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The sufficient condition for certain generalized class of non-Bazilevič functions

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Abstract

In this paper introduced certain new generalized class of Multivalent non-Bazilevič functions in the unit disc. We obtain some sufficient conditions.

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1 Introduction

Let $\mathcal{H}_n(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad (n, p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1}$$

which are analytic and multivalent in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathcal{H}_n(p)$ is said to be in the class $\mathcal{S}_n^*(p, \alpha)$ of multivalent starlike functions of order α in $\mathcal{U}$ if it satisfies the following inequality:

$$\mathfrak{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > \alpha, \quad 0 \leq \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}. \tag{2}$$

On the other hand, a function $f \in \mathcal{H}_n(p)$ is said to be in the class $\mathcal{C}_n(p,\alpha)$ of multivalent close-to-convex functions of order α in $\mathcal{U}$ if it satisfies the following inequality:

$$\mathfrak{Re}(z^{1-p}f'(z)) > \alpha, \quad 0 \le \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}. \tag{3}$$

We observe that $\mathcal{S}_1^*(1,\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}_1(1,\alpha) = \mathcal{C}(\alpha)$, where $\mathcal{S}^*(\alpha)$ are the usual subclasses of $\mathcal{H}_1(1)$ consisting of functions which are starlike of order $\alpha(0 \le \alpha < 1)$ and close-to-convex of order $\alpha(0 \le \alpha < 1)$ in $\mathcal{U}$, respectively (see, for details, [1, 2]).

Recently Frasin (see [3]) introduced and studied the following class of analytic and multivalent functions defined as follows (see also [4]).

A function $f \in \mathcal{H}_n(p)$ is said to be a member of the class $\mathcal{B}_n(p,\mu,\alpha)$ if and only if

$$\left|\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p}f'(z) - p\right| < p - \alpha, \ (p \in \mathbb{N}) \tag{4}$$

for some $\mu \ge 0, \alpha(0 \le \alpha < p), z \in \mathbb{U}$.

Note the condition (4) implies that

$$\mathfrak{Re}\left(\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p}f'(z)\right) > \alpha, \ z \in \mathcal{U}. \tag{5}$$

The class $\mathcal{B}_1(1,1,\alpha) = \mathcal{B}(\alpha)$ is the class which has been introduced and studied by Frasin and Darus [5] (see also [6]).

In this paper,we introduced and studied the following class of analytic and multivalent functions defined as follows .

Definition 1.1. *A function $f \in \mathcal{H}_n(p)$ is said to be a member of the class $\mathcal{N}_n(p,\mu,\alpha)$ if and only if*

$$\left|\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\mu}\right| > \alpha, \ (p \in \mathbb{N}) \tag{6}$$

for some $\mu \ge 0, \alpha(0 \le \alpha < 1), z \in \mathbb{U}$.

Note the condition (5) implies the condition (6). To prove our main result, we need the following Lemma:

Lemma 1.1 (see [7]). *Let the function $w(z)$ be(nonconstant) analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attsts its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then*

$$z_0 w'(z_0) = k w(z_0), \tag{7}$$

where $k \ge 1$ is a real number.

2 Main Results

Lemma 2.1. *Let $\varphi(z)$ be analytic in the unit disc $\mathbb{U}$ and $0 < \varphi(0) < 1$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\mathfrak{Re}\varphi(z) > 0 \ for \ |z| < |z_0| \tag{8}$$

and

$$\mathfrak{Re}\varphi(z_0) = 0. \tag{9}$$

Then we have

$$\frac{z_0\varphi'(z_0)}{1+\varphi(z_0)} + \frac{z_0\varphi'(z_0)}{1-\varphi(z_0)} < -\varphi(0). \tag{10}$$

Proof. Let us put

$$p(z) = \frac{\varphi(0)-\varphi(z)}{\varphi(0)+\varphi(z)}. \tag{11}$$

Then $\varphi(z)$ is analytic in $\mathbb{U}$, $p(0) = 0$, $|p(z)| < 1$ for $|z| < |z_0|$ and $p(z_0) = 1$. Therefore, applying Lemma 1.1, we have that

$$1 \leq k = \frac{z_0 p'(z_0)}{p(z_0)} = -\frac{2\varphi(0)z_0\varphi'(z_0)}{(\varphi(0))^2 + |\varphi(z_0)|^2}. \tag{12}$$

Thus

$$\begin{aligned} \frac{z_0\varphi'(z_0)}{1+\varphi(z_0)} + \frac{z_0\varphi'(z_0)}{1-\varphi(z_0)} &= 2\tfrac{z_0\varphi'(z_0)}{1-(\varphi(z_0))^2} \\ &= 2\tfrac{z_0\varphi'(z_0)}{1+|\varphi(z_0)|^2} \\ &= -k\tfrac{(\varphi(0))^2+|\varphi(z_0)|^2}{\varphi(0)(1+|\varphi(z_0)|^2)} \\ &< -\varphi(0). \end{aligned} \tag{13}$$

Lemma 2.2. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that $|p(z)| > \alpha$ for $|z| < |z_0|$ and $|p(z_0)| = \alpha$, where $0 < \alpha < 1$. Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} \leq -\frac{1-\alpha}{1+\alpha}. \tag{14}$$

Proof. Let us put

$$\varphi(z) = \frac{p(z)-\alpha}{p(z)+\alpha} \ for \ |z| < |z_0|. \tag{15}$$

Then $\varphi(z)$ is analytic in $\{z \in \mathbb{C} : |z| < |z_0|\}$ and

$$0 < \varphi(0) = \frac{1-\alpha}{1+\alpha} < 1. \tag{16}$$

By the hypothesis of the theorem 2.1, we have

$$\alpha < |p(z)| = \alpha\Big|\frac{1+\varphi(z)}{1-\varphi(z)}\Big| \ for \ |z| < |z_0|. \tag{17}$$

Thus

$$\Big|\frac{1+\varphi(z)}{1-\varphi(z)}\Big| > 1 \ for \ |z| < |z_0|. \tag{18}$$

So that

$$\mathfrak{Re}\varphi(z) > 0 \ for \ |z| < |z_0|. \tag{19}$$

On the other hand, we have

$$\alpha = |p(z_0)| = \alpha\Big|\frac{1+\varphi(z_0)}{1-\varphi(z_0)}\Big|. \tag{20}$$

So that

$$\mathfrak{Re}\varphi(z_0) = 0. \tag{21}$$

And

$$\frac{z_0p'(z_0)}{p(z_0)} = \frac{z_0\varphi'(z_0)}{1+\varphi(z_0)} + \frac{z_0\varphi'(z_0)}{1-\varphi(z_0)}. \tag{22}$$

By Lemma 2.1, we completes the proof of Lamme 2.2.

Lamme 2.3. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and suppose that*

$$\mathfrak{Re}\Big\{\frac{zp'(z)}{p(z)}\Big\} > -\frac{1-\alpha}{1+\alpha}, \ 0 < \alpha < 1, \ z \in \mathbb{U}. \tag{23}$$

Then we have $|p(z)| > \alpha$ in $\mathbb{U}$.

Proof. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that $|p(z)| > \alpha$ for $|z| < |z_0|$ and $|p(z_0)| = \alpha$, where $0 < \alpha < 1$. Then by Lemma 2.2, we have

$$\frac{z_0p'(z_0)}{p(z_0)} \leq -\frac{1-\alpha}{1+\alpha}. \tag{24}$$

It contradicts the hypothesis (21) and it completes the proof Lamme 2.3.

Making use of Lemma 2.3, we first prove the following the Theorem.

Theorem 2.1. *Let $f(z) \in \mathcal{H}_n(p)$. And suppose that for some $\mu \geq 0, \alpha(0 \leq \alpha < 1)$, such that*

$$\mathfrak{Re}\Big\{1 + \mu p + \frac{zf''(z)}{f'(z)} - (\mu+1)\frac{zf'(z)}{f(z)}\Big\} > -\frac{1-\alpha}{1+\alpha}, z \in \mathbb{U}, \tag{25}$$

then $f \in \mathcal{N}_n(p, \mu, \alpha)$.

Proof. Let us put

$$p(z) = \frac{zf'(z)}{pf(z)}\Big(\frac{z^p}{f(z)}\Big)^{\mu}, \tag{26}$$

then $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathbb{U}$ which satisfies the conditions of Lemma 2.2.

Making use of (26), it follows that

$$\frac{z_0 f'(z_0)}{pg(z_0)}\Big(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)}\Big) = z_0 p'(z_0). \tag{27}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions Lemma 2.2, therefore in view of (27), we obtain

$$\mathfrak{Re}\Big\{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)}\Big\} \leq -\frac{1-\alpha}{1+\alpha}. \tag{28}$$

This is a contradiction with (24) and therefore proof of the Theorem 2.1 is completed.

Taking $\mu = 0$ in Theorem 2.1, we have the following Corollary 2.1.

Corollary 2.1. *Let $f(z) \in \mathcal{H}_n(p)$. And suppose that for some $\alpha(0 \leq \alpha < 1)$, such that*

$$\mathfrak{Re}\Big\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\Big\} > -\frac{1-\alpha}{1+\alpha}, z \in \mathbb{U}, \tag{29}$$

then $f(z)$ satisfies the following inequality:

$$\Big|\frac{zf'(z)}{pf(z)}\Big| > \alpha, \ (p \in \mathbb{N}) \tag{30}$$

for some $\alpha(0 \leq \alpha < 1), z \in \mathbb{U}$.

Taking $\mu = 0, p = 1, n = 1$ in Theorem 2.1, we have the following Corollary 2.2.

Corollary 2.2. *Let $f(z) \in \mathcal{H}$. And suppose that for some $\alpha(0 \leq \alpha < 1)$, such that*

$$\mathfrak{Re}\Big\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\Big\} > -\frac{1-\alpha}{1+\alpha}, z \in \mathbb{U}, \tag{31}$$

then $f(z)$ satisfies the following inequality:

$$\Big|\frac{zf'(z)}{f(z)}\Big| > \alpha, \tag{32}$$

for some $\alpha(0 \leq \alpha < 1), z \in \mathbb{U}$.

3 Open Problem

With regards to the problems solved, the this work can also be applied to other classes. For example, can the same problem be applied for following classes.

Definition 3.1. *A function $f \in \mathcal{H}_n(p)$ is said to be a member of the class $\mathcal{B}_n(p, \alpha, \mu, \lambda)$ if and only if*

$$\Big|(1-\lambda)\Big(\frac{z}{f(z)}\Big)^{\mu} + \lambda\frac{zf'(z)}{pf(z)}\Big(\frac{z}{f(z)}\Big)^{\mu}\Big| > \alpha, \ (p \in \mathbb{N}, z \in \mathbb{U}) \tag{33}$$

for some $\mu \geq 0, 0 \leq \alpha < 1, \lambda \in \mathbb{C}$.

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