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Majorization Properties for Subclasses of Analytic p-Valent Functions Defined by Generalized Differintegral Operator

R. M. El-Ashwah

Department of Mathematics
Faculty of Science
Damietta University 34517, Egypt
e-mail: r_elashwah@yahoo.com

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Abstract

The object of the present paper is to investigate the majorization properties of certain subclasses of analytic and p-valent functions defined by the generalized differintegral operator.

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1 Introduction

Let f and g be analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.
We say that f is majorized by g in U (see [16]) and write

$$f(z) << g(z) \quad (z \in U), \tag{1}$$

if there exists a function φ, analytic in U such that

$$|\varphi(z)| \leq 1 \quad and \quad f(z) = \varphi(z)g(z) \quad (z \in U). \tag{2}$$

It may be noted that (1) is closely related to the concept of quasi-subordination between analytic functions.

For $f(z)$ and $g(z)$ are analytic functions in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as follows:

$$f \prec g \ \ or\ f(z) \prec g(z)\ ,$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. Further, if the function $g(z)$ is univalent in U, then we have the following equivalent (see [17, p.4])

$$f(z) \prec g(z) \ \Leftrightarrow f(0) = g(0) \ and\ f(U) \prec g(U).$$

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}) \tag{3}$$

which are analytic and p-valent in U. In [6] Catas extended the multiplier transformations and defined the operator $I_p^m(\lambda, \ell)f(z)$ on $A(p)$ by the following infinite series

$$I_p^m(\lambda, \ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p + \ell + \lambda(k-p)}{p+\ell}\right]^m a_k z^k$$

$$(\lambda \geq 0; \ell \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U). \tag{4}$$

We note that:

$$I_p^0(1, 0)f(z) = f(z) \quad and \quad I_p^1(1, 0)f(z) = \frac{zf'(z)}{p}\ .$$

By specializing the parameters λ, ℓ, p and m, we obtain the following operators studied by various authors:

(i) $I_p^m(1, \ell)f(z) = I_p(m, \ell)f(z)$ (see Kumar et al. [15] and Srivastava et al. [26]);

(ii) $I_p^m(1, 0)f(z) = D_p^m f(z)$ (see Aouf and Mostafa [4], Kamali and Orhan [14] and Orhan and Kiziltunc [20]);

(iii) $I_1^m(1, \ell)f(z) = I_\ell^m f(z)$ (see Cho and Kim [7] and Cho and Srivastava [8]);

(iv) $I_1^m(1, 0)(z) = D^m f(z)$ (see Salagean [24]);

(v) $I_1^m(\lambda, 0)(z) = D_\lambda^m(z)$ (see Al-Oboudi [1]);

(vi) $I_1^m(1, 1)(z) = I^m f(z)$ (see Uralegaddi and Somanatha [28]);

(vii) $I_p^m(\lambda, 0)(z) = D_{\lambda,p}^m f(z)$ (see El-Ashwah and Aouf [9]).

In [10] El-Ashwah and Aouf defined the integral operator $J_p^m(\lambda, \ell)f(z)$ on $A(p)$ by the following infinite series

$$J_p^m(\lambda,\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell}{p+\ell+\lambda(k-p)}\right]^m a_k z^k \quad (m \in \mathbb{N}_0). \tag{5}$$

From (4) and (5), we observe that $J_p^{-m}(\lambda,\ell)f(z) = I_p^m(\lambda,\ell)f(z)$ $(m > 0)$, so the operator $J_p^m(\lambda,\ell)f(z)$ is well-defined for $\lambda \geq 0, \ell \geq 0, p \in \mathbb{N}$ and $m \in \mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$, and it is easy to verify that:

$$\lambda z(J_p^{m+1}(\lambda,\ell)f(z))' = (\ell+p)J_p^m(\lambda,\ell)f(z) - [\ell+p(1-\lambda)]\, J_p^{m+1}(\lambda,\ell)f(z) \; (m \in \mathbb{Z}, \lambda > 0). \tag{6}$$

Also the operator $J_p^m(\lambda,\ell)f(z)$ was studied by Srivastava et al. [26] and Aouf et al. [5].

We note that:

(i)$J_1^m(\lambda,0)f(z) = I_\lambda^{-m}f(z)$ (see Patel [21])

$$= \left\{ f(z) \in A(1) : I_\lambda^{-m} f(z) = z + \sum_{k=2}^{\infty} [1+\lambda(k-1)]^{-m} a_k z^k, m \in \mathbb{N}_0 \right\};$$

(ii) $J_1^\alpha(1,1)f(z) = I^\alpha f(z)$ (see Jung et al. [13]);

$$= \left\{ f(z) \in A(1) : I^\alpha f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^\alpha a_k z^k; \alpha > 0; z \in U \right\};$$

(iii) $J_p^\alpha(1,1)f(z) = I_p^\alpha f(z)$ (see Shams et al. [25]);

$$= \left\{ f(z) \in A(p) : I_p^\alpha f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+1}{k+1}\right)^\alpha a_k z^k; \alpha > 0; z \in U \right\};$$

(iv) $J_p^m(1,1)f(z) = D^m f(z)$ (see Patel and Sahoo [22]);

$$= \left\{ f(z) \in A(p) : D^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+1}{k+1}\right)^m a_k z^k; m \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\};\ z \in U \right\};$$

(v) $J_1^m(1,1)f(z) = I^m f(z)$ (see Flett [11]);

$$= \left\{ f(z) \in A(1) : I^m f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^m a_k z^k; m \in \mathbb{N}_0;\ z \in U \right\};$$

(iv) $J_1^m(1,0)f(z) = I^m f(z)$ (see Salagean [24])

$$= \left\{ f(z) \in A(1) : I^m f(z) = z + \sum_{k=2}^{\infty} k^{-m} a_k z^k; m \in \mathbb{N}_0; z \in U \right\}.$$

Also we note that:
(i)$J_p^m(1,0)f(z) = J_p^m f(z)$

$$= \left\{ f(z) \in A(p) : J_p^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{k}\right)^m a_k z^k ; m \in \mathbb{N}_0; z \in U \right\};$$

(ii) $J_p^m(1,\ell)f(z) = J_p^m(\ell)f(z)$

$$= \left\{ f(z) \in A(p) : J_p^m(\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\ell}{k+\ell}\right)^m a_k z^k ; m \in \mathbb{N}_0; \ell \geq 0; z \in U \right\};$$

(iii) $J_p^m(\lambda,0)f(z) = J_{\lambda,p}^m f(z)$

$$= \left\{ f(z) \in A(p) : J_{\lambda,p}^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{p+\lambda(k-p)}\right)^m a_k z^k ; m \in \mathbb{N}_0; \lambda \geq 0; z \in U \right\}.$$

Definition 1.1 *Let* $-1 \leq B < A \leq 1, p \in \mathbb{N}, m \in \mathbb{Z}, j \in \mathbb{N}_0, \lambda > 0, \ell \geq 0, \gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, \left|\gamma(A-B) + \left(\frac{p+\ell}{\lambda}\right) B\right| < \left(\frac{p+\ell}{\lambda}\right)$ *and* $f \in A(p)$*. Then* $f \in S_{p,\lambda,\ell}^{m,j}(\gamma; A, B)$ *the class of p-valent functions of complex order* γ *in* U *if and only if*

$$\left\{ 1 + \frac{1}{\gamma} \left(\frac{z\left(J_p^m(\lambda,\ell)f(z)\right)^{(j+1)}}{\left(J_p^m(\lambda,\ell)f(z)\right)^{(j)}} - p + j \right) \right\} \prec \frac{1+Az}{1+Bz}. \tag{7}$$

Clearly, we have the following relationships:
(i) $S_{p,\lambda,\ell}^{m,j}(\gamma; 1, -1) = S_{p,\lambda,\ell}^{m,j}(\gamma)$;
(ii) $S_{p,1,0}^{m,j}(\gamma; 1, -1) = S_p^{m,,j}(\gamma)$;
(iii)$S_{p,1,0}^{0,j}(\gamma; 1, -1) = S_p^j(\gamma)$;
(iv) $S_{1,0,0}^{0,0}(\gamma; 1, -1) = S(\gamma)$ (see Nasr and Aouf [18] and Wiatrowski [29]);
(v) $S_{p,1,0}^{-1,j}(\gamma; 1, -1) = K_p^j(\gamma)$ (see Altintas and Srivastava [2]);
(vi)$S_{1,1,0}^{-1,0}(\gamma; 1, -1) = K(\gamma)$ (see Nasr and.Aouf [18] and Wiatrowski [29]);
(vii) $S_{1,0,0}^{0,0}(1-\alpha; 1, -1) = S^*(\alpha)$ $(0 \leq \alpha < 1)$(see Robertson [23]).
We shall need the following lemma.

Lemma 1.2 *[2]. Let* $\gamma \in \mathbb{C}^*$ *and* $f \in K_p^j(\gamma)$*. Then* $f \in S_p^j(\frac{1}{2}\gamma)$*, that is,*

$$K_p^j(\gamma) \subset S_p^j(\frac{1}{2}\gamma) \quad (\gamma \in \mathbb{C}^*). \tag{8}$$

An majorization problem for the class $S(\gamma)$ has recently been investigated by Altintas et al. [2]. Also, majorization problem for the class $S^* = S^*(0)$ has been investigated by MacGregor [14]. In this paper we investigate majorization problem for the class $S_{p,\lambda,\ell}^{m,j}(\gamma; A, B)$.

2 Main Results

Unless otherwise mentioned we shall assume throughout the paper that $-1 \leq B < A \leq 1, \gamma \in \mathbb{C}^*, \lambda > 0, \ell \geq 0, m \in \mathbb{Z}$ and $p \in \mathbb{N}$.

Theorem 2.1 *Let the function $f \in A(p)$ and suppose that $g \in S^{m,j}_{p,\lambda,\ell}(\gamma; A, B)$. If $\left(J^m_p(\lambda,\ell)f(z)\right)^{(j)}$ is majorized by $\left(J^m_p(\lambda,\ell)g(z)\right)^{(j)}$ in U, then*

$$\left|\left(J^{m-1}_p(\lambda,\ell)f(z)\right)^{(j)}\right| \leq \left|\left(J^{m-1}_p(\lambda,\ell)g(z)\right)^{(j)}\right| \qquad (|z| < r_0), \tag{9}$$

where $r_0 = r_0(p, \gamma, \lambda, \ell, A, B)$ is the smallest positive root of the equation

$$\left|\gamma(A-B) + \left(\frac{p+\ell}{\lambda}\right) B\right| r^3 - \left[2|B| + \left(\frac{p+\ell}{\lambda}\right)\right] r^2 - \left[2 + \left|\gamma(A-B) + \left(\frac{p+\ell}{\lambda}\right) B\right|\right] r + \left(\frac{p+\ell}{\lambda}\right) = 0. \tag{10}$$

Since $g \in S^{m,j}_{p,\lambda,\ell}(A, B; \gamma)$ we find from (7) that

$$1 + \frac{1}{\gamma}\left(\frac{z\left(J^m_p(\lambda,\ell)g(z)\right)^{(j+1)}}{\left(J^m_p(\lambda,\ell)g(z)\right)^{(j)}} - p + j\right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{11}$$

where w is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$. From (11), we have

$$\frac{z\left(J^m_p(\lambda,\ell)g(z)\right)^{(j+1)}}{\left(J^m_p(\lambda,\ell)g(z)\right)^{(j)}} = \frac{(p-j) + [\gamma(A-B) + (p-j)B]\, w(z)}{1 + Bw(z)}. \tag{12}$$

From (6), we have

$$z\left(J^m_p(\lambda,\ell)g(z)\right)^{(j+1)} = \left(\frac{p+\ell}{\lambda}\right)\left(J^{m-1}_p(\lambda,\ell)g(z)\right)^{(j)} - \left[\left(\frac{p+\ell}{\lambda}\right) + j - p\right]\left(J^m_p(\lambda,\ell)g(z)\right)^{(j)}$$
$$(0 \leq j \leq p; p \in \mathbb{N}; \lambda > 0; z \in U). \tag{13}$$

Also from (12) and (13), we have

$$\left|\left(J^m_p(\lambda,\ell)g(z)\right)^{(j)}\right| \leq \frac{\left(\frac{p+\ell}{\lambda}\right)(1 + |B|\,|z|)}{\left[\left(\frac{p+\ell}{\lambda}\right)\right] - \left|\gamma(A-B) + \left(\frac{p+\ell}{\lambda}\right) B\right| |z|}\left|\left(J^{m-1}_p(\lambda,\ell)g(z)\right)^{(j)}\right|. \tag{14}$$

Next, since $\left(J_p^m(\lambda,\ell)f(z)\right)^{(j)}$ is majorized by $\left(J_p^m(\lambda,\ell)g(z)\right)^{(j)}$ in U, from (2), we have

$$\left(J_p^m(\lambda,\ell)f(z)\right)^{(j)} = \varphi(z)\left(J_p^m(\lambda,\ell)g(z)\right)^{(j)}. \tag{15}$$

Differentiating (15) with respect to z and multiplying by z, we have

$$z\left(J_p^m(\lambda,\ell)f(z)\right)^{(j+1)} = z\varphi'(z)\left(J_p^m(\lambda,\ell)g(z)\right)^{(j)} + z\varphi(z)\left(J_p^m(\lambda,\ell)g(z)\right)^{(j+1)}, \tag{16}$$

using (13) in (16), we have

$$\left(J_p^{m-1}(\lambda,\ell)f(z)\right)^{(j)} = \frac{z\varphi'(z)}{\left(\frac{p+\ell}{\lambda}\right)}\left(J_p^m(\lambda,\ell)g(z)\right)^{(j)} + \varphi(z)\left(J_p^{m-1}(\lambda,\ell)g(z)\right)^{(j)}. \tag{17}$$

Thus, by noting that $\varphi(z)$ satisfies the inequality (see [19]),

$$\left|\varphi'(z)\right| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z\in U), \tag{18}$$

and making use of (14) and (18) in (17), we have

$$\left|\left(J_p^{m-1}(\lambda,\ell)f(z)\right)^{(j)}\right| \leq$$

$$\left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}.\frac{(1+|B|\,|z|)\,|z|}{\left(\frac{p+\ell}{\lambda}\right) - \left|\gamma(A-B)+\left(\frac{p+\ell}{\lambda}\right)B\right|\,|z|}\right)\left|\left(J_p^{m-1}(\lambda,\ell)g(z)\right)^{(j)}\right|, \tag{19}$$

which upon setting

$$|z| = r \quad and \quad |\varphi(z)| = \rho \quad (0\leq\rho\leq 1),$$

leads us to the inequality

$$\left|\left(J_p^{m-1}(\lambda,\ell)f(z)\right)^{(j)}\right| \leq$$

$$\frac{\Psi(\rho)}{(1-r^2)\left[\left(\frac{p+\ell}{\lambda}\right) - \left|\gamma(A-B)+\left(\frac{p+\ell}{\lambda}\right)B\right| r\right]}\left|\left(J_p^{m-1}(\lambda,\ell)g(z)\right)^{(j)}\right|,$$

where

$$\begin{aligned}\Psi(\rho) &= -r\,(1+|B|\,r)\,\rho^2 + (1-r^2)\left[\left(\frac{p+\ell}{\lambda}\right) - \left|\gamma(A-B)+\left(\frac{p+\ell}{\lambda}\right)B\right| r\right]\rho \\ &\quad + r\,(1+|B|\,r),\end{aligned} \tag{20}$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(p, \gamma, \lambda, \ell, A, B)$, where $r_0(p, \gamma, \lambda, \ell, A, B)$ is given by (10), then the function $\Phi(\rho)$ defined by

$$\begin{aligned} \Phi(\rho) &= -\sigma\,(1 + |B|\,\sigma)\,\rho^2 + (1 - \sigma^2)\left[\left(\frac{p+\ell}{\lambda}\right) - \left|\gamma(A - B) + \left(\frac{p+\ell}{\lambda}\right)B\right|\sigma\right]\rho \\ &\quad +\sigma\,(1 + |B|\,\sigma) \end{aligned}$$

(21)

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\begin{aligned} \Phi(\rho) &\leq \Phi(1) = (1 - \sigma^2)\left[\left(\frac{p+\ell}{\lambda}\right) - \left|\gamma(A - B) + \left(\frac{p+\ell}{\lambda}\right)B\right|\sigma\right] \\ &\quad (0 \leq \rho \leq 1; 0 \leq \sigma \leq r_0(p, \gamma, \lambda, \ell, A, B))\,. \end{aligned} \tag{22}$$

Hence upon setting $\rho = 1$ in (21), we conclude that (9) holds true for $|z| \leq r_0 = r_0(p, \gamma, \lambda, \ell, A, B)$, where $r_0(p, \gamma, \lambda, \ell, A, B)$, is the smallest positive root of (10). This completes the proof of Theorem 2.1.
Putting $A = 1$ and $B = -1$ in Theorem 1, we obtain the following result.

Corollary 2.2 *Let the function $f \in A(p)$ and suppose that $g \in S^{m,j}_{p,\lambda,\ell}(\gamma)$. If $\left(J^m_p(\lambda, \ell)f(z)\right)^{(j)}$ is majorized by $\left(J^m_p(\lambda, \ell)g(z)\right)^{(j)}$ in U, then*

$$\left|\left(J^{m-1}_p(\lambda, \ell)f(z)\right)^{(j)}\right| \leq \left|\left(J^{m-1}_p(\lambda, \ell)g(z)\right)^{(j)}\right| \qquad (|z| < r_0)\,,$$

where $r_0 = r_0(p, \gamma, \lambda, \ell)$ is given by

$$r_0 = r_0(p, \gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4\left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|\left(\frac{p+\ell}{\lambda}\right)}}{2\left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|},$$

where $(k = 2 + \left(\frac{p+\ell}{\lambda}\right) + \left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|, \lambda > 0, \ell \geq 0, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Putting $A = 1$, $B = -1, \lambda = 1, \ell = 0$ and $m = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.3 *[2, Theorem 1]. Let the function $f \in A(p)$ and suppose that $g \in S^j_p(\gamma)$. If $f^{(j)}(z)$ is majorized by $g^{(j)}(z)$ in U, then*

$$\left|f^{(j+1)}(z)\right| \leq \left|g^{(j+1)}(z)\right| \qquad (|z| < r_0)\,,$$

where $r_0 = r_0(p, j, \gamma)$ *is given by*

$$r_0 = r_0(p, j, \gamma) = \frac{k - \sqrt{k^2 - 4\left|2\gamma - (p-j)\right|(p-j)}}{2\left|2\gamma - (p-j)\right|},$$

where $(k = 2 + (p-j) + \left|2\gamma - (p-j)\right|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

By using Lemma 2.1 and Corollary 2.2, we obtain the following result.

Corollary 2.4 *[2, Theorem 2]. Let the function* $f \in A(p)$ *and suppose that* $g \in K_p^j(\gamma)$. *If* $f^{(j)}(z)$ *is majorized by* $g^{(j)}(z)$ *in* U, *then*

$$\left|f^{(j+1)}(z)\right| \le \left|g^{(j+1)}(z)\right| \qquad (|z| < r_0),$$

where $r_0 = r_0(p, \gamma, j)$ *is given by*

$$r_0 = r_0(p, \gamma, j) = \frac{k - \sqrt{k^2 - 4(p-j)\left|\gamma - (p-j)\right|}}{2\left|\gamma - (p-j)\right|},$$

where $(k = 2 + (p-j) + \left|\gamma - (p-j)\right|, p \in \mathbb{N}, j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*)$.

Putting $A = 1$, $B = -1, p = 1, j = 0, \lambda = 1, \ell = 0$ and $m = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.5 *[3,12]. Let the function* $f \in A$ *and suppose that* $g \in S(\gamma)$. *If* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$\left|f'(z)\right| \le \left|g'(z)\right| \qquad (|z| < r_0),$$

where $r_0 = r_0(\gamma)$ *is given by*

$$r_0 = r_0(\gamma) = \frac{k - \sqrt{k^2 - 4\left|2\gamma - 1\right|}}{2\left|2\gamma - 1\right|},$$

where $(k = 3 + \left|2\gamma - 1\right|, \gamma \in \mathbb{C}^*)$.

Putting $\gamma = 1$ in Corollary 2.4, we obtain the following result.

Corollary 2.6 *[16,12]. Let the function* $f \in A$ *and suppose that* $g \in S^*$. *If* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$\left|f'(z)\right| \le \left|g'(z)\right| \qquad (|z| < r_0),$$

where r_0 *is given by*

$$r_0 = 2 - \sqrt{3}.$$

Corollary Putting $\lambda = 1$ and $\ell = 0$ in Corollary 2.1, we obtain the following result.

Corollary 2.7 . *Let the function* $f \in A(p)$ *and suppose that* $g \in S_p^{m,j}(\gamma)$. *If* $\left(J_p^m f(z)\right)^{(j)}$ *is majorized by* $\left(J_p^m g(z)\right)^{(j)}$ *in* U, *then*

$$\left|\left(J_p^{m-1} f(z)\right)^{(j)}\right| \leq \left|\left(J_p^{m-1} g(z)\right)^{(j)}\right| \qquad (|z| < r_0)\,,$$

where $r_0 = r_0(p, \gamma)$ *is given by*

$$r_0 = r_0(p, \gamma) = \frac{k - \sqrt{k^2 - 4p\,|2\gamma - p|}}{2\,|2\gamma - p|},$$

where $(k = 2 + p + |2\gamma - p|\,, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Putting $m = 1$ in Corollary 2.6, we obtain the following result.

Corollary 2.8 . *Let the function* $f \in A(p)$ *and suppose that* $g \in S_p^{1,j}(\gamma)$. *If* $\left(J_p^1 f(z)\right)^{(j)}$ *is majorized by* $\left(J_p^1 g(z)\right)^{(j)}$ *in* U, *then*

$$\left|f^{(j)}(z)\right| \leq \left|g^{(j)}(z)\right| \qquad (|z| < r_0)\,,$$

where $r_0 = r_0(p, \gamma)$ *is given by*

$$r_0 = r_0(p, \gamma) = \frac{k - \sqrt{k^2 - 4p\,|2\gamma - p|}}{2\,|2\gamma - p|},$$

where $(k = 2 + p + |2\gamma - p|\,, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Remark 2.9

(i) *Putting* $\lambda = 1$ *in Corollary 2.1, we obtain the corresponding result for the operator* $J_p^m(\ell)f(z)$;
(ii) *Putting* $\ell = 0$ in *Corollary* 2.1, *we obtain the corresponding result for the operator* $J_{p,\lambda}^m f(z)$.

3 Open Problem

The author suggest to solve the majorization problem for the meromorphic p-valent functions $f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_n z^{n-p}$ using the analogues operator

$$\varphi_p^m(\lambda, \ell)f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{\ell + \lambda n}{\ell}\right)^m a_n z^{n-p}$$

$$(\lambda \geq 0; \ell > 0; m \in \mathbb{Z}; z \in U).$$

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