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Some Sandwich-Type Results on Starlike and Convex Functions

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Abstract

In the present paper, we find certain sufficient conditions for starlike and convex functions in terms of certain differential subordinations. We also establish the corresponding results for differential superordination and consequently get some sandwich-type results.

Keywords: *Analytic function, Starlike function, Convex function, Differential subordination, Differential superordination.*

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1 Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the open disk $\mathbb{E} = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}$[a,n] be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

Let $\mathcal{A}$ denote the class of all analytic functions f which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore the functions of the class $\mathcal{A}$ are of the following form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function $f \in \mathcal{A}$ is starlike of order α if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \leq \alpha < 1, z \in \mathbb{E}.$$

The class of starlike functions of order α is denoted by $\mathcal{S}^*(\alpha)$. Write $\mathcal{S}^*(0) = \mathcal{S}^*$, the class of starlike functions. A necessary and sufficient condition for a function $f \in \mathcal{A}$ to be convex is that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{E}$$

The class of convex functions is denoted by $\mathcal{K}$. A special subclass of $\mathcal{K}$ is the class of convex functions of order α, with $0 \leq \alpha < 1$, which is analytically defined as

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{E} \right\}.$$

Clearly $\mathcal{K}(0) = \mathcal{K}$.
Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let p be an analytic function in $\mathbb{E}$ with $\Phi(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0) \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ and for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of the differential subordination (1). The best dominant is unique up to a rotation of $\mathbb{E}$.
Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in $\mathbb{C}^2 \times \mathbb{E}$, h be analytic in $\mathbb{E}$, p be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called a solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), h(0) = \Psi(p(0), 0; 0) \tag{2}$$

An analytic function q is called a subordinant of the differential superordination (2), if $q \prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2) is said to be the best subordinant

of (2). The best subordinant is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be strongly starlike of order α, $0 < \alpha \leq 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{E},$$

or, equivalently

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha},\ z \in \mathbb{E}.$$

Let $\overline{\mathcal{S}}(\alpha)$ denote the class of such functions. Note that $\overline{\mathcal{S}}(1) \equiv \mathcal{S}^*$. The class $\overline{\mathcal{S}}(\alpha)$ was introduced and studied independently by Brannan and Kirwan [2] and Stankiewicz [9].
A number of sufficient conditions for $f \in \mathcal{A}$ to be starlike have been established in the literature of univalent functions. e.g. Miller, Mocanu and Reade [7] studied the class of α-convex functions and proved that if a function $f \in \mathcal{A}$ satisfies the differential inequality

$$\Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,\ z \in \mathbb{E},$$

where α is any real number, then f is starlike in $\mathbb{E}$.
In 1976, Lewandowski et al. [6] proved that if a function $f \in \mathcal{A}$ satisfies the differential inequality

$$\Re\left(\frac{zf'(z)}{f(z)} + \frac{z^2f''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{E},$$

then $f \in \mathcal{S}^*$. Many such sufficient conditions are available in the literature of univalent functions.
Billing [1] obtained certain such sufficient conditions for normalized analytic functions to be starlike and also found some sandwich-type results. In 2002, H. Imrak et al. [4, 5] studied certain classes involving the differential operator

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}$$

where $f \in \mathcal{A}_p, z \in \mathbb{E}, 0 \leq \lambda < 1, 0 \leq \alpha < p, p \in \mathbb{N}$ and the class $\mathcal{A}_p$ consists of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}),$$

which are analytic and p-valent in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. They obtained certain sufficient conditions for starlikeness and convexity of multivalent analytic functions. The main objective of the present paper is to obtain certain sufficient conditions for starlike and convex functions in terms of certain differential subordinations involving the above differential operator. We also obtain the corresponding results for superordination and consequently get some sandwich type results.

2 Preliminaries

To prove the main results we shall use the following definition and lemmas of Miller-Mocanu and Bulboaca:

Definition 2.1 *([8], Definition 2, p. 817) Denote by $\mathbb{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\overline{\mathbb{E}} \setminus \mathbb{E}(f)$, where*

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{E} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\overline{\mathbb{E}} \setminus \mathbb{E}(f)$.

Lemma 2.1 *([8]) Let q be univalent in the unit disk $\mathbb{E}$ and let θ and ϕ be analytic in domain $\mathbb{D}$ containing $q(\mathbb{E})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q_1(z)$ and suppose that*
(i) either h is convex or Q_1 is starlike in $\mathbb{E}$, and
(ii) $\Re \dfrac{zh'(z)}{Q_1(z)} > 0, z \in \mathbb{E}$.
If p is analytic in $\mathbb{E}$ with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[(p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.2 *([3]) Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q_1(z)$ and suppose that*
(i) Q_1 is starlike in $\mathbb{E}$ and
(ii) $\Re \dfrac{\theta'(q(z))}{\phi(q(z))} > 0, z \in \mathbb{E}$.
If $p \in \mathcal{H}[q(0), 1)] \cap Q$ with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\phi(p(z))$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi(p(z)),$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3 Main Results

Theorem 3.1 *Let α be a non-zero complex number and let q(z) be a univalent function such that*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > \max\left\{0, -\Re\left(\frac{q(z)}{\alpha}\right)\right\}, z \in \mathbb{E}. \qquad (3)$$

If $f \in \mathcal{A}$ satisfies the subordination

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+\alpha\left[\frac{(zf'(z)+\lambda z^2 f''(z))'}{f'(z)+\lambda zf''(z)}-\frac{z[(1-\lambda)f(z)+\lambda zf'(z)]'}{(1-\lambda)f(z)+\lambda zf'(z)}\right]$$
$$\prec q(z)+\alpha\frac{zq'(z)}{q(z)}, \qquad (4)$$

then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec q(z), z \in \mathbb{E}.$$

and $q(z)$ is the best dominant.

Proof. Define the function p by $p(z)=\dfrac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}$. With a little calculation, from (4), we have

$$p(z)+\alpha\frac{zp'(z)}{p(z)} \prec q(z)+\alpha\frac{zq'(z)}{q(z)}. \qquad (5)$$

Define the functions θ and ϕ as $\theta(w)=w,\quad \phi(w)=\dfrac{\alpha}{w}$. Clearly ϕ is analytic in domain in $\mathbb{D}=\mathbb{C}\setminus\{0\}$. Set $Q_1(z)=\alpha\dfrac{zq'(z)}{q(z)}$ and $h(z)=q(z)+\alpha\dfrac{zq'(z)}{q(z)}$. On differentiation, we obtain:

$$\frac{zQ_1'(z)}{Q_1(z)}=1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q_1(z)}=1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{q(z)}{\alpha}.$$

In view of condition (3), we have $Q_1(z)$ is starlike and $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right)>0$. The proof, now, follows from Lemma 2.1.

Remark 3.1 *Consider the dominant $q(z)=\dfrac{1+(1-2\beta)z}{1-z}, 0\le\beta<1$, we obtain*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)=\Re\left(\frac{1+(1-2\beta)z^2}{[1+(1-2\beta)z](1-z)}\right)>0$$

Also for $\alpha>0$,

$$\max\left\{0,-\Re\left(\frac{q(z)}{\alpha}\right)\right\}=0.$$

Clearly the condition in (3) holds and consequently, we get the following result from Theorem 3.1.

Theorem 3.2 *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies,*

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} + \alpha\left[\frac{(zf'(z)+\lambda z^2 f''(z))'}{f'(z)+\lambda zf''(z)} - \frac{z[(1-\lambda)f(z)+\lambda zf'(z)]'}{(1-\lambda)f(z)+\lambda zf'(z)}\right]$$
$$\prec \frac{1+2z[1-2\beta+\alpha(1-\beta)]+(1-2\beta)^2z^2}{1-2\beta z-(1-2\beta)z^2}, z \in \mathbb{E}.$$

then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec \frac{1+(1-2\beta)z}{1-z}, 0 \le \beta < 1.$$

Select $\lambda = 0$ in above theorem, we get the following result.

Corollary 3.1 *Let $\alpha > 0$ and $0 \le \beta < 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+2z[1-2\beta+\alpha(1-\beta)]+(1-2\beta)^2z^2}{1-2\beta z-(1-2\beta)z^2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}, z \in \mathbb{E}, \quad i.e. \quad f \in \mathcal{S}^*(\beta).$$

Select $\lambda = 1$ in Theorem 3.2, we get the following result.

Corollary 3.2 *Let $\alpha > 0$ and $0 \le \beta < 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right) + \alpha\left(\frac{(zf'(z)+z^2f''(z))'}{f'(z)+zf''(z)}\right)$$
$$\prec \frac{1+2z\left[1-2\beta+\alpha(1-\beta)\right]+(1-2\beta)^2z^2}{1-2\beta z-(1-2\beta)z^2}$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\beta)z}{1-z}, z \in \mathbb{E}, \quad i.e. \quad f \in \mathcal{K}(\beta).$$

Remark 3.2 *When we select the dominant $q(z) = \left(\frac{1+z}{1-z}\right)^\delta, 0 < \delta \le 1$ in Theorem 3.2. We see that*

$$\Re\left(1+\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1+z^2}{1-z^2}\right) > 0$$

and for $\alpha > 0$

$$\max\left\{0, -\frac{q(z)}{\alpha}\right\} = 0.$$

Obviously the condition (3) of Theorem 3.1 holds and we obtain the following result.

Theorem 3.3 *For $\alpha > 0$, If $f \in \mathcal{A}$ satisfies,*

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} + \alpha\left[\frac{(zf'(z)+\lambda z^2 f''(z))'}{f'(z)+\lambda zf''(z)} - \frac{z[(1-\lambda)f(z)+\lambda zf'(z)]'}{(1-\lambda)f(z)+\lambda zf'(z)}\right]$$

$$\prec \left(\frac{1+z}{1-z}\right)^{\delta} + 2\alpha\delta\left(\frac{z}{1-z^2}\right), z \in \mathbb{E},$$

then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, 0 < \delta \leq 1.$$

Select $\lambda = 0$ in in above theorem, we obtain the following result for strongly starlikeness.

Corollary 3.3 *Let $\alpha > 0$ and $0 < \delta \leq 1$. If $f \in \mathcal{A}$ satisfies,*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\delta} + 2\alpha\delta\left(\frac{z}{1-z^2}\right), z \in \mathbb{E}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \quad i.e. \quad f \in \overline{\mathcal{S}}(\delta).$$

Select $\lambda = 1$ in Theorem 3.3, we get following result.

Corollary 3.4 *Let $\alpha > 0$ and $0 < \delta \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right) + \alpha\left(\frac{(zf'(z)+z^2 f''(z))'}{f'(z)+zf''(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\delta} + 2\alpha\delta\left(\frac{z}{1-z^2}\right)$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, z \in \mathbb{E}.$$

Remark 3.3 *When we select the dominant $q(z) = e^z$, we have*

$$\Re\left(1+\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0.$$

Also

$$\max\left\{0, -\Re\left(\frac{q(z)}{\alpha}\right)\right\} = 0$$

for $\alpha > 0$. Thus (3) of Theorem 3.1 holds and consequently, we get the following result.

Theorem 3.4 *Let $\alpha > 0$ be real and let $f \in \mathcal{A}$ satisfy*

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+\alpha\left[\frac{z(zf'(z)+\lambda z^2 f''(z))'}{zf'(z)+\lambda z^2 f''(z)}-\frac{z[(1-\lambda)f(z)+\lambda zf'(z)]'}{(1-\lambda)f(z)+\lambda zf'(z)}\right] \prec e^z+\alpha z$$

then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec e^z, z \in \mathbb{E}.$$

Select $\lambda = 0$ in above theorem, we obtain:

Corollary 3.5 *For $\alpha > 0$, if $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec e^z+\alpha z$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z, z \in \mathbb{E} \quad i.e. \quad f \in \mathcal{S}^*.$$

Select $\lambda = 1$ in Theorem 3.4, we get following result.

Corollary 3.6 *Suppose that $\alpha > 0$ and $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(\frac{(zf'(z)+z^2 f''(z))'}{f'(z)+zf''(z)}\right) \prec e^z+\alpha z$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec e^z, z \in \mathbb{E}, \quad i.e. \quad f \in \mathcal{K}.$$

Remark 3.4 *For the dominant $q(z) = 1+az, 0 \leq a \leq 1$, we obtain*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1}{1+az}\right) > 0.$$

Moreover for $\alpha > 0$, $\max\left\{0, -\frac{q(z)}{\alpha}\right\} = 0$. *Clearly the condition (3) of Theorem 3.1 holds and consequently we have the following result.*

Theorem 3.5 *Let $\alpha > 0$ and $0 \leq a \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+\alpha\left[\frac{z(zf'(z)+\lambda z^2 f''(z))'}{zf'(z)+\lambda z^2 f''(z)}-\frac{z[(1-\lambda)f(z)+\lambda zf'(z)]'}{(1-\lambda)f(z)+\lambda zf'(z)}\right] \prec 1+az+\alpha\left(\frac{az}{1+az}\right)$$

then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec 1+az, z \in \mathbb{E}.$$

Select $\lambda = 0$ in above theorem, we get the following result.

Corollary 3.7 *Let $\alpha > 0$ and $0 \leq a \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + az + \alpha\left(\frac{az}{1+az}\right)$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + az, z \in \mathbb{E}, \quad i.e. \quad f \in \mathcal{S}^*.$$

Select $\lambda = 1$ in Theorem 3.5, we obtain the following result.

Corollary 3.8 *Let $\alpha > 0$ and $0 \leq a \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\left(1 + \frac{zf''(z)}{f'(z)}\right) + \alpha\left(\frac{(zf'(z) + z^2f''(z))'}{f'(z) + zf''(z)}\right) \prec 1 + az + \alpha\left(\frac{az}{1+az}\right)$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + az, z \in \mathbb{E}, \quad i.e. \quad f \in \mathcal{K}.$$

Theorem 3.6 *Let α be a non-zero complex number and let $q(z)$ be univalent such that*
(i) $\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0$ *and*
(ii) $\Re\left(\frac{q(z)}{\alpha}\right) > 0, z \in \mathbb{E}$.
Suppose

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \in \mathcal{H}[q(0), 1)] \cap Q,$$

and

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} + \alpha\left[\frac{z(zf'(z) + \lambda z^2 f''(z))'}{zf'(z) + \lambda z^2 f''(z)} - \frac{z[(1-\lambda)f(z) + \lambda zf'(z)]'}{(1-\lambda)f(z) + \lambda zf'(z)}\right]$$

is univalent in $\mathbb{E}$. If $f \in \mathcal{A}$ satisfies the superordination

$$q(z) + \alpha\frac{zq'(z)}{q(z)} \prec$$

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} + \alpha\left[\frac{z(zf'(z) + \lambda z^2 f''(z))'}{zf'(z) + \lambda z^2 f''(z)} - \frac{z[(1-\lambda)f(z) + \lambda zf'(z)]'}{(1-\lambda)f(z) + \lambda zf'(z)}\right] \tag{6}$$

then

$$q(z) \prec \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)}, z \in \mathbb{E},$$

and q is the best subordinant.

Proof. Define the function p as $p(z) = \dfrac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}$. Using (6), a little calculation yields that

$$q(z) + \alpha \frac{zq'(z)}{q(z)} \prec p(z) + \alpha \frac{zp'(z)}{p(z)}$$

Define the functions θ and ϕ as $\theta(w) = w$, $\phi(w) = \dfrac{\alpha}{w}$. Clearly ϕ is analytic in domain in $\mathbb{D} = \mathbb{C} \setminus \{0\}$. Set $Q_1(z) = \alpha \dfrac{zq'(z)}{q(z)}$ and $h(z) = q(z) + \alpha \dfrac{zq'(z)}{q(z)}$. On differentiation, we obtain:

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}$$

$$\frac{\theta'(q(z))}{\phi(q(z))} = \frac{zh'(z)}{Q_1(z)} - \frac{zQ_1'(z)}{Q_1(z)} = \frac{q(z)}{\alpha}.$$

In view of conditions (i) and (ii), we have $Q_1(z)$ is starlike and $\Re \dfrac{\theta'(q(z))}{\phi(q(z))} > 0$
The proof, now, follows from Lemma 2.2.
On combing Theorem 3.1 and Theorem 3.6, we obtain the following sandwich-type theorem.

Theorem 3.7 *Let α be a non zero complex number and let q_1, q_2 be univalent functions such that*
(i) $\Re \left(1 + \dfrac{zq_i''(z)}{q_i'(z)} - \dfrac{zq_i'(z)}{q_i(z)}\right) > 0, z \in \mathbb{E}, i = 1, 2$ *and*
(ii) $\Re \left(\dfrac{q_i(z)}{\alpha}\right) > 0, z \in \mathbb{E}, i = 1, 2.$
Suppose

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \in \mathcal{H}[q(0), 1)] \cap Q,$$

and

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} + \alpha \left[\frac{(zf'(z) + \lambda z^2 f''(z))'}{f'(z) + \lambda z f''(z)} - \frac{z[(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)f(z) + \lambda z f'(z)}\right]$$

is univalent in $\mathbb{E}$. If $f \in \mathcal{A}$ satisfies

$$q_1(z) + \alpha \frac{zq_1'(z)}{q_1(z)} \prec$$
$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} + \alpha \left[\frac{(zf'(z) + \lambda z^2 f''(z))'}{f'(z) + \lambda z f''(z)} - \frac{z[(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)f(z) + \lambda z f'(z)}\right]$$
$$\prec q_2(z) + \alpha \frac{zq_2'(z)}{q_2(z)}$$

then

$$q_1(z) \prec \frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)} \prec q_2(z), z \in \mathbb{E},$$

where q_1 *and* q_2 *are the best subordinant and the best dominant respectively.*

When we select $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\delta_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\delta_2}$ in Theorem 3.7, we obtain the following result.

Theorem 3.8 *Let* $\alpha > 0$ *and* $0 < \delta_1 < \delta_2 \leq 1$*. If* $f \in \mathcal{A}$ *satisfies,*

$$\left(\frac{1+z}{1-z}\right)^{\delta_1} + 2\alpha\delta\left(\frac{z}{1-z^2}\right) \prec$$

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)} + \alpha\left[\frac{(zf'(z)+\lambda z^2 f''(z))'}{f'(z)+\lambda z f''(z)} - \frac{z[(1-\lambda)f(z)+\lambda z f'(z)]'}{(1-\lambda)f(z)+\lambda z f'(z)}\right]$$

$$\prec \left(\frac{1+z}{1-z}\right)^{\delta_2} + 2\alpha\delta\left(\frac{z}{1-z^2}\right)$$

then

$$\left(\frac{1+z}{1-z}\right)^{\delta_1} \prec \frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta_2}, z \in \mathbb{E}.$$

Select $\lambda = 0$ in above theorem, we have the following result.

Corollary 3.9 *Let* $\alpha > 0$ *and* $0 < \delta_1 < \delta_2 \leq 1$*. If* $f \in \mathcal{A}$ *satisfies,*

$$\left(\frac{1+z}{1-z}\right)^{\delta_1} + 2\alpha\delta\left(\frac{z}{1-z^2}\right) \prec (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)$$

$$\prec \left(\frac{1+z}{1-z}\right)^{\delta_2} + 2\alpha\delta\left(\frac{z}{1-z^2}\right)$$

then

$$\left(\frac{1+z}{1-z}\right)^{\delta_1} \prec \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta_2}, z \in \mathbb{E}.$$

Select $\lambda = 1$ in Theorem 3.8, we get the following result.

Corollary 3.10 *Let* $\alpha > 0$ *and* $0 < \delta_1 < \delta_2 \leq 1$*. If* $f \in \mathcal{A}$ *satisfies,*

$$\left(\frac{1+z}{1-z}\right)^{\delta_1} + 2\alpha\delta\left(\frac{z}{1-z^2}\right)$$

$$\prec (1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right) + \alpha\left(\frac{(zf'(z)+z^2 f''(z))'}{f'(z)+zf''(z)}\right)$$

$$\prec \left(\frac{1+z}{1-z}\right)^{\delta_2} + 2\alpha\delta\left(\frac{z}{1-z^2}\right)$$

then

$$\left(\frac{1+z}{1-z}\right)^{\delta_1} \prec 1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta_2}, z \in \mathbb{E}.$$

When we select $q_1(z) = 1 + az,\ q_2(z) = 1 + bz,\ \ 0 < a < b$ in Theorem 3.7, we obtain:

Theorem 3.9 *Let $\alpha > 0$ and $0 < a < b \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$1 + az + \alpha\left(\frac{az}{1+az}\right) \prec$$

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} + \alpha\left[\frac{(zf'(z) + \lambda z^2 f''(z))'}{f'(z) + \lambda zf''(z)} - \frac{z[(1-\lambda)f(z) + \lambda zf'(z)]'}{(1-\lambda)f(z) + \lambda zf'(z)}\right]$$

$$\prec 1 + bz + \alpha\left(\frac{bz}{1+bz}\right)$$

then

$$1 + az \prec \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \prec 1 + bz, z \in \mathbb{E}.$$

Select $\lambda = 0$ in above theorem, we get following result.

Corollary 3.11 *Let $\alpha > 0$ and $0 < a < b \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$1 + az + \alpha\left(\frac{az}{1+az}\right) \prec (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

$$\prec 1 + bz + \alpha\left(\frac{bz}{1+bz}\right)$$

then

$$1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz, z \in \mathbb{E}.$$

Select $\lambda = 1$ in Theorem 3.9, we get the following result.

Corollary 3.12 *Let $\alpha > 0$ and $0 < a < b \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$1 + az + \alpha\left(\frac{az}{1+az}\right) \prec (1-\alpha)\left(1 + \frac{zf''(z)}{f'(z)}\right) + \alpha\left(\frac{(zf'(z) + z^2 f''(z))'}{f'(z) + zf''(z)}\right)$$

$$\prec 1 + bz + \alpha\left(\frac{bz}{1+bz}\right)$$

then

$$1 + az \prec 1 + \frac{zf''(z)}{f'(z)} \prec 1 + bz, z \in \mathbb{E}.$$

Selecting $\alpha = a = \frac{1}{2}, b = 1$ in Corollary 3.11, we obtain:

Example 3.1 *If $f \in \mathcal{A}$ satisfies,*

$$2 + z + \frac{z}{2+z} \prec 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} \prec 2(1+z) + \frac{z}{1+z}$$

then

$$1 + \frac{1}{2}z \prec \frac{zf'(z)}{f(z)} \prec 1 + z, z \in \mathbb{E}.$$

Selecting $\alpha = a = \frac{1}{2}, b = 1$ in Corollary 3.12, we get:

Example 3.2 *If $f \in \mathcal{A}$ satisfies,*

$$2 + z + \frac{z}{2+z} \prec 1 + \frac{zf''(z)}{f'(z)} + \frac{(zf'(z) + z^2 f''(z))'}{f'(z) + zf''(z)} \prec 2(1+z) + \frac{z}{1+z}$$

then

$$1 + \frac{1}{2}z \prec 1 + \frac{zf''(z)}{f'(z)} \prec 1 + z, z \in \mathbb{E}.$$

Remark 3.5 *In Figure 1, we plot the images of the unit disk $\mathbb{E}$ under the functions $h_1(z) = 2 + z + \frac{z}{2+z}$ and $h_2(z) = 2(1+z) + \frac{z}{1+z}$ and in Figure 2, the images of the unit disk $\mathbb{E}$ under the functions $q_1(z) = 1 + \frac{1}{2}z$ and $q_2(z) = 1 + z$ are plotted. Therefore, from Example 3.1, we notice that the differential operator $\frac{zf'(z)}{f(z)}$ takes values in the light shaded region of Figure 2 when the differential operator $1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)}$ takes values in the light shaded region of Figure 1. Hence the function f is starlike in $\mathbb{E}$. Similarly, in Example 3.2, we see that the differential operator $1 + \frac{zf''(z)}{f'(z)}$ takes values in the light shaded region of Figure 2 when the differential operator $1 + \frac{zf''(z)}{f'(z)} + \frac{(zf'(z) + z^2 f''(z))'}{f'(z) + zf''(z)}$ takes values in the light shaded region of Figure 1 and hence the function f is convex in $\mathbb{E}$.*

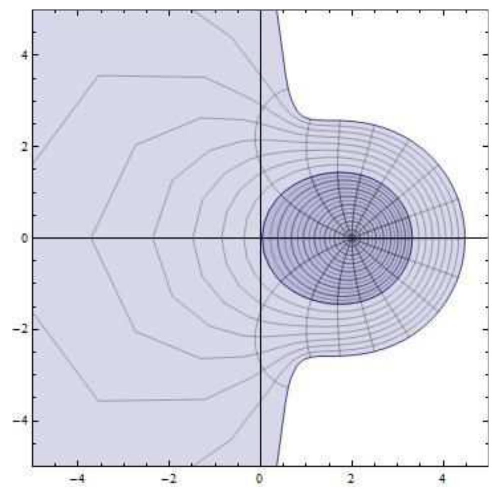

Figure 1

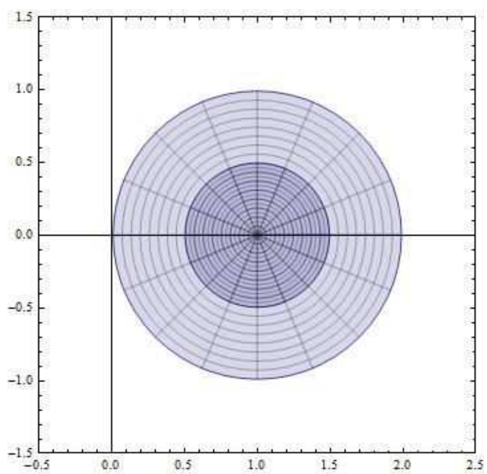

Figure 2

4 Open Problem

The results obtained in this paper hold for $\alpha > 0$. One may find such dominants that the results hold for $\alpha < 0$.

References

[1] Billing, S. S., "Differential sandewich type results and criteria for starlikenees", *Rend. sem. Mat. Univ. Politec. Torino*, **69**(1)(2011), 57-71.

[2] Brannan, D. A. and Kirwan, W. E., “On some classes of bounded univalent functions”, *J. London Math. Soc.*, **2**(1)(1969), 431-443.

[3] Bulboca, T., “Differential Subordinations and Superordinations- Recent Results”, *House of Science Book Publ. Cluj-Nupoca*, (2005).

[4] Imrak, H. and Raina, R. K., “The Starlikeness and convexity of multivalent functions involving certain inequalities”, *Revista Matematica Complutense*, **16**(2) (2003), 391-398.

[5] Imrak, H., Raina, R. K. and Owa, S., “Certain Results Involving Inequalities on Analytic and Univalent Functions”, *Far East J. Math Sci.(FJMS)* **10**(3)(2003), 359-366.

[6] Lewandowski, Z., Miller S. S. and Zlotkiewicz, E., Generating functions for some classes of univalent functions, Proc. Amer. Math. Soc., 56(1976), 111-117.

[7] Miller, S. S., Mocano, P. T. and Reade, M. O., “All alpha-covex functions are univalent and starlike”, *Proc. Amer. Math. Soc.*, **37**(1973), 553-554.

[8] Miller, S. S. and Mocanu, P. T., “Differential Subordination: theory and Applications”, *Marcel Dekker, new York and Basel*, (2000).

[9] Stankiewicz, J., “Some remarks concerning starlike functions”, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phy.*, **18**(1970), 143-146.