that $\eta_2(a) = \eta_2(b) = 0$, we obtain:

$$\begin{aligned}&\int_a^b \partial_2 K_\Delta[\hat{y}](t)\eta_2^\sigma(t)\Delta t = \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\eta_2(t)|_{t=a}^{t=b}\\ &-\int_a^b\left(\int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t = -\int_a^b\left(\int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t\end{aligned}$$

and

$$\begin{aligned}&\int_a^b \partial_2 K_\nabla\{\hat{y}\}(t)\eta_2^\rho(t)\nabla t = \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\eta_2(t)|_{t=a}^{t=b}\\ &-\int_a^b\left(\int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t = -\int_a^b\left(\int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t.\end{aligned}$$

Therefore,

$$\begin{aligned}\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} &= \mathcal{K}_\nabla(\hat{y})\int_a^b\left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t\\ &\quad + \mathcal{K}_\Delta(\hat{y})\int_a^b\left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t. \qquad (15)\end{aligned}$$

Let

$$f(t) = \mathcal{K}_\nabla(\hat{y}) \left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \right)$$

and

$$g(t) = \mathcal{K}_\Delta(\hat{y}) \left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \right).$$

We can then write equation (15) in the form

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = \int_a^b f(t)\eta_2^\Delta(t)\Delta t + \int_a^b g(t)\eta_2^\nabla(t)\nabla t. \tag{16}$$

Transforming the delta integral in (16) to a nabla integral by means of (6) we obtain

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = \int_a^b f^\rho(t)(\eta_2^\Delta)^\rho(t)\nabla t + \int_a^b g(t)\eta_2^\nabla(t)\nabla t$$

and by (4)

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = \int_a^b \left(f^\rho(t) + g(t)\right)\eta_2^\nabla(t)\nabla t.$$

As $\hat{y}$ is a normal extremizer we conclude, by Lemma 2.4 and equation (12), that there exists η_2 such that $\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} \neq 0$. Since $\bar{K}(0,0) = 0$, by the implicit function theorem we conclude that there exists a function ε_2 defined in the neighborhood of zero, such that $\bar{K}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0$, i.e., we may choose a subset of variations $\bar{y}$ satisfying the isoperimetric constraint.

Let us now consider the real function

$$\bar{L}(\varepsilon_1, \varepsilon_2) = \mathcal{L}(\bar{y}) = \left(\int_a^b L_\Delta[\bar{y}](t)\Delta t\right)\left(\int_a^b L_\nabla\{\bar{y}\}(t)\nabla t\right).$$

By hypothesis, $(0,0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K} = 0$ and $\nabla\bar{K}(0,0) \neq \mathbf{0}$. By the Lagrange multiplier rule, there exists some real λ such that $\nabla(\bar{L}(0,0) - \lambda\bar{K}(0,0)) = \mathbf{0}$. Having in mind that $\eta_1(a) = \eta_1(b) = 0$, we can write

$$\begin{aligned}\left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)} &= \mathcal{L}_\nabla(\hat{y}) \int_a^b \left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau \right)\eta_1^\Delta(t)\Delta t \\ &\quad + \mathcal{L}_\Delta(\hat{y}) \int_a^b \left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau \right)\eta_1^\nabla(t)\nabla t \end{aligned} \tag{17}$$

and

$$\begin{aligned}\left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)} &= \mathcal{K}_\nabla(\hat{y}) \int_a^b \left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \right)\eta_1^\Delta(t)\Delta t \\ &\quad + \mathcal{K}_\Delta(\hat{y}) \int_a^b \left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \right)\eta_1^\nabla(t)\nabla t. \end{aligned} \tag{18}$$

Let

$$m(t) = \mathcal{L}_\nabla(\hat{y})\left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right)$$

and

$$n(t) = \mathcal{L}_\Delta(\hat{y})\left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right).$$

Then equations (17) and (18) can be written in the form

$$\left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)} = \int_a^b m(t)\eta_1^\Delta(t)\Delta t + \int_a^b n(t)\eta_1^\nabla(t)\nabla t$$

and

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)} = \int_a^b f(t)\eta_1^\Delta(t)\Delta t + \int_a^b g(t)\eta_1^\nabla(t)\nabla t.$$

Transforming the delta integrals in the above equalities to nabla integrals by means of (6) and using (4) we obtain

$$\left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)} = \int_a^b \left(m^\rho(t) + n(t)\right)\eta_1^\nabla(t)\nabla t$$

and

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)} = \int_a^b \left(f^\rho(t) + g(t)\right)\eta_1^\nabla(t)\nabla t.$$

Therefore,

$$\int_a^b \eta_1^\Delta(t)\left\{m^\rho(t) + n(t) - \lambda\left(f^\rho(t) + g(t)\right)\right\}\nabla t = 0. \tag{19}$$

Since (19) holds for any η_1, by Lemma 2.4 we have

$$m^\rho(t) + n(t) - \lambda\left(f^\rho(t) + g(t)\right) = c$$

for some $c \in \mathbb{R}$ and all $t \in [a, b]_\kappa$. Hence, condition (13) holds. In a similar way we can obtain equation (14). In that case we use relationships (5) and (7), and [5, Lemma 4.1]. ■

In the particular case $L_\nabla \equiv \frac{1}{b-a}$ we get from Theorem 3.3 the main result of [11]:

Corollary 3.4 (Theorem 3.4 of [11]) *Suppose that*

$$J(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t$$

has a local minimum at y_* subject to the boundary conditions $y(a) = y_a$ and $y(b) = y_b$ and the isoperimetric constraint

$$I(y) = \int_a^b g(t, y^\sigma(t), y^\Delta(t))\Delta t = k\,.$$

Assume that y_* is not an extremal for the functional I. Then, there exists a Lagrange multiplier constant λ such that y_* satisfies the following equation:

$$\partial_3 F^\Delta(t, y_*^\sigma(t), y_*^\Delta(t)) - \partial_2 F(t, y_*^\sigma(t), y_*^\Delta(t)) = 0 \quad \text{for all} \quad t \in [a,b]^{\kappa^2},$$

where $F = L - \lambda g$ and $\partial_3 F^\Delta$ denotes the delta derivative of a composition.

One can easily cover abnormal extremizers within our result by introducing an extra multiplier λ_0.

Theorem 3.5 *If $\hat{y} \in C_\diamond^1$ is an extremizer for the isoperimetric problem (8)–(10), then there exist two constants λ_0 and λ, not both zero, such that $\hat{y}$ satisfies the following delta-nabla integral equations:*

$$\begin{aligned}\lambda_0 \Bigg\{ \mathcal{L}_\nabla(\hat{y}) \left(\partial_3 L_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau \right) \\ + \mathcal{L}_\Delta(\hat{y}) \left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \Bigg\} \\ - \lambda \Bigg\{ \mathcal{K}_\nabla(\hat{y}) \left(\partial_3 K_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \right) \\ + \mathcal{K}_\Delta(\hat{y}) \left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \Bigg\} = const \quad \forall t \in [a,b]_\kappa\,; \end{aligned} \tag{20}$$

$$\begin{aligned}\lambda_0 \Bigg\{ \mathcal{L}_\nabla(\hat{y}) \left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau \right) \\ + \mathcal{L}_\Delta(\hat{y}) \left(\partial_3 L_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \Bigg\} \\ - \lambda \Bigg\{ \mathcal{K}_\nabla(\hat{y}) \left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \right) \\ + \mathcal{K}_\Delta(\hat{y}) \left(\partial_3 K_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \Bigg\} = const \quad \forall t \in [a,b]^\kappa\,. \end{aligned} \tag{21}$$

Proof Following the proof of Theorem 3.3, since $(0,0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K} = 0$, the extended Lagrange multiplier rule (see for instance [18, Theorem 4.1.3]) asserts the existence of reals λ_0 and λ, not both zero, such that $\nabla(\lambda_0 \bar{L}(0,0) - \lambda \bar{K}(0,0)) = \mathbf{0}$. Therefore,

$$\int_a^b \eta_1^\Delta(t) \left\{\lambda_0 \left(m^\rho(t) + n(t)\right) - \lambda \left(f^\rho(t) + g(t)\right)\right\} \nabla t = 0. \tag{22}$$

Since (22) holds for any η_1, by Lemma 2.4, we have

$$\lambda_0 \left(m^\rho(t) + n(t)\right) - \lambda \left(f^\rho(t) + g(t)\right) = c$$

for some $c \in \mathbb{R}$ and all $t \in [a,b]_\kappa$. This establishes equation (20). Equation (21) can be shown using a similar technique. ■

Remark 3.6 *If $\hat{y} \in C_\diamond^1$ is an extremizer for the isoperimetric problem* (8)–(10)*, then we can choose $\lambda_0 = 1$ in Theorem 3.5 and obtain Theorem 3.3. For abnormal extremizers, Theorem 3.5 holds with $\lambda_0 = 0$. The condition $(\lambda_0, \lambda) \neq \mathbf{0}$ guarantees that Theorem 3.5 is a useful necessary optimality condition.*

In the particular case $L_\Delta \equiv \frac{1}{b-a}$ we get from Theorem 3.5 the main result of [2]:

Corollary 3.7 (Theorem 2 of [2]) *If y is a local minimizer or maximizer for*

$$I[y] = \int_a^b f(t, y^\rho(t), y^\nabla(t)) \nabla t$$

subject to the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ and the nabla-integral constraint

$$J[y] = \int_a^b g(t, y^\rho(t), y^\nabla(t)) \nabla t = \Lambda \,,$$

then there exist two constants λ_0 and λ, not both zero, such that

$$\partial_3 K^\nabla \left(t, y^\rho(t), y^\nabla(t)\right) - \partial_2 K \left(t, y^\rho(t), y^\nabla(t)\right) = 0$$

for all $t \in [a,b]_\kappa$, where $K = \lambda_0 f - \lambda g$.

4 An Example

Let $\mathbb{T} = \{1, 2, 3, \ldots, M\}$, where $M \in \mathbb{N}$ and $M \geq 2$. Consider the problem

$$\text{minimize} \quad \mathcal{L}(y) = \left(\int_0^M (y^\Delta(t))^2 \Delta t\right) \left(\int_0^M \left(y^\nabla(t))^2 + y^\nabla(t)\right) \nabla t\right) \tag{23}$$
$$y(0) = 0, \quad y(M) = M,$$

subject to the constraint

$$\mathcal{K}(y) = \int_0^M t y^{\Delta}(t)\Delta t = 1. \tag{24}$$

Since

$$L_\Delta = (y^\Delta)^2, \quad L_\nabla = (y^\nabla)^2 + y^\nabla, \quad K_\Delta = t y^\Delta, \quad K_\nabla = \frac{1}{M}$$

we have

$$\partial_2 L_\Delta = 0, \quad \partial_3 L_\Delta = 2y^\Delta, \quad \partial_2 L_\nabla = 0, \quad \partial_3 L_\nabla = 2y^\nabla + 1,$$

and

$$\partial_2 K_\Delta = 0, \quad \partial_3 K_\Delta = t, \quad \partial_2 K_\nabla = 0, \quad \partial_3 K_\nabla = 0.$$

As

$$\mathcal{K}_\nabla(\hat{y})\left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right) + \mathcal{K}_\Delta(\hat{y})\left(\partial_3 K_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = t$$

there are no abnormal extremals for the problem (23)–(24). Applying equation (14) of Theorem 3.3 we get the following delta-nabla differential equation:

$$2Ay^\Delta(t) + B + 2By^\nabla(\sigma(t)) - \lambda t = C, \tag{25}$$

where $C \in \mathbb{R}$ and A, B are the values of functionals $\mathcal{L}_\nabla$ and $\mathcal{L}_\Delta$ in a solution of (23)–(24), respectively. Since $y^\nabla(\sigma(t)) = y^\Delta(t)$ (5), we can write equation (25) in the form

$$2Ay^\Delta(t) + B + 2By^\Delta - \lambda t = C. \tag{26}$$

Observe that $B \neq 0$ and $A > 2$. Hence, solving equation (26) subject to the boundary conditions $y(0) = 0$ and $y(M) = M$ we get

$$y(t) = \left[1 - \frac{\lambda\,(M-t)}{4(A+B)}\right] t\,. \tag{27}$$

Substituting (27) into (24) we obtain $\lambda = -\frac{(A+B)(M-2)}{12M(M-1)}$. Hence,

$$y(t) = \frac{(4\,M^2 - 7\,M - 3M\,t + 6\,t)\,t}{M\,(M-1)}$$

is an extremal for the problem (23)–(24).

5 Conclusion

Minimization of functionals given by the product of two integrals were considered by Euler himself, and are now receiving an increase of interest due to their nonlocal properties and applications to economics [8, 14]. In this paper we obtained general necessary optimality conditions for isoperimetric problems of the calculus of variations on time scales. Our results extend the ones with delta derivatives proved in [11] and analogous nabla results [2] to more general variational problems described by the product of delta and nabla integrals.

6 Open Problems

The results here obtained can be generalized in different ways: (i) to variational problems involving higher-order delta and nabla derivatives, unifying and extending the higher-order results on time scales of [10] and [16]; (ii) to problems of the calculus of variations with a functional which is the composition of a certain scalar function H with the delta integral of a vector valued field f_Δ and a nabla integral of a vector field f_∇, i.e., of the form

$$H\left(\int_a^b f_\Delta(t, y^\sigma(t), y^\Delta(t))\Delta t\,, \int_a^b f_\nabla(t, y^\rho(t), y^\nabla(t))\nabla t\right).$$

It remains to prove Euler-Lagrange equations and natural boundary conditions for such problems on time scales, with or without constraints.

Sufficient optimality conditions for delta-nabla problems of the calculus of variations is a completely open question. It would be also interesting to study direct optimization methods, extending the results of [15] to the more general delta-nabla setting.

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On Some Nonlinear Integral Equation at the Boundary in the Potential Method

H. Saker[1] **and A. Djellit**[2]

L.M.A. Dep. of Mathematics, Fac. of Sciences, Univ. of Annaba
Annaba 23000, Algeria.

e-mail[1]: h_ saker@yahoo.fr
e-mail[2]: A_ Djellit@hotmail.com

Abstract

In the present work, we apply the potential methods to the Laplacian with the nonlinear boundary conditions in a bounded domain with smooth boundary. A nonlinear boundary integral equation is obtained. We will interpret that as pseudo-differential operators, and via the symbolic computation of these operators, we can give the properties of these operators which allow us to used the fixed point theorem of Krasnosel'skii for establish the existence of the solution.

Keywords: *Boundary Integral equations methods, Fixed point theorems, Nonlinear Hammerstein equations, Pseudodifferential operator.*

1 Introduction

The resolution of boundary value problem for partial differential operators with nonlinear boundary conditions bay the method of integral equations, in recent years much attention has been paid to this problem in many directions (we quote, for instance, the works for example in Ruotssalainen and Wendland [7] and in Kendall E. Atkinson [1]).
In various applications, however, the problems involve nonlinearities. Also

some electromagnetic problems contain nonlinearities in the boundary condition, for instance problems, where the electrical conductivity of the boundary is variable. Further applications arise in heat transfer and potential problems [1, 2].
Motivated by the above applications we study here a nonlinear integral equation associated to the Laplacian equation with nonlinear data of the form:

$$(P) \qquad \begin{cases} \Delta u = 0 \quad , \quad x \in \Omega \\ \frac{\partial u(x)}{\partial n} = g\left(x, [u(x)]\right) \quad , \quad x \in \Gamma \end{cases} \tag{1.1}$$

Where Ω is an open bounded region in $\mathbb{R}^2$ with a smooth boundary $\Gamma = \partial\Omega$ and $g\left(x, [u(x)]\right)$ is a measurable function. If the solution of problem (1.1) is represented by a potential of double layer, we obtain a nonlinear integral equation on the boundary of the form

$$Tw = N_g w \tag{1.2}$$

with $w = [u(x)] = u|_{inter} - u|_{exter}$.
Where T is linear hyper singular integral operator and N_g is a Nemytskii operator.

Unfortunately, the integral operator T^{-1} is not continuous. To surmount this difficulty, we will transform (1.2) to in the form:

$$w = Aw + Bw \tag{1.3}$$

where A is compact linear operator and B is a strict contraction nonlinear operator. The existence result will be a consequence of the Krasnosel'skii fixed point theorem [1, 2].

2 Definitions and notations

In what follows, we denote by $F[.]$ the Fourier transform.

Definition 2.1

1. *Let $m \in \mathbb{N}$, we denote by $H^m(\Omega)$ the Sobolev space:*

$$H^m(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), |\alpha| \leq m\}$$

2. *Let $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^n)$ the Sobolev space:*

$$H^s(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n); (1 + |\xi|^2)^{\frac{s}{2}} |F[u]| \in L^2(\mathbb{R}^n)\right\}.$$

and the associated norm:

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |F[u]|^2 d\xi\right)^{\frac{1}{2}}.$$

3. Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $\Gamma := \partial\Omega$, we define

$$H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^n)\}, s \in \mathbb{R}$$

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+\frac{1}{2}}(\mathbb{R}^n)\}, s > 0 \\ L^2(\Gamma), \quad s = 0 \\ (H^{-s}(\Gamma))' \quad (\textit{ dual space}), s < 0 \end{cases}$$

3 The Integral equation at the boundary

3.1 Representative formula and boundary operator

We need the fundamental solution $E(x)$ of operator $\Delta : \Delta E = \delta$, defined by:

$$E(x) = \frac{1}{2\pi} \log|x| \tag{2.1}$$

We note: $E(x,y) = E(x-y)$

We know that the solution of problem (P) can be represented by different way (see [1,2,3,5]):

1. Representation by potential of double layer is obtained if we extend the solution $\frac{\partial u}{\partial n}$ of our problem continuously to $\mathbb{R}^2$:

$$g(x, w(x)) = -\int_\Gamma w(y) \, .\partial_{nx}\partial_{ny} E(x,y) \, ds_y$$

with $w(x) = u|_{int\Gamma} - u|_{ext\Gamma}$

2. Representation by potential of single layer is obtained if we extend u continuously to $\mathbb{R}^2$:

$$g(x, w(x)) = \frac{[\frac{\partial u}{\partial n}]}{2} + \int_\Gamma [\frac{\partial u}{\partial n}].\partial_{ny} E(x,y) \, ds_y$$

with $[\frac{\partial u}{\partial n}] = \frac{\partial u}{\partial n}|_{int\Gamma} - \frac{\partial u}{\partial n}|_{ext\Gamma}$

In this paper we are concerned to the first case.

Definition 3.1 *Let $u \in C^\infty(\Gamma)$. We define the following operator:*

$$D_\Omega u(x) = \int_\Gamma u(y) \, .\partial_{ny} E(x,y) \, ds_y \quad , \; x \in \Omega \tag{2.2}$$

This definition will yield for an arbitrary distribution u on Γ since for $x \notin \Gamma$, the kernel of the operator (2.2) are C^∞ function onΓ.
The operator in (2.2) gives the following representative formula:

Lemma 3.2 *For $u \in H^1(\Omega)$ with $w \in H^{\frac{1}{2}}(\Gamma)$ and for $x \in \Omega$ we have:*

$$\partial_n u(x) = \partial_n D_\Omega w(x) \tag{2.3}$$

Proof If the solution of the problem (P) is represented by the potential of the double layer (convolution of the elementary solution with a double layer$(-\partial_n w \delta_\Gamma)$ with δ (the Dirac measure), by differential we obtain the representative formula of $w = [u]$. ■
In present time, in order to formulate the integral equation, we define the following operator at the boundary:

Definition 3.3 *For $x \in \Gamma$ let $u \in C^\infty(\Gamma)$. We define the operator:*

$$Tu(x) = -\int_\Gamma u(y)\, \partial_{nx}\partial_{ny} E(x,y)\, ds_y \tag{2.4}$$

The extension to distribution has a meaning since the defined operator above is a pseudo-differential operator.

Lemma 3.4 *The operator Tu defined by (2.4) is an pseudo-differential operator of order:* 1 *and it is continuous:*

$$T : H^{\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{-1}{2}}(\Gamma) \tag{2.5}$$

Proof We have: Γ is a curve which can be given by a regular parametric representation:

$$\begin{cases} x = (x_1(t), x_2(t)) = (r.\sin(t/r), r.\cos(t/r)) \\ y = (y_1(t), y_2(t)) = (r.\sin(t/r), r.\cos(t/r)) \end{cases}$$

with $r = |x - y|$.
According to the Taylor formula, we can write, with

$$\begin{cases} \tau = t_0 - t \\ \dot{x}_1^2 + \dot{x}_2^2 = 1 \end{cases}$$

So,

$$\partial_{nx}\partial_{ny} \log|x(t) - y(t_0)| = -\tau^{-2} - \left(\frac{2}{r^2} + \frac{1}{2r^2}\right)\tau + \cdots + R_1(t,\tau)$$

Let $\chi(|\tau|) = 1$ in neighborhood of zero. Then,

$$Tu(t) = -\frac{1}{\pi}\int_{\Gamma} \chi|\tau|(-|\tau|^{-2})u(t_0)\,dt_0 + Ru(t)$$

where $Ru(t)$ is a C^{∞} kernel operator, which means of order $(-\infty)$.

Now, let the inverse Fourier transform of $u(t_0)$:

$$u(t_0) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \exp(it_0\xi)\,F[u(\xi)]d\xi$$

let's replace this representation of $u(t_0)$ in $Tu(t)$ we obtain:

$$Tu(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} P_T(t,\xi)\exp(it_0\xi)\,F[u(\xi)]d\xi + Ru(t)$$

where

$$P_T(t,\xi) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \exp(it_0\xi)\,\chi(|\tau|)\,|\tau|^{-2}\,d\tau.$$

In addition we have:

$$\chi(|\tau|) = \chi(0) + \tau\chi^{'}(0) + \frac{\tau^2}{2!}\chi^{''}(0) + \circ(\tau^n).$$

Thus, the main symbol will be the fourier transform of $\chi(0)|\tau|^{-2}$

$$\sigma_T(t,\xi) = -\frac{1}{\pi}F[(\chi(0)|\tau|^{-2})] = |\xi|$$

then $Tu(t)$ is a pseudo-differential operator of order $(+1)$ on Γ.
We know that a pseudo-differential operator is continuous from [11]:

$$H^s(\Gamma) \longrightarrow H^{s-2\alpha}(\Gamma)$$

or from

$$H^{s+\alpha}(\Gamma) \longrightarrow H^{s-\alpha}(\Gamma)$$

for all $s \in \mathbb{R}$. 2α is the order of the operator which gives (2.5). ■

Now, let's come back to lemma 2.1 and let's cite the following second representative formula:

Lemma 3.5 *Let $w \in H^{+\frac{1}{2}}(\Gamma)$. Then we have:*

$$\partial_n u(x) = Tw(x),\, x \in \Gamma \tag{2.6}$$

Proof To show (2.6) we consider the boundary process in (2.3) in which x approaches from the interior in the domain Ω, an arbitrary point x on the boundary Γ and with the continuity of the normal derivative of double layer potential $\partial_n D_\Omega$. We can write the integral equation on the boundary Γ (2.6). ■

3.2 Representation of the problem (P) as integral equation at the boundary (Γ).

We consider $u \in H^1(\Omega)$ satisfying the boundary condition of the problem (P) where $g(.,.) \in H^{-\frac{1}{2}}(\Gamma)$. Now if we introduce in (2.6) the boundary condition on Γ and the unknown functions on Γ such that:

$$\begin{cases} \partial_n u(x)|_\Gamma = g(x, w(x)) \\ [u(x)] = w(x) \end{cases} , x \in \Gamma \tag{2.7}$$

we have on the boundary Γ then the nonlinear integral equation

$$-\frac{1}{2\pi}\int_\Gamma w(y)\,\partial_{nx}\partial_{ny}\log|x-y|ds_y = g(x, w(x)), x \in \Gamma \tag{2.8}$$

under the equivalent form:

$$Tw(x) = g(x, w(x)), x \in \Gamma \tag{2.9}$$

Also introduce the nonlinear operator (Nemytskii operator)

$$N_g w(x) = g(x, w(x)), x \in \Gamma \tag{2.10}$$

Remark 3.6 *Note that the nonlinear integral equation (2.9) may be written in the form:*

$$Aw(x) + Bw(x) = w(x) \quad , w(x) \in H^{1/2}(\Gamma) \tag{2.11}$$

Where A and B are nonlinear operators.

In some special cases a useful tool for solving problems in the form (2.11) is the following fixed point theorem due to Krasnosel'skii[7,8, 10]].

Theorem 3.7 *(Krasnosel'skii[7])*
Let $\mathfrak{D}$ be a nonempty closed convex subset of a Banach space X and A and B be two maps from $\mathfrak{D}$ into X such that:

1. *A is compact and continuous,*
2. *B is a strict contraction mapping,*

3. $A\mathfrak{D} + B\mathfrak{D} \subset \mathfrak{D}$.

Then $A + B$ *has at least one fixed point in* $\mathfrak{D}$.

The Krasnosel'skii proof combines both the Banach contraction mapping principle and the Schauder fixed point theorem. In fact, under the hypotheses of Theorem , the problem may be transformed into the following one: $(I - B)^{-1}Aw = w$ which may be solved via the Schauder fixed point theorem since $(I - B)^{-1}A$ is a continuous compact map on $\mathfrak{D}$.
There are various problems arising in mathematical physics and population dynamics which may be written in the form $Aw(x) + Bw(x) = w(x)$ but, in general, A and B do not satisfy the hypotheses of The Krasnosel'skii theorem.

Lemma 3.8 *The hyper singular linear integral operator in (2.9) may be written in the form*

$$Tw(x) = T_0w(x) - T_1w(x) \quad , w(x) \in H^{1/2}(\Gamma) \tag{2.12}$$

Where T_0 *is a positive definite operator and* T_1 *is compact.*

Proof We use on the boundaries Γ the Fourier transform $F[.]$. We deduce the situation in the case $\Gamma = \mathbb{R}$. According to Parseval formula [11] we have:

$$\langle Tw, w\rangle_{L^2(\mathbb{R})} = \int |\xi|^{+1} |F[w]|^2 \, d\xi$$

Let T_0 a pseudo-differential operator of main symbol:

$$\sigma_{T_0}(x, \xi) = \left(1 + |\xi|^{\frac{1}{2}}\right)^2.$$

Then we have,

$$T = T_0 - T_0 + T = T_0 - T_1$$

with T_1 is 0 order operator. Hence it is compact from $H^{\frac{1}{2}}(\Gamma)$ in $H^{-\frac{1}{2}}(\Gamma)$.
Let's show that T_0 is a positive definite operator: there exists a constant $c > 0$ such that

$$c\left(1 + |\xi|^2\right)^{\frac{1}{2}} \leq \left(1 + |\xi|^{\frac{1}{2}}\right)^2$$

which means

$$\langle T_0w, w\rangle_{L^2(\mathbb{R})} = \int \left(1 + |\xi|^{\frac{1}{2}}\right)^2 |F[w]|^2 \, d\xi \geq c \int \left(1 + |\xi|^2\right)^{\frac{1}{2}} |F[w]|^2 \, d\xi$$
$$\geq c\|w\|^2_{H^{\frac{1}{2}}(\Gamma)}$$

According to the Garding inequality we have

$$\langle Tw,w\rangle_{L^2(\Gamma)} \geq c\,\|w\|^2_{H^{\frac{1}{2}}(\Gamma)} - \langle T_1 w,w\rangle_{L^2(\Gamma)}\,. \quad \blacksquare$$

The nonlinear integral equation (2.9) can now be written symbolically as:

$$Aw + Bw = w \quad\quad , w \in H^{1/2}(\Gamma) \tag{2.13}$$

where

$$A = T_0^{-1}T_1 \ \ \textit{and} \ \ B = T_0^{-1}N_g$$

First, we write precisely our hypotheses on A and B:

3.3 Hypotheses

(H1) The function $g(.,.) : \Gamma \times H^{\frac{1}{2}}(\Gamma) \longrightarrow H^{-\frac{1}{2}}(\Gamma)$
satisfies the condition with a constant

$$0 < \alpha < \frac{1 - \|T_0^{-1}\|.\|T_1\|}{\|T_0^{-1}\|}$$

with respect to the second variable

$$\|g(x,\omega) - g(x,\mu)\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \alpha\|\omega - \mu\|_{H^{\frac{1}{2}}(\Gamma)}$$

for all $x \in \Gamma$ and $\omega, \mu \in H^{\frac{1}{2}}(\Gamma)$

Remark 3.9 *Using the hypothesis* ***(H1)*** *we get*

$$\|g(x,\omega)\|_{H^{-\frac{1}{2}}(\Gamma)} \leq a(x) + \alpha\|\omega\|_{H^{\frac{1}{2}}(\Gamma)}$$

where $a(x) = \|g(x,0)\|_{H^{-\frac{1}{2}}(\Gamma)}$

Remark 3.10 *Using the hypothesis* ***(H1)*** *on* α *we get*

$$0 < \alpha.\|T_0^{-1}\| < 1$$

and

$$1 - \alpha\|T_0^{-1}\| - \|T_0^{-1}\|.\|T_1\| > 0$$

4 Main result

Now we are in a position to state our main result:

Theorem 4.1 *The problem (2.9) with the conditions **(H1)**, has at least one solution in* $H^{\frac{1}{2}}(\Gamma)$.

Proof The proof is based to the theorem 2.1:

1. first we show that $A = T_0^{-1}T_1$ is continuous:

   we have T_1 is compact from $H^{\frac{1}{2}}(\Gamma)$ in $H^{-\frac{1}{2}}(\Gamma)$ (lemma 2.4).

   And T_0^{-1} is continuous from $H^{-\frac{1}{2}}(\Gamma)$ in $H^{\frac{1}{2}}(\Gamma)$:

$$\|T_0 w\|^2_{H^{-\frac{1}{2}}(\Gamma)} = \int_R (1+|\xi|^2)^{-\frac{1}{2}} |F[T_0 w]|^2 d\xi$$

$$= \int_R (1+|\xi|^2)^{-\frac{1}{2}} \left((1+|\xi|^{\frac{1}{2}})^2 |F[w]|\right)^2 d\xi$$

$$\geq c^2 \int_R (1+|\xi|^2)^{-\frac{1}{2}} \left((1+|\xi|^2)^{\frac{1}{2}}\right)^2 |F[w]|^2 d\xi$$

$$\geq c^2 \int_R (1+|\xi|^2)^{\frac{1}{2}} |F[w]|^2 d\xi$$

$$\geq c^2 \|w\|^2_{H^{\frac{1}{2}}(\Gamma)}.$$

2. In the second we show that $B = T_0^{-1}N_g$ is a strict contraction mapping. Let $\omega, \mu \in H^{\frac{1}{2}}(\Gamma)$ it follows from the assumption (H1) that:

$$\|B\omega - B\mu\|_{H^{\frac{1}{2}}(\Gamma)} = \|T_0^{-1}N_g\omega - T_0^{-1}N_g\mu\|_{H^{\frac{1}{2}}(\Gamma)}$$

$$= \|T_0^{-1}\|.\|N_g\omega - N_g\mu\|_{H^{-\frac{1}{2}}(\Gamma)}$$

$$\leq \alpha\|T_0^{-1}\|.\|\omega - \mu\|_{H^{\frac{1}{2}}(\Gamma)}$$

   so, B is a strict contraction mapping on $H^{\frac{1}{2}}(\Gamma)$.

3. Finally we have:

$$\|A\omega + B\mu\|_{H^{\frac{1}{2}}(\Gamma)} = \|T_0^{-1}T_1\omega + T_0^{-1}N_g\mu\|_{H^{\frac{1}{2}}(\Gamma)}$$

$$\leq \|T_0^{-1}\| \left(\|T_1\|\|\omega\|_{H^{\frac{1}{2}}(\Gamma)} + \|a\| + \alpha\|\mu\|_{H^{\frac{1}{2}}(\Gamma)}\right)$$

letting r_0 be a real number defined by

$$r_0 = \frac{\|a\|.\|T_0^{-1}\|}{1-\alpha\|T_0^{-1}\| - \|T_0^{-1}\|.\|T_1\|}$$

the hypothesis (H1) ensure that $r_0 > 0$. Clearly, the last estimate guarantees that for all ω, μ in $\mathcal{B}_r$ (the closed boll in $H^{\frac{1}{2}}(\Gamma)$ centered at 0 with radius r) we have

$$\|A\omega + B\mu\|_{H^{\frac{1}{2}}(\Gamma)} \leq r$$

provided that $r \geq r_0$.
Accordingly, for $r \geq r_0$ we have

$$A\mathcal{B}_r + B\mathcal{B}_r \subseteq \mathcal{B}_r.$$

Now the result follows from theorem 2.1. ■

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